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MARTINGALES OF VECTOR VALUED SET FUNCTIONS

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This paper is concerned with the norm convergence of Banach space valued martingales in Orlicz spaces whose underlying measure is (possibly) only finitely additive. Because of the possible incompleteness of these Orlicz spaces of measurable point functions, this subject will be treated in the setting of Orlicz spaces of set functions V^{Φ} rather than the corresponding spaces L^{Φ} of measurable point functions. First, a conditional expectation P_B, operating on finitely additive set functions, is introduced and related to the usual conditional expectation E^B operating on L^1 by the equality

$$(*) \qquad P_B(F)(E) = \int_E E^B(f)d\mu \qquad E \in \Sigma$$

where (Ω, Σ, μ) is a measure space, B is a sub σ-field of Σ and $F(E) = \int_E f d\mu$ for $E \in \Sigma$.

Then, with the use of P_B martingales of set functions are defined and their convergence in appropriate V^{Φ} spaces is investigated. In addition, in the countably additive case, the results obtained for martingales of set functions are related to martingales of measurable point functions and extensions of certain results of Scalora, Chatterji, and Helms are obtained.

The study of finitely additive set functions appears to have begun during the close of the last century with such notions as Jordan Content. Through the first half of this century, with the introduction of the Lebesgue theory, most effort was concentrated on countably additive set functions. Recently, however, certain work, such as representations of linear functionals of the space of bounded functions has demanded the employment of finitely additive set functions. More important is the fact that finitely additive set functions provide considerable flexibility in applications and are sometimes no more untractable than their countably additive counterparts.

In their new approach to probability theory, Dubins and Savage [6] have noted that countable additivity is sometimes unnecessarily restrictive and have dropped it. In the study of the classical function spaces L^p, Bochner [1] and Leader [12] find it "natural to consider" the L^p spaces of finitely additive set functions. More recently in [16, 17] Bochner and Leader's groundwork was placed in the Orlicz space setting. In various ways, each of these papers present the argument that certain classical results can be handled more easily with the set function approach and perhaps more importantly, that new results may be obtained by employing this approach.

The purpose of the present paper is to treat the theory of norm convergence of martingales in Orlicz spaces, not in the classical manner, but rather to treat this theory in the setting of finitely additive set functions in the context of [16] and [17]. Here again, the goal will be to reduce to a minimum the limiting processes needed in the study of mean martingale convergence.

In the first section, preliminaries including relevant facts about the $V^{\Phi}(X)$[16, 17] spaces are given and collected for ready reference. The second section introduces a generalized conditional expectation operator which operates on vector valued finitely additive set functions. Properties of this generalized conditional expectation are exploited in the third section where martingales of finitely additive set functions are defined and studied. Here extensions of certain known results of Scalora [15], Chatterji [4], Helms [9], and Krickeberg and Pauc [11] are obtained.

1. Preliminaries. Throughout this paper Ω is a point set: Σ is a field of subsets of Ω, and μ is a finitely additive (extended) real valued nonnegative set function defined on Σ. $\Sigma_0 \subset \Sigma$ is the ring of sets of finite μ-measure. X is a Banach space. Φ is a Young's function [18] with complementary function Ψ.

By $V^{\Phi}(\Sigma, X)$ is meant the linear space of all finitely additive, μ-continuous[1], X-valued set functions F defined on Σ_0 which satisfy

$$I_{\Phi}(F/k) = \sup_{\pi} \sum_{\pi} \Phi(\| F(E_n) \| / k\mu(E_n))\mu(E_n) \leqq 1 \tag{1.1}$$

for some $k > 0$, where the supremum is taken over all partitions π consisting of a finite collection $\{E_n\}$ of disjoint members Σ_0 and the convention $0/0 = 0$ is observed. Upon the introduction of the norm N_{Φ} defined for $F \in V^{\Phi}(\Sigma, X)$ by

$$N_{\Phi}(F) = \inf \{k > 0 \colon I_{\Phi}(F/k) \leqq 1\}\ , \tag{1.2}$$

$V^{\Phi}(\Sigma, X)$ becomes a Banach space [16, 17].

A partition π is a finite collection $\{E_n\}$ of disjoint members of Σ_0. The partitions are partially ordered by defining $\pi_1 \leqq \pi_2$ if each members of π_1 can be written as a union of members of π_2. Corresponding to each to each $F \in V^{\Phi}(\Sigma, X)$ and each partition $\pi = \{E_n\}$ is the function

* F is μ-continuous if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\mu(E) < \delta$ implies $\| F(E) \| < \varepsilon$.

(1.3) $$F_\pi = \sum_\pi \frac{F(E_n)}{\mu(E_n)} \mu \cdot E_n$$

where $\mu \cdot E_n$ is the set function defined for $E \in \Sigma_0$ by $\mu \cdot E_n(E) = \mu(E \cap E_n)$. A set function of the form E_π will be termed a step function. The introduction of F_π allows us to single out a (possibly proper) closed subspace of $V^\Phi(\Sigma, X)$. By $S^\Phi(\Sigma, X)$ is meant the collection of all $F \in V^\Phi(\Sigma, X)$ such that $\lim_\pi N_\Phi(F_\pi - F) = 0$ where the limit is taken in the Moore-Smith sense through all partitions π.

THEOREM 1.1. *If X is reflexive and Φ obeys the Δ_2-condition ($\Phi(2x) \leqq K\Phi(x)$ for some K and all x), then $S^\Phi(\Sigma, X) = V^\Phi(\Sigma, X)$.*

The proof of this theorem may be found in [17, IV. 7]. If, $\Phi(x) = |x|$, then the corresponding $V^\Phi(\Sigma, X)$ and $S^\Phi(\Sigma, X)$ will be denoted by $V^1(\Sigma, X)$ and $S^1(\Sigma, X)$ respectively.

