PACIFIC JOURNAL OF MATHEMATICS
Vol. 30, No. 3, 1969

GALOIS THEORY FOR BANACH ALGEBRAS

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This paper deals with the classical Galois theory in the context of the Arens-Hoffman extension $B = A[x]/(\alpha(x))$ of a commutative Banach algebra A (with identity over the complex field $\mathscr{C}$) with respect to a monic polynomial $\alpha(x)$ over A with an invertible discriminant. We show that the fundamental theorem of the Galois theory for commutative rings [S.U. Chase, D.K. Harrison, and A. Rosenberg, *Galois theory and cohomology of commutative rings*, Memoirs, Amer. Math. Soc. No. 52 (1965)] applies to our situation. The fixed algebras of the subgroups of the Galois group are then characterized for the case where A is semi-simple. The techniques are primarily topological and consist in examining the relationships between Φ_B and Φ_A, where the Φ's denote the respective carrier spaces of the Banach algebras A and B together with the usual weak * topology.

The topological techniques referred to in the abstract, stem from two results. The first is by J.A. Lindberg, Jr. [12, Proposition 1.3] which shows that Φ_B possesses a property similar to that possessed by a covering space of Φ_A. The other result is by G.A. Heuer [6, Th. 3.5] which shows that under certain connectedness assumptions, Φ_B is a covering space (in the sense of Chevalley [5]) of Φ_A. This classical notion of a covering space will not be used since we do not wish to limit ourselves to working with a connected and locally connected space Φ_A. At the other extreme, we do not need the full generality used by S. Lubkin in [13]. We will therefore modify Lubkin's definition and work with our own version of a covering space. The first section of this paper develops all aspects of this notion which we will be using.

We present some of the basic notions of the Arens-Hoffman extension in §2 and then show that in a very special case (which includes the case where Φ_A is a connected space), the fundamental theorem of the Galois theory for commutative rings [4] can be applied.

In §3 and §4, we characterize the fixed algebras of the subgroups of the Galois group in terms of our notion of a covering space. This will be done under the assumptions that A is semi-simple and that the generating polynomial $\alpha(x)$ factors completely over B. After obtaining results for the special case dealt with in §2, we prove corresponding results for the general situation.

This paper builds upon the work of G.A. Heuer [6], J.A. Lindberg, Jr. [11] and [12] and Heuer and Lindberg [7].

1. DEFINITION. Let X and Y be topological spaces, and let p be a continuous mapping of X onto Y. We will say that (X, p) is a *local covering space* of Y if card $(p^{-1}(y))$ (= cardinality of the set $\{p^{-1}(y)\}$) is a finite constant for each $y \in Y$ (this constant may depend on $y \in Y$) and if for each $y \in Y$ with $p^{-1}(y) = \{x_1, x_2, \cdots, x_k\}$, there exists disjoint neighborhoods $U_1, U_2, \cdots, U_k$ in X of $x_1, x_2, \cdots, x_k$ respectively such that p restricted to each U_i is a homeomorphism of U_i onto $p(U_1)$ and $p^{-1}(p(U_1)) = \bigcup_{i=1}^k U_i$.

In the event (X, p) is a local covering space of Y with the property that card $(p^{-1}(y))$ is a constant independent of $y \in Y$, we will say that (X, p) is a *covering space* of Y.

We give a necessary and sufficient condition that (X, p) be a local covering space of Y in the case where both X and Y are compact Hausdorff spaces. We note that the compactness of X implies that the cardinality of each fiber $p^{-1}(y)$ in X must not only be finite, but must also be bounded by some integer.

THEOREM 1.1. *Assume X and Y are compact Hausdorff spaces and p is a continuous mapping of X onto Y. Then (X, p) is a local covering space (covering space) of Y if and only if p is an open mapping with the property that for each $x \in X$, there exists a neighborhood V_x in X of x such that p restricted to V_x is one-to-one (and card $(p^{-1}(y))$ is a finite constant independent of $y \in Y$).*

Proof. Assume (X, p) is a local covering space of Y. Let V be an open subset in X and select any point $y_0 \in p(V)$ with $p^{-1}(y_0) = \{x_1, \cdots, x_k\}$. Let $U_1, U_2, \cdots, U_k$ be disjoint open neighborhoods in X of $x_1, x_2, \cdots, x_k$ respectively such that $U_1 \cap V \neq \phi$, p restricted to each U_i is a homeomorphism of U_i onto $p(U_1)$ and $p^{-1}(p(U_1)) = \bigcup_{i=1}^k U_i$.

For each i, set $V_i = U_i \cap p^{-1}(p(U_1 \cap V))$. It follows that the V_i's are mutually disjoint, $p(V_i) = p(U_1 \cap V)$ for each i and $p^{-1}(p(V_1)) = \bigcup_{i=1}^k V_i$. Since p restricted to each U_i is a homeomorphism, each V_i is open with respect to the relative topology of U_i. Since the latter set is open, each V_i is open in X.

Since $X - p^{-1}(p(V_1))$ is a compact subset of X, $p(X - p^{-1}(p(V_1))) = Y - p(V_1)$ is a compact, and therefore closed, subset of Y. Therefore $p(V_1) = p(U_1 \cap V)$ is an open subset of Y which contains y_0 and which is contained in $p(V)$. Thus, $p(V)$ is an open subset of Y. Consequently, p is an open mapping.

To prove the converse, assume p is an open mapping, which satisfies the given condition. Let $y_0 \in Y$ and set $p^{-1}(y_0) = \{x_1, \cdots, x_k\}$. Since X is a Hausdorff space, there exist mutually disjoint open neighborhoods $V_1, V_2, \cdots, V_k$ in X of $x_1, x_2, \cdots, x_k$ respectively. We assume that p restricted to each V_i is one-to-one (if not, replace each V_i by $V_i \cap$

(interior of V_{x_i})). Let $U_0 = \bigcap_{i=1}^k p(V_i)$ and let $U_i = V_i \cap p^{-1}(U_0)$ for each $i = 1, 2, \cdots, k$. Thus $\bigcup_{i=1}^k U_i \subset p^{-1}(U_0)$. Since p is a continuous open mapping, U_0 is an open neighborhood in Y of y_0 and each U_i is an open neighborhood in X of x_i. The U_i's are mutually disjoint and each of them is mapped onto U_0 by p.

Now let $W = U_0 \cap p(X - \bigcup_{i=1}^k U_i)$. This subset of Y is closed with respect to the relative topology of U_0. If W is the empty set, then $p^{-1}(U_0) \subset \bigcup_{i=1}^k U_i$ and therefore $p^{-1}(U_0) = \bigcup_{i=1}^k U_i$. Also $p \mid U_i$ (p restricted to U_i) is one-to-one for $i = 1, 2, \cdots, k$. Finally $p \mid U_i$ is a homeomorphism since each U_i is open and p is an open mapping. This completes the proof for the case $W = \phi$. If, on the other hand, there is a point $y \in Y$ such that $y \in W$, then there is a set W_0 in Y which is open with respect to the relative topology of U_0 and which contains y_0 but is disjoint from W. This means that $p^{-1}(W_0) \subset \bigcup_{i=1}^k U_i$. Since U_0 is open in Y and since $y_0 \in W_0$, W_0 is an open neighborhood in Y of y_0. For each i, let $W_i = U_i \cap p^{-1}(W_0)$. This set is an open set in X which contains x_i. It follows that $p(W_i) = W_0$ for $i = 1, 2, \cdots, k$ and $p^{-1}(W_0) = \bigcup_{i=1}^k W_i$. The proof is now completed by applying to the W_i's the argument used on the U_i's in the case $W = \phi$.

From now on, when we write the phrase "(V, r) is a local covering space (or covering space) of U", we will assume both V and U are compact Hausdorff spaces and r is a continuous mapping of V onto U. Note that a subset of a compact Hausdorff space is closed if and only if it is compact, and a continuous mapping between two such spaces is a closed mapping. Also, if r and s are two mappings such that the composite mapping $r \circ s$ is defined, we will write rs for $r \circ s$.

LEMMA 1.2. *Let (X, p) and (Z, q) be two local covering spaces of Y and assume w is a continuous mapping of X into Z such that $p = qw$. Then w is an open mapping.*

Proof. Let U be an open subset of X and suppose $z \in w(U)$. Select a point $x \in w^{-1}(z) \cap U$. Since (Z, q) is a local covering space of Y, there exists an open neighborhood V in Z of z such that $q \mid V$ is one-to-one. We assume V is so small that $q(V) \subset p(U)$. There also exists an open neighborhood U_0 in X of x such that $U_0 \subset w^{-1}(V) \cap U$. By replacing U_0 by $U_0 \cap p^{-1}(p(U_0) \cap q(V))$ and V by $U_0 \cap q^{-1}(p(U_0) \cap q(V))$ if need be, we may and do assume that $p(U_0) = q(V)$.

Now let $z_0 \in V$. There is a point $x_0 \in U_0$ such that $p(x_0) = q(z_0)$. Therefore $qw(x_0) = p(x_0) = q(z_0)$. Since $U_0 \subset w^{-1}(V)$, $w(x_0) \in V$. But since $q \mid V$ is one-to-one, $w(x_0) = z_0$. This means $z_0 \in w(U_0)$. Thus $V \subset w(U_0) \subset w(U)$. The fact that $z \in V$ and V is open implies that $w(U)$ is an open subset of Z. This completes the proof.

If (X, p) is a local covering space of Y, denote by $E(X: Y)$ the group of all homeomorphisms ϕ of X onto itself with the property that $p\phi = p$.

LEMMA 1.3. *Assume* (X, p) *and* (Z, q) *are two local covering spaces of* Y. *Suppose* w *and* u *are two continuous mappings of* X *into* Z *such that* $p = qw = qu$. *Then* $H = \{x \in X: w\phi(x) = u(x)$ *for any* $\phi \in E(X: Y)\}$ *is an open and closed subset of* X.

Proof. Since $w\phi$ and u are both continuous mappings of X into the Hausdorff space Z, H is closed in X. (See, for example, [10, Problem C, p. 100]).

To show H is also open, let $x_0 \in H$ and set $z_0 = u(x_0) = w\phi(x_0)$. Let U be any open neighborhood in Z of z_0 such that $q \mid U$ is one-to-one and let V be any open neighborhood in X of x_0 such that $u(V) \cap w\phi(V) \subset U$. If $x \in V$, then $u(x)$ and $w\phi(x)$ are both in U, and furthermore, $qu(x) = p(x) = p\phi(x) = qw\phi(x)$. Since $q \mid V$ is one-to-one, $u(x) = w\phi(x)$. This means $x \in H$. We have thus shown that $V \subset H$ which implies that H is an open subset. This completes the proof.

The next result enables us to work first with components of spaces and then to extend to open and closed subsets. An equivalent form of the following lemma can be found in [8, p. 47].

LEMMA 1.4. *Let* C *be a component in a compact Hausdorff space* X *and let* U *be any open set in* X *which contains* C. *Then there exists an open and closed set* K *in* X *which contains* C *and which is contained in* U.

LEMMA 1.5. *Assume* (X, p) *is a local covering space of* Y *and let* K *be an open and closed subset of* X. *Then for any integer* s, $K_s = \{y \in Y: \text{card}\,(p^{-1}(y) \cap K) = s\}$ *is an open and closed subset of* Y.

Proof. Let $y \in K_s$ and let $p^{-1}(y) \cap K = \{x_1, x_2, \cdots, x_s\}$. Select open neighborhoods V_i in X of x_i for $i = 1, 2, \cdots, s$ such that (i) each V_i is contained in K (ii) p restricted to each V_i is a homeomorphism of V_i onto a set V_0 in Y, and (iii) $p^{-1}(V_0) \cap K = \bigcup_{i=1}^{s} V_i$. This choice of the V's is possible since p is an open mapping. (The details are essentially the same as in the proof of Theorem 1.1).

V_0 is an open neighborhood in Y of y. It follows from (ii) and (iii) that V_0 is contained in K_s. This means K_s is an open set. But K_s must also be closed since $p(K)$ is the finite union of $K_i (i = 1, 2, \cdots, N)$ and the K_i's are mutually disjoint. (Recall the comment prior to Theorem 1.1 that there exists an integer N such that $\text{card}\,(p^{-1}(y)) \leqq N$ for all $y \in Y$).

Our main interest in local covering spaces is to obtain properties of the group $E(X: Y)$ of "covering homeomorphisms". The next result is the first of our two main technical lemmas which deal with this situation.