As usual, the Orlicz space $L^\Phi(\Sigma, X)$ is the space of all totally μ-measurable X-valued functions f which satisfy

(1.4) $$\int_\Omega \Phi(\|f\|/k)d\mu \leqq 1$$

for some k where the integral here and thoughout this paper is that of [7, Chap. III]. With functions which differ only on a μ-null set [7, Chap. III] identified, $L^\Phi(\Sigma, X)$ becomes a normed linear space (complete if μ is countably additive) under the norm N_Φ defined for $f \in L^\Phi(\Sigma, X)$ by

(1.5) $$N_\Phi(f) = \inf\left\{k > 0: \int_\Omega \Omega(\|f\|/k)d\mu \leqq 1\right\}.$$

The use of identical symbols for the $V^\Phi(\Sigma, X)$ and the $L^\Phi(\Sigma, X)$ norms will be justified in the next result. No confusion should arise since set functions will normally be denoted by upper case letters, while point functions will be denoted by lower case letters.

A nonnegative set function G defined on Σ is said to have the finite subset property if $E \in \Sigma, G(E) = \infty$ implies the existence of $E_0 \subset E, E_0 \in \Sigma$ such that $0 < G(E_0) < \infty$.

THEOREM 1.2. *Suppose μ has the finite subset property. The mapping $\lambda: L^\Phi(\Sigma, X) \to V^\Phi(\Sigma, X)$ defined for $f \in L^\Phi(\Sigma, X)$ by $\lambda f(E) = \int_E f d\mu, E \in \Sigma_0$, is an isometric injection of $L^\Phi(\Sigma, X)$ into $V^\Phi(\Sigma, X)$. If μ is countably additive, Σ is a σ-field, X is reflexive, and Φ obeys the Δ_2-condition, then the range of λ is all of $V^\Phi(\Sigma, X)$.*

The proof of this theorem may be synthesized from [17, II. 5], [17, IV. 8], and the fact that if μ is countably additive, $L^{\Phi}(\Sigma, X)$ is complete.

2. A generalized conditional expectation. The purpose of this section is to define and explore the properties of a generalized conditional expectation operator operating on finitely additive set functions. An attempt will be made to relate this operator and its properties to the usual conditional expectation [15] operating on point functions.

DEFINITION 2.1. A class of sets $B \subset \Sigma$ is a subfield of Σ if and only if B is a ring and $\Omega \in B$. A partition π_B $\{E_n\}$ is a B-partition if $\{E_n\} \subset B$.

DEFINITION 2.2. Let B be a subfield of Σ. A set function $F \in V^{\Phi}(\Sigma, X)$ is termed B-measurable if for each $E \in \Sigma_0$,

$$F(E) = \lim F_{\pi_B}(E) \tag{2.1}$$

where the limit is taken in the Moore-Smith sense through all B-partitions π_B.

The following result establishes the existence of an operator analogous to the usual conditional expectation [13, 15].

THEOREM 2.3. *Let Φ obey the Δ_2-condition and B be a subfield of Σ. Then for each $F \in S^{\Phi}(\Sigma, X)$ there exists a B-measurable set function $F_B \in S^{\Phi}(\Sigma, X)$ such that*

(i) $F_B(E) = F(E)$ *for all* $E \in B \cap \Sigma_0$,

(ii) $N_\Phi(F_B) \leqq N_\Phi(F)$,

and

(iii) $\lim_{\pi_B} N_\Phi(F_B - (F_B)_{\pi_B}) = 0$,

where the limit is taken in the Moore-Smith sense through all B-partitions π_B.

Proof. Let $F \in S^{\Phi}(\Sigma, X)$ be arbitrary and consider the mapping $\theta: S^{\Phi}(\Sigma, X) \to V^{\Phi}(B, X)$ defined by $\theta(F) = F \mid B$ where $F I B$ is the restriction of F to $B \cap \Sigma_0$. The linearity of θ is clear. Moreover for any $k > 0$,

$$\begin{aligned} I_\Phi(\theta F/K) &= \sup_{\pi B} \sum_{\pi B} \Phi(\| F(E_n) \|/k\mu(E_n))\mu(E_n) \\ &\leqq \sup_{\pi} \sum_{\pi} \Phi(\| F(E_n) \|/k\mu(E_n))\mu(E_n) \leqq I_\Phi(F/k) \,. \end{aligned} \tag{2.2}$$

From inequality (2.2) and the definition of the N_Φ norm, it follows immediately that $N_\Phi(\theta F) \leqq N_\Phi(F)$. Thus θ is a linear contraction.

Next we shall show that the range of Φ is contained in $S^{\varphi}(B, X)$ (which is possibly strictly contained in $V^{\varphi}(\Sigma, X)$). Because θ is a contradiction, it suffices to show θ maps step functions (i.e., functions of the form F_{π}) into $S^{\varphi}(B, X)$. From the linearity of θ, we can infer that this reduces to showing that $\theta(x\mu\cdot E) \in S^{\varphi}(B, X)$ for each $x \in X$ and all $E \in \Sigma_0$. Thus, let $x \in X$, $E \in \Sigma_0$, and $\pi_B = \{E_n\}$ be a B-partition. A brief computation yields

$$(2.3)\quad N_{\Phi}(\theta(x\mu\cdot E) - (\theta(x\mu\cdot E))_{\pi_B}) = \|x\| N_{\Phi}(\mu\cdot E \mid B - (\mu\cdot E \mid B)_{\pi_B})$$

where the last norm is taken in $V^{\varphi}(B, R)(R = \text{reals})$. Since $\mu\cdot E \mid B$ is $\mu \mid B$-continuous and satisfies $I_{\Phi}(\mu\cdot E \mid B) < \infty$, and Φ obeys the Δ_2-condition, Theorem 1.1 implies $\mu\cdot E \mid B \in S^{\varphi}(B, R)$. Thus

$$\lim_{\pi_B} N_{\Phi}(\mu\cdot E \mid B - (\mu\cdot E \mid B)_{\pi_B}) = 0 .$$

In view of this, the definition of $S^{\varphi}(B, X)$ and (2.3), we have $\theta(x\mu\cdot E) \in S^{\varphi}(B, X)$. This proves $\theta(S^{\varphi}(\Sigma, X)) \subset S^{\varphi}(B, X)$.