LEMMA 1.6. *Assume (X, p) is a local covering space of Y and assume K is an open and closed subset of X. Also suppose that x_0 and x_0' are two distinct points of K, and that r and s are two continuous mappings of X into itself such that $pr = p = ps$, $r(x_0) = x_0'$, and $s(x_0') = x_0$. Then there is a homeomorphism ϕ in $E(X: Y)$ such that $\phi(x_0) = x_0'$, $\phi(K) = K$, and ϕ is the identity homeomorphism on $X - K$.*

Proof. Let $H = \{x \in X: sr(x) = x\}$. This is a nonempty subset of X which, by Lemma 1.3 is open and closed. Since $sr \mid H$ is the identity mapping, $r \mid H$ is one-to-one. Both mappings are closed since X is a compact Hausdorff space. Moreover by Lemma 1.2, both mappings are also open.

Let C and D be the components in X such that $x_0 \in C$ and $x_0' \in D$. Then $C \subset K \cap H, D \subset K, r(C) \subset D$ and $s(D) \subset C$. Therefore, $C = sr(C) \subset s(D) \subset C$. Consequently, $s(D) = C$. Furthermore $r(C) = D$ since $D \subset r(H)$ and $rs \mid r(H)$ is the identity mapping. If C and D are distinct components, then there exists disjoint open sets U and V in X such that $C \subset U \subset K \cap H, D \subset V \subset K \cap r(H)$, and $r(U) \subset V$. By Lemma 1.4, there exists an open and closed set K_0 in X such that $C \subset K_0 \subset U$. Thus $r(K_0) \subset r(U) \subset V$. If follows that the sets $r(K_0)$ and K_0 are disjoint. Define a mapping ϕ of X into itself as follows:

$$\phi \mid K_0 = r \mid K_0 , \qquad \phi \mid r(K_0) = s \mid r(K_0) ,$$

and ϕ is the identity mapping on $X - K_0 \cup r(K_0)$. This is a one-to-one mapping of K_0 onto $r(K_0)$ and of $r(K_0)$ on K_0. Since K_0 and $r(K_0)$ are disjoint, $\phi \in E(X: Y)$. Also, $\phi(x_0) = x_0'$, and the fact that $K_0 \subset K$ implies $\phi(K) = K$ and ϕ is the identity off of K. This completes the proof for the case where C and D are distinct components.

On the other hand, if C and D are the same component, then $r(C) = C = s(C)$. Since $p(C)$ is a closed and connected subset of Y, the argument used in the proof of [12, Th. 2.4] shows that $p^{-1}(p(C))$ is a finite union of disjoint connected sets each of which is open (and therefore closed) in the relative topology of $p^{-1}(p(C))$. This means that there exists an open set U in X such that $U \cap p^{-1}(p(C)) = C$. We assume, without loss of generality, that U is a subset of $H \cap r(H) \cap K$. By Lemma 1.4, there exists an open and closed subset K_0 in X such that $C \subset K_0 \subset U$. It follows that $K_0 \cap p^{-1}(p(C)) = C$. Also, $K_0 \subset H \cap r(H)$ implies that both $r \mid K_0$ and $s \mid K_0$ are one-to-one mappings.

Let y_1 be any element of $p(C)$ and assume card $(p^{-1}(y_1) \cap C) = m$.

It follows from Lemma 1.5 that $K_m = \{y \in p(K_0): \operatorname{card}(p^{-1}(y) \cap K_0) = m\}$ is an open and closed subset of Y. Since $p(C)$ is connected and has a nonempty intersection with K_m, it must be a subset of K_m. Therefore, for any $y \in p(C)$, $\operatorname{card}(p^{-1}(y) \cap C) = \operatorname{card}(p^{-1}(y) \cap p^{-1}(p(C)) \cap K_0) = \operatorname{card}(p^{-1}(y) \cap K_0) = m$.

Now let $P = \{y \in K_m: r(p^{-1}(y) \cap K_0) \subset K_0\}$. P is a subset of Y which contains $p(C)$ and is contained in K_m. We note that $y \in P$ if and only if $y \in K_m$ and r maps $p^{-1}(y) \cap K_0$ onto itself. We will show that P is an open and closed subset of Y.

(i) P is an open set. Let $y_0 \in P$ and assume $p^{-1}(y_0) \cap K_0 = \{x_0^1, x_0^2, \cdots, x_0^m\}$. For each $i = 1, 2, \cdots, m$, let U_i be an open neighborhood in X of x_0^i such that both U_i and $r(U_i)$ are contained in the open and closed set $p^{-1}(K_m) \cap K_0$ and such that p restricted to each U_i is a homeomorphism of U_i onto a set U_0 in Y. For any point $y \in U_0$, $p^{-1}(y) \cap K_0 \subset p^{-1}(K_m) \cap K_0$. This means $\operatorname{card}(p^{-1}(y) \cap K_0) = m$. Consequently, $p^{-1}(y) \cap K_0 \subset \bigcup_{i=1}^m U_i$. Thus for $x \in p^{-1}(y) \cap K_0$, $r(x)$ is an element of $r(U_j)$ for some j. Since the latter set is a subset of K_0, $r(p^{-1}(y) \cap K_0) \subset p^{-1}(y) \cap K_0$. Therefore $U_0 \subset P$. It follows, since U_0 is open, that P is an open set.

(ii) P is a closed set. Let y_1 be a point in the closure of P and assume there exists $x_1 \in p^{-1}(y_1) \cap K_0$ such that $r(x_1) \notin K_0$. (i.e. assume $y_1 \notin P$.) Let V be any open neighborhood in X of x_1 such that $V \subset p^{-1}(K_m) \cap V_0$ and $r(V) \cap K_0 = \phi$. Since $p(V)$ is an open neighborhood in Y of y_1, there exists a point y in $p(V) \cap P$. But for any x in $p^{-1}(y) \cap V$, $r(x)$ must be a point in $p^{-1}(y) \cap K_0$. Thus $r(V) \cap K_0$ is nonempty. This is a contradiction. Therefore any x_1 in $p^{-1}(y_1) \cap K_0$ has the property that $r(x_1) \in K_0$. This means $y_1 \in P$ and thus P is a closed set.

Finally, consider the set $Q = p^{-1}(P) \cap K_0$. This is an open and closed set in X which contains C. Since $pr = p$, and $r(p^{-1}(P)) \subset p^{-1}(P)$, r maps Q into itself. But the fact that $r \mid K_0$ is one-to-one and $\operatorname{card}(p^{-1}(y) \cap K_0) = m$ for all $y \in P$ implies that r must map Q onto itself.

Define a mapping ϕ of X into itself by $\phi \mid Q = r \mid Q$ and ϕ is the identity elsewhere. Since $K_0 \subset K$ and $r \mid K_0$ is one-to-one, $\phi \mid K$ is a one-to-one mapping of K onto itself. It follows that $\phi(x_0) = x_0'$ and $\phi \in E(X: Y)$. This completes the proof for the case where C and D are the same component. Therefore the proof of Lemma 1.6 is complete.

We have the following corollary to the proof of the above lemma.

COROLLARY 1.7. *Assume (X, p) is a local covering space of Y and let C be a component of X. Then* $\operatorname{card}(p^{-1}(y) \cap C)$ *is a constant,* *say m, independent of $y \in p(C)$. Furthermore, if K is any open and closed set in X which contains C and if r is a continuous mapping of X into itself such that $pr = p$ and $r(C) = C$, then there exists an open and closed set Q in X with the following properties*: (i) $C \subset Q \subset K$, (ii) $r(Q) = Q$, (iii) $\operatorname{card}(p^{-1}(y) \cap Q) = m$ *for all* $y \in p(Q)$, *and* (iv) $p^{-1}(p(C)) \cap Q = C$.

We also remark that if C and D are distinct components such that $p(C) = p(D)$, then the corresponding open and closed sets Q_C and Q_D given by the above corollary can be chosen such that they are disjoint and such that $p(Q_C) = p(Q_D)$.

The above two results now yield a structure theorem for local covering spaces.

COROLLARY 1.8. *(X, p) is a local covering space of Y if and only if there exists a finite covering of Y by mutually disjoint, open and closed sets $Y_1, \cdots, Y_k$ with the property that if $X_i = p^{-1}(Y_i)$ and $p_i = p \mid X_i$, then (X_i, p_i) is a covering space of Y_i for $i = 1, 2, \cdots, k$.*

Proof. Assume (X, p) is a local covering space of Y and let R be a component of Y. Then, as in the second half of the proof of Lemma 1.6, $p^{-1}(R) = \bigcup_{j=1}^{q} R_j$, where the R_j's are mutually disjoint, connected, open and closed with respect to the relative topology of $p^{-1}(R)$, and $p(R_j) = p(R_1)$ for each $j = 1, 2, \cdots, q$. It follows that each R_j is a component of X. By applying the above corollary to each of the R_j's with r equal to the identity mapping in each case, there exists open and closed sets $Q_1, Q_2, \cdots, Q_q$ in X containing $R_1, R_2, \cdots, R_q$ respectively such that if $m_j = \operatorname{card}(p^{-1}(y) \cap R_j)$ for each $y \in R$, then $m_j = \operatorname{card}(p^{-1}(y) \cap Q_j)$ for each $y \in p(Q_j)$. We assume, without loss of generality, that the Q_j's are mutually disjoint and that p maps each Q_j onto a set Q_0 in Y. This latter set is open and closed in Y. Thus for each $y \in Q_0$, $\operatorname{card}(p^{-1}(y)) = \sum_{j=1}^{q} m_j$. This means that $(p^{-1}(Q_0), p \mid p^{-1}(Q_0))$ is a covering space of Q_0. This part of the proof is completed by using the fact that Y is a compact space. Since the converse is immediate, the proof is complete.

DEFINITION. Assume (X, p) is a local covering space of Y and K is any open and closed set in X. We will say that a subset E of $E(X: Y)$ is *(simply) transitive on the fibers of* K if for any $y \in p(K)$ and for any two points x_1 and x_2 in $p^{-1}(y) \cap K$, there exists a (unique) homeomorphism ϕ in E such that $\phi(x_1) = x_2$.

If for any $y \in p(K)$, $\operatorname{card}(p^{-1}(y) \cap K) = m$, then a subset E of $E(X: Y)$ is simply transitive on the fibers of K if and only if E contains exactly m elements no two of which agree at any point of K.

We have an additional corollary to Lemma 1.6.

COROLLARY 1.9. *Assume* (X, p) *is a local covering space of* Y *and let* K *be any open and closed set in* X. *If* $E(X: Y)$ *is transitive on the fibers of* X, *then* $E(K: p(K))$ *is transitive on the fibers of* K.

For the remainder of this section, we will deal exclusively with covering spaces. (Our application of local covering spaces does not occur until the last section of this paper.) The next result is the second of our two major lemmas in this section.

LEMMA 1.10. *Assume* (X, p) *is a covering space of* Y. *Then* $E(X: Y)$ *is transitive on the fibers of* X *if and only if there exists a subset* E *of* $E(X: Y)$ *which is simply transitive on the fibers of* X.

Proof. Assume there exists elements $\phi_1, \phi_2, \cdots, \phi_n$ in $E(X: Y)$ ($n = \text{card}\,(p^{-1}(y))$ for all $y \in Y$) such that for any $x \in X$ and $i \neq j$ $\phi_i(x) \neq \phi_j(x)$. Let $y \in Y$ and let x_1 and x_2 be any two points in $p^{-1}(y)$. Since $\text{card}\,(\{\phi_i(x_1): i = 1, 2, \cdots, n\}) = n = \text{card}\,(p^{-1}(y))$, there exists a j such that $\phi_j(x_1) = x_2$. Thus $E(X: Y)$ is transitive on the fibers of X.

Assume $E(X: Y)$ is transitive on the fibers of X and let D be a component in Y. We will construct n elements of $E(X: Y)$ no two of which agree at any point of $p^{-1}(D)$. As in the proof of Corollary 1.8, $p^{-1}(D) = \bigcup_{j=1}^{m} X_j$ where the X_j's are mutually disjoint components of X which are open and closed with respect to the relative topology of $p^{-1}(D)$. If X_i and X_j are any two distinct components, then there is at least one homeomorphism ϕ in $E(X: Y)$ such that $\phi(X_i) \cap X_j$ is nonempty. But the latter set is open and closed with respect to the relative topology of $p^{-1}(D)$. Thus $\phi(X_i) = X_j$. Consequently, there is a constant k independent of $j = 1, 2, \cdots, m$ such that $\text{card}\,(p^{-1}(y) \cap X_j) = k$ for all $y \in D$. (Note that $km = n$).