Leaving, for the time being, the problem of projecting $S^{\varphi}(\Sigma, X)$ into $S^{\varphi}(B, X)$, we shall now consider the opposite problem: the extension of members of $S^{\varphi}(B, X)$ to members of $S^{\varphi}(\Sigma, X)$. Let $G \in S^{\varphi}(B, X)$ and π_B be a B-partition. Then for $E \in B \cap \Sigma_0$,

$$G_{\pi_B}(E) = \sum_{\pi_B} \frac{G(E_n)}{\mu \mid B(E_n)} (\mu\cdot E_n \mid B)(E) .$$

Clearly G_{π_B} has a "natural" extension to all of Σ_0-namely

$$\sum_{\pi_B} \frac{G(E_n)}{\mu(E_n)} \mu\cdot E_n ,$$

which is defined for all $E \in \Sigma_0$. Denote this extension by $\rho(G_{\pi_B})$. Then evidently $\rho(G_{\pi_B}) \in S^{\varphi}(\Sigma, X)$, and clearly ρ is linear. Moreover, as a brief computation shows, $N_{\Phi}(G_{\pi_B}) = N_{\Phi}(\rho(G_{\pi_B}))$.

Now, for $G \in S_{\Phi}(B, X)$, we have

$$\lim_{\Delta_B, \pi_B} N_{\Phi}(G_{\Delta_B} - G_{\pi_B}) = 0 .$$

Hence

$$\lim_{\Delta_B, \pi_B} N_{\Phi}(\rho(G_{\Delta_B}) - \rho(G_{\pi_B})) = 0 .$$

This and the completeness of $S^{\varphi}(\Sigma, X)$ assures the existence of $\rho(G) \in S^{\varphi}(\Sigma, X)$ such that

$$(2.4)\qquad \lim_{\pi_B} N_{\Phi}(\rho(G) - \rho(G_{\pi_B})) = 0 .$$

Moreover,

$$N_\Phi(G) = \lim_{\pi_B} N_\Phi(G_{\pi_B}) = \lim_{\pi_B} N_\Phi(\rho(G_{\pi_B})) = N_\Phi(\rho(G)) \ .$$

Also note that if $E \in B \cap \Sigma_0$, then $G(E) = \lim_{\pi_B} G_{\pi_B}(E) = \lim_{\pi_B} \rho(G_{\pi_B})(E)$, by the definition of $\rho(G_{\pi_B}) = \rho(G)(E)$, since norm convergence in V^Φ implies setwise convergence for sets in Σ_0. Therefore we have

$$\rho(G) \mid B = G \ , \tag{2.5}$$

$$\rho(G)_{\pi_B} = \rho(G_{\pi_B}) \tag{2.6}$$

and

$$\lim_{\pi_B} N_\Phi(\rho(G) - \rho(G)_{\pi_B}) = 0 \ , \tag{2.7}$$

Now, to prove the theorem, let $F \in S^\Phi(\Sigma, X)$ and consider $F_B = \rho(\theta(F))$. By the definition of θ and (2.5) (with $\theta(F) = G$), $F_B(E) = F(E), E \in B \cap \Sigma_0$, and (i) is satisfied. Since θ is a contraction and ρ is norm preserving, $N_\Phi(F_B) \leqq N_\Phi(F)$, and (ii) is satisfied. (iii) follows immediately from (2.7).

A corollary of the proof of Theorem 2.3 is given below for use later.

COROLLARY 2.4. *Let Φ obey the Δ_2-condition and B be a subfield of Σ. Then there exists a "natural" isometric embedding ρ of $S^\Phi(B, X)$ into $S^\Phi(\Sigma, X)$. The image of $S^\Phi(B, X)$ under ρ consists* of *all B-measurable members of $S^\Phi(\Sigma, X)$.*

Proof. The assertions of this corollary are all clear form the proof of Theorem 2.3 with the possible exception of the linearity of ρ. It is clear from ρ's definition that ρ is linear on the step functions. Since step functions are dense in $S^\Phi(B, X)$, ρ is linear on all of $S^\Phi(B, \Sigma)$.

The above corollary allows us to think of $S^\Phi(B, X)$ as a subspace of $S^\Phi(\Sigma, X)$ in very much the same way that the B-measurable members of $L^\Phi(B, X)$ constitute a subspace of $L^\Phi(\Sigma, X)$. With the aid of theorem 2.3, an operator P_B which will be called a generalized conditional expectation and which is a genuine generalization of Kolmogrorov's classical concept of probability theory (cf. Theorem 2.7) can be defined.

DEFINITION 2.5. Let Φ obey the Δ_2-condition and B be a subfield of Σ. For $F \in S^\Phi(\Sigma, X)$, the operator P_B is defined by

$$P_B(F) = F_B \tag{2.8}$$

where $F \to F_B$ in the sense of Theorem 2.3.

The following theorem is an immediate consequence of Theorem 2.3.

THEOREM 2.6. *If Φ obeys the Δ_2-condition and B is a subfield of Σ, then*

(i) P_B *on* $S^{\Phi}(\Sigma, X)$ *is linear and contractive,*

(ii) $P_B(F) \mid B = F \mid B$,

and

(iii) $\lim_{\pi_B} N_\Phi(P_B(F) - F_{\pi_B}) = 0$.

The relationship between the operator P_B and the usual conditional expectation operator E^B[15, pp. 353–356] which operates on point functions is clarified in the next result.

THEOREM 2.7. *Suppose Σ is a σ-field, B is a sub-σ-field of Σ, and μ is a countably additive finite measure on Σ. If E^B is the usual conditional expectation operator on $L^1(\Omega, \Sigma, \mu, X)$ then $\lambda E^B(f) = P_B(\lambda f)$ for all f $L^1(\Omega, \Sigma, \mu, X)$ or equivalently,*

$$\int_E E^B(f)d\mu = P_B\Big(\int_{(\cdot)} fd\mu\Big)(E)$$

for all $E \in \Sigma$ where λ is the isometric isomorphism of L^1 into V^1 of Theorem 1.2.