For any $y \in D$, arbitrarily label the points of $p^{-1}(y) \cap X_j$ by x_{1j}, $x_{2j}, \cdots, x_{kj}$. Since $E(X: Y)$ is transitive on the fibers of X, for each $i = 1, 2, \cdots, k$ there exists an element p_{ij} in $E(X: Y)$ such that $p_{ij}(x_{1j}) = x_{ij}$. An argument used in a previous paragraph of this proof shows that each p_{ij} maps X_j onto itself. By Lemma 1.3, no two p_{ij}'s agree at any point of X_j. Upon applying Corollary 1.7 for each p_{ij} and then taking the intersection of all of the open and closed sets in X obtained, there exists an open and closed set Q_j in X which contains X_j and has the following properties: $\text{card}\,(p^{-1}(y) \cap Q_j) = k$ for all $y \in p(Q_j)$; $p_{ij}(Q_j) = Q_j$ for each $i = 1, 2, \cdots, k$; and $p^{-1}(D) \cap Q_j = X_j$. For any x in Q_j; with $p(x) = y$, $\{p_{ij}(x): i = 1, 2, \cdots, k\} = p^{-1}(y) \cap Q_j$. Consequently, no two p_{ij}'s agree at any point of Q_j.

By replacing each Q_j by $Q_j \cap (\bigcap_{j=1}^{m} p(Q_j))$ if need be, we may and do assume that p maps each Q_j onto a set Q_0 in Y. Furthermore, we assume the Q_j's are mutually disjoint since we can select m mutually disjoint open and closed sets $K_1, \cdots, K_m$ in X containing $X_1, \cdots, X_m$ respectively and then (Corollary 1.7) pick the Q_j's such that in addition to all of the above properties, $X_j \subset Q_j \subset K_j$ for each j. It follows from these assumptions that $p^{-1}(Q_0) = \bigcup_{j=1}^m Q_j$.

The next step in the proof is to show that without the loss of generality, we may assume that for each $j = 1, 2, \cdots, m$, $E_j = \{p_{ij}: i = 1, 2, \cdots, k\}$ is a subgroup of $E(Q_j: Q_0)$. This part of the proof is accomplished in two steps.

(i) Corresponding to each integer $i (1 \leqq i \leqq k)$, there exists an integer $r (1 \leqq r \leqq k)$ such that $p_{rj}(x_{1j}) = p_{ij}^{-1}(x_{1j})$. By Lemma 1.3, $H_i = \{x \in X: p_{rj}(x) = p_{ij}^{-1}(x)\}$ is an open and closed subset of X. Thus $X_j \subset H_i$ for $i = 1, 2, \cdots, k$. Let $T_j = Q_j \cap (\bigcap_{i=1}^k H_i)$. This set is open and closed in X, $X_j \subset T_j \subset Q_j$ and the elements of E_j restricted to T_j are closed with respect to the operation of taking inverse mappings. We now modify the T_j's so that p maps each T_j onto a subset T_0 in Y and $p^{-1}(T_0) = \bigcup_{j=1}^k T_j$. Therefore we may and do assume $T_j = Q_j$ for $j = 1, 2, \cdots, k$.

(ii) For any two distinct integers r and s $(1 \leqq r, s \leqq k)$ there exists an integer q such that $p_{qj}(x_{1j}) = p_{rj}(p_{sj}(x_{1j}))$. An application of Lemma 1.3 yields an open and closed set in X which contains X_j and on which $p_{qj} = p_{rj}p_{sj}$. We now repeat this argument for each of the k^2 elements in $E_j \times E_j$. Upon taking the intersection of all of the sets obtained, we get an open and closed set O_j in X which contains X_j. Set $Z_j = Q_j \cap O_j$. This is an open and closed set such that the elements of E_j restricted to Z_j are closed with respect to the operation of taking composite mappings. We modify the sets Z_j as above. Therefore we may and do assume $Z_j = Q_j$ for each j. We have shown, therefore, that without loss of generality, $E_j = \{p_{ij}: i = 1, 2, \cdots, k\}$ can be assumed to be a subgroup of $E(Q_j: Q_0)$ for each $j = 1, 2, \cdots, m$. Indeed, E_j is a k^{th} order subgroup which is simply transitive on the fibers of $Q_j (j = 1, 2, \cdots, m)$. (That is, for each y in Q_0 and any two points x and x' in $p^{-1}(y) \cap Q_j$, there exists a unique i such that $p_{ij}(x) = x'$.) For each j, we redefine each $p_{ij} (i = 1, 2, \cdots, k)$ to be the identity mapping on $X - Q_j$. Let $E_0 = E_1 \times E_2 \times \cdots \times E_m$. E_0 is an n^{th} order subgroup of $E(p^{-1}(Q_0): Q_0)$ which is simply transitive on the fibers of $p^{-1}(Q_0)$. We also view E_0 as a subgroup of $E(X: Y)$.

To summarize the proof thus far, corresponding to each component D in Y, there exists an open and closed set Q_0 in Y containing D and there also exists an n^{th} order subgroup E_0 of $E(p^{-1}(Q_0): Q_0)$ which is simply transitive on the fibers of $p^{-1}(Q_0)$.

Using the compactness of Y, we extract a finite covering of Y by open and closed sets $Y_1, Y_2, \cdots, Y_s$ such that corresponding to each

Y_r, there exists an n^{th} order subgroup $E_r = \{\phi_{1r}, \phi_{2r}, \cdots, \phi_{nr}\}$ of $E(p^{-1}(Y_r): Y_r)$ which is simply transitive on the fibers of $p^{-1}(Y_r)$. Also, each of the ϕ_{ir}'s is the identity mapping off of $p^{-1}(Y_r)$. Finally, we may and do assume the Y_r's are mutually disjoint since if not, set $Y_1' = Y$, and for $2 \leqq r \leqq s$, set $Y_r' = Y_r - \bigcup_{i=1}^{r-1} Y_i'$.

For each $i = 1, 2, \cdots, n$, define the mapping ϕ_i of X into itself by $\phi_i \mid p^{-1}(Y_r) = \phi_{ir} \mid p^{-1}(Y_r)$ for $r = 1, 2, \cdots, s$. Since each ϕ_{ir} maps $p^{-1}(Y_r)$ onto itself and is the identity mapping off of $p^{-1}(Y_r)$, each ϕ_i is a well-defined homeomorphism in $E(X: Y)$. Also the set $\{\phi_1, \phi_2, \cdots, \phi_n\}$ has the property that no two of its elements agree at any point of X. That is, the set $E = \{\phi_1, \phi_2, \cdots, \phi_n\}$ is simply transitive on the fibers of X. This completes the proof of Lemma 1.10.

We emphasize that the set E given by the above lemma does not in general form a subgroup of $E(X: Y)$. Also, the elements in the set E are not unique. The proof of Lemma 1.10 yields the following corollary, which will be exploited in the final section of this paper.

COROLLARY 1.11. *Assume (X, p) is a covering space of Y and also assume $E(X: Y)$ is transitive on the fibers of X. Then there exists a finite covering of Y by mutually disjoint open and closed sets $Y_1, Y_2, \cdots, Y_s$ with the property that for each r, there exists an n^{th} order subgroup E_r of $E(p^{-1}(Y_r): Y_r)$ which is simply transitive on the fibers of $p^{-1}(Y_r)$ and each of whose elements is the identity mapping off of $p^{-1}(Y_r)$.*

There is one case where the set $\{\phi_1, \cdots, \phi_n\}$ given by Lemma 1.10 does form a subgroup of $E(X: Y)$.

COROLLARY 1.12. *Assume (X, p) is a covering space of a connected space Y. Then $E(X: Y)$ is transitive on the fibers of X if and only if there exists an n^{th} order subgroup of $E(X: Y)$ which is simply transitive on the fibers of X. Also, if X is connected, then $E(X: Y)$ is transitive on the fibers of X if and only if it is simply transitive on the fibers of X. (This implies that $E(X: Y)$ is transitive on the fibers of X if and only if its order is n).*

The final topic in this section deals with the "orbit space" of a finite subgroup of $E(X: Y)$. We assume (X, p) is a covering space of Y and let E be a finite subgroup of $E(X: Y)$. Let $R = \{(x, x') \in X \times X$: for some $\phi \in E, \phi(x) = x'\}$. R is a closed subset of $X \times X$ since $R = \bigcup_{\phi \in E} \{(x, x') \in X \times X: \phi(x) = x'\}$. Denote by X/E the space of equivalence classes determined by the equivalence relation defined by R. We endow this space with the quotient topology. That is, the largest topology (greatest number of open sets) which makes the projection map

$P: X \to X/E$ continuous. Thus a set U is open in X/E if and only if $P^{-1}(U)$ is an open subset of X (see [10, p. 94] for details). If A is an open subset of X, $R[A] = \{x \in X\text{: for some } x' \in A, (x, x') \in R\} = \{x \in X\text{: for some } x' \in A \text{ and } \phi \in E, \phi(x) = x'\} = \bigcup_{\phi \in E} \phi(A)$. Thus $R[A]$ is an open subset of X. Therefore, P is an open mapping of X onto X/E [10, p. 97]. It follows from [10, p. 98] that X/E is a Hausdorff space. Moreover, since P is a continuous mapping, X/E is a compact space. Consequently, P is also a closed mapping.

To summarize, if E is a finite subgroup of $E(X\colon Y)$, then the "orbit space" X/E is a compact Hausdorff space with respect to the quotient topology. Also the projection mapping P of X onto X/E is a continuous, open and closed mapping. If follows from Theorem 1.1 that (X, P) is a covering space of X/E if and only if $\operatorname{card}(P^{-1}(P(x)))$ is a finite constant independent of $P(x) \in X/E$. We also note that the subgroup E of $E(X\colon Y)$ is transitive on the fibers of (X, P).

LEMMA 1.13. *Assume (X, p) is a covering space of Y and assume E is a finite subgroup of $E(X\colon Y)$ with the property that no two elements of E agree at any point of X. Then (X, P) is a covering space of X/E. Conversely, if (X, P) is a covering space of X/E, then E possesses a set of k $(= \operatorname{card}(P^{-1}(P(x)))$ for all $P(x) \in X/E)$ elements which is simply transitive on the fibers of (X, P).*

Proof. From the above remarks, we need only show that card $(P^{-1}(P(x)))$ is a constant independent of $P(x) \in X/E$. Since $P^{-1}(P(x)) = \bigcup_{\phi \in E} \{x' \in X\colon \phi(x) = x'\}$, the assumption on E implies that for each $P(x) \in X/E$, $\operatorname{card}(P^{-1}(P(x))) =$ order of E.

Since E is transitive on the fibers of (X, P), the converse follows from Lemma 1.10. This completes the proof.

2. After presenting some basic facts concerning Arens-Hoffman extensions, we give a necessary and sufficient condition for the generating polynomial to factor into monic linear factors over B. The final objective of this section is to apply the fundamental theorem of the Galois theory for commutative rings [4].

We assume A is a commutative Banach algebra over the complex field $\mathscr{C}$ and we assume A possesses an identity element e. Let $\alpha(x) = x^n + \sum_{i=0}^{n-1} \alpha_i x^i$ be a monic polynomial in $A[x]$. Denote by $(\alpha(x))$ the principal ideal in $A[x]$ generated by $\alpha(x)$. Then R. Arens and K. Hoffman have shown in [2] that $B = A[x]/(\alpha(x))$ possesses a family of equivalent norms with respect to which B is a Banach algebra with the property that the natural embedding of A into B is an isometric isomorphism of A onto a closed subalgebra of B. The family of norms is given by

$$\left\| \sum_{i=0}^{n-1} a_i x^i + (\alpha(x)) \right\| = \sum_{i=0}^{n-1} \| a_i \| t^i \qquad (\text{for } a_i \in A)$$

where t is any positive number such that $\sum_{i=0}^{n-1} \| \alpha_i \| t^i \leqq t^n$. Since such a t always exists, we take $t = 1$. We refer to B as the Arens-Hoffman extension of A with respect to $\alpha(x)$.

We denote the coset $x + (\alpha(x))$ in B by $\mathfrak{x}$ and the coset $a + (\alpha(x))$ in B by a for any $a \in A$. Thus any element in B is uniquely expressible in the form $\sum_{i=0}^{n-1} a_i \mathfrak{x}^i$ for $a_i \in A$. Also, the norm of such an element is $\| \sum_{i=0}^{n-1} a_i \mathfrak{x}^i \| = \sum_{i=0}^{n-1} \| a_i \|$ where the latter norm is the given norm in A.

If $\beta(x) = \sum_{i=0}^{m} \beta_i x^i \in A[x]$, we view x as an indeterminant over $\hat{A}$ (= Gelfand representation of A) and $\mathscr{C}$ as well as an indeterminant over A and we let $\hat{\beta}(x)$ and $\beta_h(x)$ denote, respectively, the polynomials $\sum_{i=0}^{m} \hat{\beta}_i x^i$ in $\hat{A}[x]$ and $\sum_{i=0}^{m} \hat{\beta}_i(h) x_i$ in $\mathscr{C}[x]$ for $h \in \Phi_A$. Arens and Hoffman have shown that if $B = A[x]/(\alpha(x))$, then Φ_B is identifiable with the set $\{(h, \lambda) \in \Phi_A \times \mathscr{C}: \alpha_h(\lambda) = 0\}$ together with the relative topology of $\Phi_A \times \mathscr{C}$[2, Th. 4.2]. (We note that the carrier space $\Phi_{\mathfrak{A}}$ of any commutative Banach algebra $\mathfrak{A}$ over $\mathscr{C}$ with identity is a compact Hausdorff space.) The projection mapping π of Φ_B onto Φ_A defined by $\pi(h, \lambda) = h$ for $(h, \lambda) \in \Phi_B$ is a continuous open mapping [12, §1].