Proof. Since simple functions are dense in $L^1(\Omega, \Sigma, \mu, X)(L^1(\Sigma, X))$, it suffices to prove the statement for all simple functions f. The linearity of E^B, P_B, and the integral allow us to reduce this problem to the problem of showing $E^B(f) = P_B(\lambda f)$ for all f of the form $f = x\chi_E$ where $x \in X, E \in \Sigma$, and χ_E is the characteristic or indicator function of E. The definition of E^B on $L^1(\Sigma, X)$ [15] and [7, IV. 8. 17] imply

$$E^B(x\chi_E) = xE^B(\chi_E) = \lim_{\pi_B} \sum_{B} \frac{\int_{E_n} xE^B(\chi_E)d\mu}{\mu(E_n)} \chi_{E_n}$$

strongly in $L^1(B, X)$ and therefore in $L^1(\Sigma, X)$. Hence by the continuity of λ,

$$\lambda(E^B(x\chi_E)) = \lim_{\pi_B} \lambda\left(\sum_{\pi_B} \frac{\int_{E_n} xE^B(\chi_E)d\mu}{\mu(E_n)} \chi_{E_n}\right),$$

$$\lim_{\pi_B} \sum_{\pi_B} \frac{\int_{E_n} x\chi_E d\mu}{\mu(E_n)} \mu\cdot E_n\ , \quad \text{since} \quad E_n \in B\ ,$$

$$\lim_{\pi_B} \sum_{\pi_B} \frac{\lambda(x\lambda_E)(E_n)}{\mu(E_n)} \mu \cdot E_n ,$$

$$\lim_{\pi_B} \sum_{\pi_B} \frac{\lambda f(E_n)}{\mu(E_n)} \mu \cdot E_n = P_B(\lambda f) ,$$

strongly in $V^1(\Sigma, X)$ by Theorem 2.6.

The crux of Theorem 2.7 is that the operator P_B is a genuine extension of the classical conditional expectation operator E^B. Indeed, the definition of P_B does not depend directly on the Radon-Nikodym theorem (which is not available in usable form), while the definition of E^B depends crucially on the Radon-Nikodym theorem. But, as shown above, P_B coincides with E^B whenever the Radon-Nikodym theorem is applicable. Another property common to P_B and E^B is contained in the next result.

THEOREM 2.8. *Let Φ obey the Δ_2-condition. If $B_1 \subset B_2$ are subfields of Σ, then*

$$P_{B_1}(P_{B_2}) = P_{B_2}(P_{B_1}) = P_{B_1}$$

on $S^\Phi(\Sigma, X)$. Consequently P_B is a contractive projection of $S^\Phi(\Sigma, X)$ into $S^\Phi(\Sigma, X)$.

Proof. If $G \in S^\Phi(\Sigma, X)$ is arbitrary, then according to Theorem 2.6,

$$P_{B_1}(G) = \lim_{\pi_{B_1}} G_{\pi_{B_1}} \quad \text{strongly in} \quad V^\Phi(\Sigma, X) .$$

Hence if $F \in S^\Phi(\Sigma, X)$,

$$P_{B_1}(P_{B_2}(F)) = \lim_{\pi_{B_1}} (P_{B_2}(F))_{\pi_{B_1}} \qquad \text{in } N_\Phi \text{ norm} .$$

Since $P_{B_2}(F)$ agrees with F on B_2-sets and $B_1 \subset B_2$, $P_{B_2}(F)$ agrees with F on B_1-sets. Therefore

$$(P_{B_2}(F))_{\pi_{B_1}} = F_{\pi_{B_1}}$$

for each B_1-partition π_{B_1}, and

$$P_{B_1}(P_{B_2}(F)) = \lim_{\pi_{B_1}} (P_{B_2}(F))_{\pi_{B_1}} = \lim_{\pi_{B_1}} F_{\pi_{B_1}} = P_{B_1}(F)$$

strongly in $V^\Phi(\Sigma, X)$. Hence $P_{B_1}(P_{B_2}) = P_{B_1}$.

To prove $P_{B_2}(P_{B_1}) = P_{B_1}$, the boundedness of P_{B_1} and P_{B_2} will be used. If $F \in S^\Phi(\Sigma, X)$, then Theorem 2.6 implies $P_{B_1}(F) = \lim_{\pi_{B_1}} F_{\pi_{B_1}}$ strongly in $U^\Phi(\Sigma, X)$. The boundedness of P_{B_2} yields

$$P_{B_2}(P_{B_1}(F)) = \lim_{\pi_{B_1}} P_{B_2}(F_{\pi_{B_1}}) .$$

But each B_1-partition π_{B_1} is a B_2-partition since $B_1 \subset B_2$. Therefore $P_{B_2}(F_{\pi_{B_1}}) = F_{B_1}$ and

$$P_{B_2}(P_{B_1}(F)) = \lim_{\pi_{B_1}} F_{\pi_{B_1}} = P_{B_1}(F)$$

by Theorem 2.6.

That P_B is a contractive projection follows from $P_B^2 = P_B$ and Theorem 2.6.

According to [17, I. 13], $V^{\Phi}(\Sigma, X) \subset V^1(\Sigma, X)$ for all Young's functions provided $\mu(\Omega) < \infty$, it is natural consider P_B applied to functions in $V^{\Phi}(\Sigma, X)$ which belong to $S^1(\Sigma, X)$. The (closed) subspace of such functions will be denoted by $R^{\Phi}(\Sigma, X)$. According to [17, I. 13], N_{Φ} dominates N_1, the variation norm of V^1, when $\mu(\Omega) < \infty$; hence $S^{\Phi}(\Sigma, X) \subset R^{\Phi}(\Sigma, X) \subset V^{\Phi}(\Sigma, X) \subset V^1(\Sigma, X)$. In the case $\mu(\Omega) = \infty$, as the preceding definitions and theorems show, P_B is directly defined on S^{Φ} only if Φ obeys the Δ_2-condition. Because stronger hypotheses on Φ are needed when $\mu(\Omega) = \infty$, some of the following theorems will be stated in two parts- the first part dealing with the case $\mu(\Omega) < \infty$ and the second part dealing with the case $\mu(\Omega) = \infty$.

Some properties of P_B on $R^{\Phi}(\Sigma, X)$ are collected below:

THEOREM 2.9. *Let $\mu(\Omega)$ be finite and B be a subfield of Σ: then*

(i) $P_B: R^{\Phi}(\Sigma, X) \to R^{\Phi}(\Sigma, X)$ *is a contractive projection.*

(ii) $P_B(F)(E) = \lim_{\pi_B} F_{\pi_B}(E)$ *in X for each F in $R^{\Phi}(\Sigma, X)$ and $E \in \Sigma$.*

(iii) $P_B(F) \mid B = F \mid B$ *for all $F \in R^{\Phi}(\Sigma, X)$.*

Proof. Let $F \in R^{\Phi}(\Sigma, X)$. (iii) is simply Theorem 2.6 applied to $S^1(\Sigma, X)$. (ii) follows directly from Theorem 2.6 (iii) applied to $S^1(\Sigma, X)$. To prove (i), note that P_B maps $S^1(\Sigma, X)$ into $S^1(\Sigma, X)$. Therefore, to show P_B is a contractive projection on $R^{\Phi}(\Sigma, X)$, it suffices to show $N_{\Phi}(P_B(F)) \leqq N_{\Phi}(F)$ for $F \in R^{\Phi}(\Sigma, X)$. (This has already been established in the case Φ obeys the Δ_2-condition. A separate proof is furnished for the general Φ). From (ii) and the "lower semi-continuity" of I_{Φ}[17, I. 7], it follows that for each $k > 0$,

$$I_{\Phi}(P_B(F)/k) \leqq \liminf_{\pi_B} I_{\Phi}(F_{\pi_B}/k) \leqq I_{\Phi}(F/k) .$$

Hence $N_{\Phi}(P_B(F) \leqq N_{\Phi}(F)$

3. Martingales of additive set functions. The study of martingales of point functions in the finitely additive context appears to be somewhat intractable because of the possible incompleteness of the

$L^{\Phi}(\Sigma, X)$ spaces when the underlying measure μ is only finitely additive. However, even when μ is only finitely additive, the corresponding $V^{\Phi}(\Sigma, X)$ spaces are complete. Primarily for this reason we shall deal with martingales from a set function standpoint. In addition, as will be seen later, the assumption of only finite additivity will not present any special difficulties. Using a definition equivalent to that of Krickeberg and Pauc [11] we now define a martingale of additive set functions. This is a generalization of the classical concept of Doob [5] to the setting of finitely additive set functions (cf. Theorem 3.2).

DEFINITION 3.1. Let $\{B_\tau, \tau \in T\}$ be an increasing net of subfields of Σ (i.e., $B_1 \subset B_2$ if $\tau_1 \leqq \tau_2$). A $V^{\Phi}(\Sigma, X)(S^{\Phi}(\Sigma, X), R^{\Phi}(\Sigma, X))$-martingale is a net of finitely additive set functions $\{F_\tau, B_\tau, \tau \in T\}$ such that $F_\tau \in V^{\Phi}(\Sigma, X)(S^{\Phi}(\Sigma, X)\, R^{\Phi}(\Sigma, X))$ for each $\tau \in T$ and $P_{B_\tau}(F_{\tau_2}) = F_{\tau_1}$ for $\tau_1 \leqq \tau_2$.

In some important cases, martingales of set functions and martingales of measurable point functions [15, p. 358] can be indentified under the isometric isomorphism λ: $L^{\Phi}(\Sigma, X) \to V^{\Phi}(\Sigma, X)$ of Theorem 1.2. This is made precise in the following result.

THEOREM 3.2. *Let Σ be a σ-field and $\{B_\tau, \tau \in T\}$ be an increasing net of sub-σ-fields of Σ. If μ is a countably additive finite measure on Σ, then $\{F_\tau, B_\tau, \tau \in T\}$ is an $R^{\Phi}(\Sigma, X)$-martingale if and only if $\{\lambda^{-1}(F_\tau), B_\tau, \tau \in T\}$ is a martingale (of point functions) in $L^{\Phi}(\Sigma, X)$. This means if $f_\tau = \lambda^{-1}(F_\tau)$*

(i) *Each f_τ is B_τ-measurable,*

and

(ii) $\int_E f_{\tau_1} d\mu = \int_E f_{\tau_2} d\mu$ for all $\tau_2 \geqq \tau_1$ *and* $E \in B_{\tau_1}$

Proof. (Necessity). Because of the definition of $R^{\Phi}(\Sigma, X)$, it may and will be assumed that $R^{\Phi}(\Sigma, X) = S^1(\Sigma, X)$. The hypothesis guarantees $\lambda(L^1(\Sigma, X)) = S^1(\Sigma, X)$; so that $\lambda^{-1}(F_\tau) = f_\tau \in L^1(\Sigma, X)$ for each $\tau \in T$. From Theorem 2.6 and the definition of an $R^{\Phi}(\Sigma, X)$-martingale, it follows that

$$\lim_{\pi B_\tau} N_1(F_\tau - (F_\tau)_{\pi B_\tau}) = 0 .$$

Hence

$$\lim_{B} N_1(\lambda^{-1}(F_\tau) - \lambda^{-1}((F_\tau)_{\pi B_\tau})) = 0 .$$

Since the $\lambda^{-1}(F_\tau)_{\pi B_\tau}$ are B_τ-simple functions, it follows that $f_\tau = \lambda^{-1}(F_\tau)$ is B_τ-measurable for each τ. This establishes (i). (ii) follows directly from Definition 3.1; if $\lambda^{-1}(F_\tau) = f_\tau$, then for each B_{τ_1} and $\tau_2 \geqq \tau_1$, we have

$$\int_E f_{\tau_1} d\mu = F_{\tau_1}(E) = F_{\tau_2}(E) = \int_E f_{\tau_2} d\mu ,$$

for $E \in B_\tau$.

(Sufficiency). If $\{f_\tau, B_\tau, \tau \in T\}$ is a martingale in $L^1(\Sigma, X)$ and $F_\tau(\cdot) = \lambda f_\tau(\cdot) = \int_{(\cdot)} f_\tau d\mu$ for each $\tau \in T$, (i) implies $F_{\tau_1}(E) = F_{\tau_2}(E)$ for each $E \in B_{\tau_1}$ and all $\tau_2 \geqq \tau_1$. Hence

$$F_{\tau_1} = \lambda f_{\tau_1} = \lambda E^{B_{\tau_1}}(f_{\tau_2}) = P_{B_{\tau_1}}(\lambda f_{\tau_2}) = P_{B_{\tau_1}}(F_{\tau_2})$$

by Theorem 2.7. Hence $\{F_\tau, B_\tau, \tau \in T\}$ is a martingale in $S^1(\Sigma, X)$.