If $\beta(x) \in A[x]$, the discriminant d_β of $\beta(x)$ is defined as in [15, p. 82] and is an element of A. Furthermore for $h \in \Phi_A$, $\hat{d}_\beta(h)$ is the discriminant of $\beta_h(x)$. Thus $\hat{d}_\beta(h) = 0$ if and only if $\beta_h(x) = 0$ has at least one root of multiplicity $\geqq 2$. Throughout this paper we will be assuming that the generating polynomial $\alpha(x)$ has an invertible discriminant d. Thus $\hat{d}(h) \neq 0$ for each $h \in \Phi_A$. Therefore, $\pi^{-1}(h)$ consists of precisely n (= degree of $\alpha(x)$) distinct points in Φ_B for each $h \in \Phi_A$. In order to simplify the statements of our results, we denote by $I(A)$ (or simply I if there is no confusion as to the algebra in question) the collection of all monic polynomials in $A[x]$ having an invertible discriminant in A.

For the remainder to this paper we assume A is a commutative Banach algebra over the complex field $\mathscr{C}$, A has an identity element e, and $B = A[x]/(\alpha(x))$ is the Arens-Hoffman extension of A with respect to the n^{th} degree polynomial $\alpha(x) \in I$.

From the above remarks and from Theorem 1.1, we have the following:

2.1. (Φ_B, π) is a covering space of Φ_A.

Another proof of this follows from [12, Proposition 1.3].

The following is our criterion for showing that a monic polynomial in $A[x]$ factors into monic linear factors over B.

LEMMA 2.2. *Let $\gamma(x)$ be an n^{th} degree monic polynomial in $A[x]$ and let $\mathfrak{A}$ be any Banach algebra extension of A.* (i.e., *$\mathfrak{A}$ is a Banach algebra which possesses a closed subalgebra isomorphic and isometric to A.*)

If there exists $b_1, b_2, \cdots, b_n$ in $\mathfrak{A}$ such that $\gamma(b_i) = 0$ for each i, and if $(b_i - b_j)^{-1} \in \mathfrak{A}$ for all $i \neq j$, then $\gamma(x) = \prod_{i=1}^{n}(x - b_i)$ in $\mathfrak{A}[x]$.

Proof. Let $\beta(x) = \prod_{i=1}^{n}(x - b_i)$. Since both $\gamma(x)$ and $\beta(x)$ are monic polynomials of the same degree in $\mathfrak{A}[x]$, there exists a polynomial $r(x) \in \mathfrak{A}[x]$ such that $\gamma(x) = \beta(x) + r(x)$ where either $r(x) \equiv 0$ or else the degree of $r(x)$ is less than n. We assume $r(x) \not\equiv 0$. Since $r(b_1) = 0$, $r(x) = (x - b_1)q_1(x)$. Also, since $r(b_2) = (b_2 - b_1)q_1(b_2) = 0$ and $(b_2 - b_1)^{-1} \in \mathfrak{A}$, $q_1(b_2) = 0$. Thus $r(x) = (x - b_1)(x - b_2)q_2(x)$. Continuing in this fashion, $r(x) = \beta(x)q_n(x)$. Thus the degree of $r(x)$ is at least n. This is a contradiction and the proof is complete.

The next theorem uses the full machinery for covering spaces which was developed in the first section.

THEOREM 2.3. *A necessary and sufficient condition for $\alpha(x)$ to factor into monic linear factors over B is that the group $E(\Phi_B\colon \Phi_A)$ is transitive on the fibers of Φ_B. Moreover, if the condition does hold, we may select the n distinct roots of $\alpha(x) = 0$ so that each of them generates B over A.*

Proof. Assume $\alpha(x) = \prod_{i=1}^{n}(x - b_i)$, where the b_i's are all distinct elements of B. Since $\alpha(x) \in I$, for each $(h, \lambda) \in \Phi_B$ and for each pair of distinct integers i and j, $\hat{b}_i(h, \lambda) \neq \hat{b}_j(h, \lambda)$. For $i = 1, 2, \cdots, n$, define the mappings p_i of Φ_B into itself by $p_i(h, \lambda) = (h, \hat{b}_i(h, \lambda))$ for $(h, \lambda) \in \Phi_B$. Each p_i is a continuous mapping, which by Lemma 1.2 is also open. (There are examples to show that the p_i's are not necessarily one-to-one.) For any $(h, \lambda) \in \Phi_B$, $\{\hat{b}_i(h, \lambda)\colon i = 1, 2, \cdots, n\}$ consists of the n distinct roots of $\alpha_h(x) = 0$. It follows that for any $h \in \Phi_A$ and $(h, \lambda) \in \pi^{-1}(h)$, $\pi^{-1}(h) = \{p_i(h, \lambda)\colon i = 1, 2, \cdots, n\}$.

To show $E(\Phi_B\colon \Phi_A)$ is transitive on the fibers of Φ_B, let $h \in \Phi_A$ and assume (h, λ) and (h, λ') are two points of $\pi^{-1}(h)$. From the above paragraph, there exists two integers j and j' such that $p_j(h, \lambda) = (h, \lambda')$ and $p_{j'}(h, \lambda') = (h, \lambda)$. By Lemma 1.6, there exists a homeomorphism $\phi \in E(\Phi_B\colon \Phi_A)$ such that $\phi(h, \lambda) = (h, \lambda')$. Thus this part of the proof is complete.