We shall now begin a study of the norm or mean convergence of martingales of set functions. The main result dealing with this problem is contained in the following theorem.

THEOREM 3.3. (i) *Let $\mu(\Omega) < \infty$ and $\{F_\tau, B_\tau, \tau \in T\}$ be an $R^\varphi(\Sigma, X)$ martingale. If $\Sigma_1 = \bigcup_\tau B_\tau$ and the function F_1 defined by $\lim_\tau F_\tau(E) = F_1(E)$ for $E \in \Sigma_1$ belongs to $S^\Phi(\Sigma_1, X)$, then the net $\{F_\tau, B_\tau, \tau \in T\}$ converges in the $V^\Phi(\Sigma, X)$ norm.*

(ii) *If $\mu(\Omega) = \infty$, and Φ obeys the Δ_2-condition, the same conclusion holds if $\{F_\tau, B_\tau, \tau \in T\}$ is a martingale in $S^\Phi(\Sigma, X)$ and the function F_1 defined by $\lim_\tau F_\tau(E) = F_1(E)$ for $E \in \Sigma_1 \cap \Sigma_0$ belongs to $S^\Phi(\Sigma_1, X)$ where Σ_0 is the ring of sets in Σ of finite μ-measure.*

Proof. (i) Since $\{B_\tau, \tau \in T\}$ is an increasing net of subfields of Σ, it is clear that $\Sigma_1 = \bigcup_\tau B_\tau$ is a subfield of Σ. From Definition 3.1, it follows immediately that $F_1(E) = \lim_\tau F_\tau(E)$ exists for each $E \in \Sigma_1$. Now consider the net $\{F_\tau \mid \Sigma_1, \tau \in T\}$ in $V^\Phi(\Sigma, X)$. By hypothesis F_1 belongs to $S^\Phi(\Sigma_1, X)$. Therefore if $\varepsilon > 0$ is given, there exists a Σ_1-partition π such that $N_\Phi(F_1 - F_{1_\pi}) < \varepsilon/2$. By virtue of the facts that $\Sigma_1 = \bigcup_\tau B_\tau$ and $\{B_\tau, \tau \in T\}$ is an increasing net of fields, there exists a B_{τ_0} such that $\pi \subset B_\tau$ for all $\tau \geqq \tau_0$. Moreover Corollary 2.4 guarantees the existence of a Σ_1-measurable function F defined on all of Σ such that $N_\Phi(F - F_\pi) < \varepsilon/2$. In addition, from the definition of P_{B_τ}, we have $P_{B_\tau}(F) = F_\tau$ for all $\tau \in T$ and $P_{B_\tau}(F_\pi) = F_\pi$ for all $\tau \geqq \tau_0$. Therefore for $\tau \geqq \tau_0$, the triangle inequality yields

$$N_\Phi(F - F_\tau) \leqq N_\Phi(F - F_\pi) + N_\Phi(F_\pi - F_{B_\tau})$$
$$N_\Phi(F - F_\pi) + N_\Phi(P_{B_\tau}(F - F_\pi)) \leqq 2N_\Phi(F - F_\pi) < \varepsilon ,$$

since P_{B_τ} is a contraction. Consequently $\lim_\tau N_\Phi(F - F_\tau) = 0$.

The proof of (ii) is the same with a few obvious modifications.

Because of its generality, the hypothesis of Theorem 3.3 may be somewhat difficult to verify. The utilization of Theorem 1.1 permits us to state some corollaries which are more easily applied.

COROLLARY 3.4. *Let X be reflexive and Φ obey the Δ_2-condition. If $\{F_\tau, B_\tau, \tau \in T\}$ is a $V^\Phi(\Sigma, X)$-martingale which satisfies*

(i) $N_\Phi(F_\tau) \leqq M < \infty$ for some M and all $\tau \in T$ and

(ii) $\nu(F_\tau, \cdot)$ are uniformly μ-continuous (but not necessarily finite) where $\nu(F_\tau, E)$ is the variation of F_τ on E for $E \in \Sigma_0$, then

(a) the net $\{F_\tau, \tau \in T\}$ converges in $V^\Phi(\Sigma, X)$ norm,

and

(b) If Ψ, the complementary function to Φ, is also continuous (ii) *of the hypothesis may be dropped.*

Proof. (a) Since Φ obeys the Δ_2-condition and X is reflexive, Theorem 1.1 implies $S^\Phi(\Sigma, X) = V^\Phi(\Sigma, X)$; therefore Theorem 3.3 is applicable. According to this theorem, it need only be shown that the set function F_1 defined on $\Sigma_1 \cap \Sigma_0$, where $\Sigma_1 = \bigcup_\tau B_\tau$, by $F_1(E) = \lim_\tau F_\tau(E)$ belongs to $S^\Phi(\Sigma, X)(= V^\Phi(\Sigma, X))$. The "lower semi-continuity" of $I_\Phi(\cdot)$ [17, I. 7] yields for each $k > M$

$$I_\Phi(F_1/k) = \liminf_\tau I_\Phi(F_\tau \mid \Sigma_1/k) \leqq \sup_\tau \{I_\Phi(F_\tau/k)\} \leqq 1 ,$$

since $N_\Phi(F_\tau) \leqq M$ for each $\tau \in T$. In addition, from the "lower semi-continuity" of the variation ν, one has $\nu(F_1, E) \leqq \liminf \nu(F_\tau, E)$ for all $E \in \Sigma_1$. Since the $\nu(F, \cdot)$ are uniformly μ-continuous, it follows that $\nu(F_1, \cdot)$ is $\mu \mid \Sigma_1$-continuous and hence that F_1 is $\mu \mid \Sigma_1$-continuous. Therefore $F_1 \in V^\Phi(\Sigma_1, X) = S(\Sigma_1, X)$. This proves (a).

(b) If Ψ is continuous, then the fact that $I_\Phi(F_1/k) < \infty$ for some k guarantees F_1 is $\mu \mid \Sigma_1$-continuous by [17, I. 17]. Hence (ii) may be dropped.

A specialization of Corollary 3.4 to martingales of point functions yields the following result.

COROLLARY 3.5. *Let Σ be a σ-field and μ be countably additive and finite on Σ. If X is reflexive, Φ satisfies the Δ_2-condition and $\{f_\tau, B_\tau, \tau \in T\}$ is a martingale in $L^\Phi(\Sigma, X)$ such that*

(i) $N_\Phi(f_\tau) \leqq M < \infty$ for some M and all $\tau \in T$,

and

(ii) the functions $\|f_\tau\|_X$ are uniformly integrable,

then

(a) the net $\{f_\tau, \tau \in T\}$ converges in the L^Φ norm.

(b) If Ψ, the complementary function to Φ, is continuous, then

(ii) *of the hypothesis may be dropped.*

Proof. (a) Let $F_\tau = \lambda f_\tau$. According to Theorem 3.2, $\{F_\tau, B_\tau, \tau \in T\}$ is a martingale of set functions. Since λ is an isometry, $N_\Phi(F_\tau) = N_\Phi(f_\tau) \leqq M$ for all $\tau \in T$. From (ii) $\nu(F_\tau, \cdot) = \int_{(\cdot)} \|f_\tau\|_X \, d\mu$ are uniformly μ-continuous because the $\|f_\tau\|_X$ are uniformly integrable. An application of Corollary 3.4 shows $\{F_\tau, \tau \in T\}$ converges in $V^\Phi(\Sigma, X)$ norm. Since the current hypothesis guarantees that $L^\Phi(\Sigma, X)$ is complete, and λ is an isometry, it follows that $\{f_\tau, \tau \in T\}$ converges in $L^\Phi(\Sigma, X)$.

(b) This follows directly from the above and Corollary 3.4.

Corollary 3.5 extends one of the main results of Chatterji [4, Th. 3] in two ways: The index set T is possibly uncountable, and the convergence is in $L^\Phi(\Sigma, X)$ while in [4], the convergence is in $L^p(\Sigma, X)$, $1 \leqq \rho < \infty$. Furthermore the methods of proof in [4] do not seem to apply to the more general setting of Corollary 3.5.

A set function martingale version of a theorem of Krickeberg and Pauc [11, Th. 6, p. 500] is contained in

THEOREM 3.6. (a) *Suppose $\mu(\Omega) < \infty$ and $\{F_\tau, B_\tau, \tau \in T\}$ is a martingale in $R^\Phi(\Sigma, X)$. Then the following are equivalent*:

(i) *$\{F_\tau, \tau \in T\}$ converges in the strong topology of $V^\Phi(\Sigma, X)$.*

(ii) *$\{F_\tau, \tau \in T\}$ converges in the weak topology of $V^\Phi(\Sigma, X)$.*

(iii) $\lim_\tau x^*(F_\tau(E)) = x^*(z_E)$ *for each E some $x_E \in X$, and all $x^* \in X^*$, the conjugate space to X. The function $F_\infty(E) = x_E$ for $E \in \Sigma$, is a member of $V^\Phi(\Sigma, X)$ and for each $\varepsilon > 0$, there exists a parameter $\tau_\varepsilon \in T$ and a B_{τ_ε}-measurable function $G_{\tau_\varepsilon} \in R^\Phi(\Sigma, X)$ such that $N_\Phi(G_{\tau_\varepsilon} - F_\infty) < \varepsilon$.*

(b) *If $\mu(\Omega) = \infty$, thetheorem remains true provided Φ obeys the Δ_2-condition, R^Φ is replaced by S^Φ and the limit in* (iii) *is taken only for $E \in \Sigma_0$.*

Proof. (i) $\longrightarrow$ (ii) is obvious. (ii) $\longrightarrow$ (iii): Let $x^* \in X^*$ and $E \in \Sigma$. It is easily seen that the functional l defined for $F \in V^\Phi(\Sigma, X)$ by $l(F) = x^*(F(E))$ is a bounded linear functional on $V^\Phi(\Sigma, X)$. By (ii), the net $\{F_\tau, \tau \in T\}$ converges in the weak topology to some $H \in V^\Phi(\Sigma, X\}$. Therefore $x^*(H(E)) = \lim_\tau x^*(F_\tau(E))$ for each $E \in \Sigma_0$ and $x^* \in X^*$. This is the first part of (iii) with $H = F_\infty$. Now let M be the collection of all $V^\Phi(\Sigma, X)$ functions which are B_τ-measurable for some $\tau \in T$. From the fact that $\{B_\tau, \tau \in T\}$ is an increasing net of subfields of Σ, it follows that M is a linear submanifold of $V^\Phi(\Sigma, X)$. But $\{F_\tau, \tau \in T\}$ converges weakly to F_∞ and each $F_\tau \in M$. Hence F_∞ belongs to the strong closure of M since the weak and strong closure of a linear manifold are identical. Consequently for each $\varepsilon > 0$, there exists a parameter $\tau \in T$ and a B_τ-measurable function G_{τ_ε} such that $N_\Phi(G_{\tau_\varepsilon} - F_\infty) < \varepsilon$. This proves (ii) $\to$ (iii).

(iii) $\to$ (i). Let $\varepsilon > 0$ be given. From (iii), there exists a $\tau_2 \in T$ and a B_{τ_ε}-measurable function G_{τ_ε} such that $N_\Phi(G_{\tau_\varepsilon} - F_\infty) < \varepsilon/2$. Since ε is arbitrary and G_{τ_ε} belongs to the closed subspace $R^\Phi(\Sigma, X)$ of $V^\Phi(\Sigma, X)$, we have $F_\infty \in R^\Phi(\Sigma, X)$. Hence $P_{B_\tau}(F_\infty)$ is defined for each $\tau \in T$. In addition, $P_{B_\tau}(F_\infty)$ and F_τ are both B_τ-measurable and agree on B_τ-sets. It follows that $P_{B_\tau}(F_\infty) = F_\tau$. Moreover since G_{τ_ε} is B_{τ_ε}-measurable and belongs to R (Σ, X), $P_{B_\tau}(G_{\tau_\varepsilon}) = G_{\tau_\varepsilon}$ for all $\tau \geqq \tau_\tau$. Thus for $\tau \geqq \tau_\varepsilon$,

$$\begin{aligned} N_\Phi(F_\infty - F_\tau) &\leqq N_\Phi(F_\infty - G_{\tau_\varepsilon}) + N_\Phi(C_{\tau_\varepsilon} - F_\tau) \\ &= N_\Phi(F_\infty - G_{\tau_\varepsilon}) + N_\Phi(P_{B_\tau}(G_{\tau_\varepsilon} - F_\infty)) \\ &\leqq 2N_\Phi(F_\infty - F_{\tau_\varepsilon}) < \varepsilon , \end{aligned}$$

since P_{B_τ} is a contraction. This proves (a).

(b) With a few evident modifications, the proof is the same.

As a corollary to Theorem 3.6, the following extension of [11, Th. 6, p. 500] for vector-valued functions can be given.

COROLLARY 3.7. *Let Σ be a σ-field and μ be a countably additive finite measure on Σ. If $\{f_\tau, B_\tau, \tau \in T\}$ is a martingale in $L^\Phi(\Sigma, X)$, the following conditions are equivalent*:

(i) *$\{f_\tau, \tau \in T\}$ converges in the strong topology of $L^\Phi(\Sigma, X)$.*

(ii) *$\{f_\tau, \tau \in T\}$ converges in the weak topology of $L^\Phi(\Sigma, X)$.*

(iii) *There exists a function $f_\infty \in L^\Phi(\Sigma, X)$* such that

$$\lim x^*\left(\int_E f_\tau d\mu\right) = x^*\left(\int_E f_\infty d\mu\right)$$

for each $E \in \Sigma$ and all $x^* \in X^*$, and for each $\varepsilon > 0$ there exists a parameter $\tau_\varepsilon \in T$ and a B_{τ_ε}-measurable function g_{τ_ε} in $L^\Phi(\Sigma, X)$ such that $N_\Phi(g_{\tau_\varepsilon} - f_\infty) < \varepsilon$.

Proof. If $\lambda: L^\Phi(\Sigma, X) \to V^\Phi(\Sigma, X)$ is the isometric isomorphism of Theorem 1.2, since $L^\Phi(\Sigma, X) \subset L^1(\Sigma, X)$, we have $\lambda(L^\Phi(\Sigma, X)) \subset R^\Phi(\Sigma, X)$. The proof now follows directly from Theorem 3.6 after an application of Theorem 3.2 and the isometric isomorphism λ.

The next theorem and its corollary are the final results of this section.

THEOREM 3.8. *Let X be reflexive, Φ and its complementary* *function Ψ be continuous and Φ obey the Δ_2-condition. If $\{F_\tau, B_\tau, \tau \in T\}$ is a $V^\Phi(\Sigma, X)$-martingale, then the following statements are equivalent.*

(i) *$\{F_\tau, B_\tau, \tau \in T\}$ converges weakly in $V^\Phi(\Sigma, X)$.*

(ii) *$\{F_\tau, B_\tau, \tau \in T\}$ converges strongly in $V^\Phi(\Sigma, X)$.*

(iii) *There exists F_∞ in $V^\Phi(\Sigma, X)$ such that $F_\tau = P_{B_\tau}(F_\infty)$ for all $\tau \in T$.*

(iv) *The set $\{F_\tau, \tau \in T\}$ is (strongly) bounded.*

Proof. (i) $\rightarrow$ (ii) is Theorem 3.6. (ii) $\rightarrow$ (iii): Let F_∞ be the strong limit of $\{F_\tau, \tau \in T\}$. Then $F_\infty \in V^\Phi(\Sigma, X)$ and clearly $F_\tau = P_{B_\tau}(F_\infty)$ for all $\tau \in T$. (iii) $\rightarrow$ (iv): Since P_{B_τ} is a contraction,

$$N_\Phi(F_\tau) = N_\Phi(P_{B_\tau}(F_\infty)) \leqq N_\Phi(F_\infty) .$$

(iv) $\rightarrow$ (i) follows from Corollary 3.4.

A formulation of Theorem 3.8 for martingales of point functions is

COROLLARY 3.9. *Let X be reflexive, Φ and its complementary function Ψ be continuous and Φ obey the Δ_2-condition. If Σ is a σ-field and μ is a countably additive finite measure on Σ and $\{f_\tau, B_\tau, \tau \in T\}$ is a martingale in $L^\Phi(\Sigma, X)$, then the following statements are equivalent.*

(i) *$\{f_\tau, \tau \in T\}$ converges in the weak topology of $L^\Phi(\Sigma, X)$.*

(ii) *$\{f_\tau, \tau \in T\}$ converges in the strong topology of $L^\Phi(\Sigma, X)$.*

(iii) *There exists $f_\infty \in L^\Phi(\Sigma, X)$ such that $f_\tau = E^{B_\tau}(f_\infty)$ for all $\tau \in T$.*

(iv) *The set $\{f_\tau, \tau \in T\}$ is (strongly) bounded.*

Proof. The proof follows from an application of the isometric isomorphism λ of $L^\Phi(\Sigma, X)$ onto (in this case) $V^\Phi(\Sigma, X)$ and theorems 2.7, 3.2 and 3.8.

It should be noted that Corollary 3.9 subsumes some of the main results of [9]. In addition, it extends these results to the vector-valued case and to the Orlicz space setting.

Furthermore, it should be noted that some of the preceding results for finitely additive set functions can be deduced from known results by use of the isomorphism theorems of [7, IV. 9]. On the other hand, the method of approach of this paper seems more direct and seems to yield more insight than the indirect method using known results and the isomorphism theorems of [7, IV. 9].

Finally, we note that the above results do not, in general, admit extensions to proving pointwise convergence theorems for martingales of point function. Indeed, the properties of the integrals of the functions involved enables these methods to work by avoiding the need for consideration of the (non-) measurability of the limits of nets of (point) functions. Thus, little information about pointwise convergence can be deduced from this approach. However, it is too much to expect to be able to deduce such results from the properties of finitely additive set functions, and a different approach is needed in such a study.

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