Conversely, assume $E(\Phi_B\colon \Phi_A)$ is transitive on the fibers of Φ_B. By Lemma 1.10 there exists a set $\{\phi_1, \phi_2, \cdots, \phi_n\}$ in $E(\Phi_B\colon \Phi_A)$ which is simply transitive on the fibers of Φ_B. Define, for each $i = 1, 2, \cdots, n$, the functions f_i of Φ_B into itself by $f_i(h, \lambda) = \hat{\mathfrak{x}}(\phi_i(h, \lambda))$ for $(h, \lambda) \in \Phi_B$. Since each f_i is a simple root of $\alpha_h(x) = 0$, by the Arens-Calderon Theorem [1, Th. 7.2], there exists elements $b_1, \cdots, b_n$ in B such that $\alpha(b_i) = 0$ and $\hat{b}_i = f_i$. The fact that each b_i seperates each fiber of Φ_B implies by [11, Th. 7.2] that each b_i generates B over A. Thus $A[b_i] = B$. Now for any two distinct integers i and j, and for any $(h, \lambda) \in \Phi_B$, $(b_i - b_j)\hat{}(h, \lambda) = \hat{\mathfrak{x}}(\phi_i(h, \lambda)) - \hat{\mathfrak{x}}(\phi_j(h, \lambda)) \neq 0$ since no two of the ϕ_j's agree at any point of Φ_B. Therefore $(b_i - b_j)^{-1} \in B$. An application of Lemma 2.2 completes the proof.

Both the necessary and the sufficient condition in the above theorem are true if the assumption that $\alpha(x) \in I$ is slightly weakened. Also, the former condition is true if A is a normed algebra. (See [3], for details.)

Denote by $G(B: A)$ the group of all automorphisms g on B such that $g(a) = a$ for all $a \in A$. The following facts enable us to translate a property of $G(B: A)$ into one of $E(\Phi_B: \Phi_A)$ and thus utilize the material in the first section of this paper. For any $g \in G(B: A)$, the dual mapping g^* of g is defined by $\hat{b}(g^*(h, \lambda)) = g(b)\hat{}(h, \lambda)$ for all $(h, \lambda) \in \Phi_B$ and $b \in B$. Since $g^* \in E(\Phi_B: \Phi_A)$ ([11, Lemma 6.2]), we have a mapping from $G(B: A)$ into $E(\Phi_B: \Phi_A)$. This mapping is order reversing and, by [11, Corollary 6.5], it is also one-to-one and onto. Thus $G(B: A)$ is anti-isomorphic to $E(\Phi_B: \Phi_A)$. For any subgroup G of $G(B: A)$, we denote by G^* the image of G under this anti-isomorphism.

For the remainder of this section, we assume $G(B: A)$ possesses an n^{th} order subgroup $G_0(B: A)$ with the property that $G_0(B: A)^* = E_0(\Phi_B: \Phi_A)$ is simply transitive on the fibers of Φ_B. Thus, for example, if Φ_A is connected, then by Corollary 1.12 and Theorem 2.3, the existence of $G_0(B: A)$ is a necessary and sufficient condition for $\alpha(x)$ to factor into monic linear factors over B. If follows from Lemma 2.2 that

2.4. If $G_0(B: A) = \{g_1, g_2, \cdots, g_n\}$, then $\alpha(x) = \prod_{i=1}^{n} (x - g_i(\mathfrak{x}))$.

LEMMA 2.5. *Let* $F_0 = \{b \in B: g(b) = b$ *for all* $g \in G_0(B: A)\}$ *be the fixed algebra of* $G_0(B: A)$. *Then* $F_0 = A$.

Proof. Assume $G_0(B: A) = \{g_1, \cdots, g_n\}$ and set $b_j = g_j(\mathfrak{x})$ for $j = 1, 2, \cdots, n$. If $b = \sum_{i=0}^{n-1} a_i \mathfrak{x}^i$ is any element in F_0, then b is a symmetric function of the b_j's. Therefore, b can be written as a polynomial in the elementary symmetric functions of the b_j's with coefficients in A. But in view of 2.4, these elementary symmetric functions are the coefficients of $\alpha(x)$. Consequently, $b \in A$. This means $F_0 \subset A$. The proof is complete since the reverse inclusion always holds.

DEFINITION. ([4, Definition 1.4 and Th. 1.3, pp. 18–20]). Let G be a finite group of automorphisms on a commutative ring S and let $R = \{s \in S: g(s) = s \text{ for all } g \in G\}$. Then S is called a *Galois extension of R with Galois group G* if for any $g \in G$ and any maximal ideal M in S, there exists an element $s \in S$ such that $s - g(s) \notin M$.

As is shown in [4], S is a Galois extension of R with Galois group G if and only if S is a separable R-algebra (i.e., S is a projective $S \otimes_R S$-module) and the elements of G are pairwise strongly distinct. (Two ring homomorphisms h_1 and h_2 from T into U (both commutative rings) are called *strongly distinct* if for any nonzero idempotent u in U there exists an element $t \in T$ such that $h_1(t)u \neq h_2(t)u$.)

LEMMA 2.6. *If $G_0(B: A)$ is a subgroup of $G(B: A)$ such that $G_0(B: A)^*$ is simply transitive on the fibers of Φ_B, then B is a Galois extension of A with Galois group $G_0(B: A)$.*

Proof. By Lemma 2.5, $F_0 = A$. It follows from the fact that $G_0(B: A)$ is simply transitive on the fibers of Φ_B that if g_1 and g_2 are two distinct automorphisms in $G_0(B: A)$, then $(g_1(\mathfrak{x}) - g_2(\mathfrak{x}))^{-1} \in B$. Thus for any $g \in G_0(B: A)$ not equal to the identity, $(\mathfrak{x} - g(\mathfrak{x}))^{-1} \in B$. In particular, $(\mathfrak{x} - g(\mathfrak{x}))$ is not in any maximal ideal of B. This completes the proof.

In the event Φ_A is connected, the above result is not new (see [9, Th. 2.2]).

DEFINITION. ([4, p. 22]). Let $\mathfrak{A}$ be any Banach algebra which contains A and which is a Galois extension of A with Galois group G. Then a subalgebra F of $\mathfrak{A}$ is called *G-strong* if the restrictions to F of any two elements of G are either equal or strongly distinct as maps from F into $\mathfrak{A}$.

THEOREM 2.7. ([4, Th. 2.3, p. 24]). *Assume $G_0(B: A)$ is a subgroup of $G(B: A)$ such that $G_0(B: A)^*$ is simply transitive on the fibers of Φ_B. Then there exists a one-to-one lattice inverting correspondence between the subgroups of $G_0(B: A)$ and the separable A-subalgebras of B which are G_0-strong. If F is a separable A-subalgebra of B which is G_0-strong, then the corresponding subgroup is $G_0(B: F) = \{g \in G_0(B: A): g(c) = c \text{ for all } c \in F\}$. Moreover, for $g \in G_0(B: A)$, $G_0(B, g(F)) = g\, G_0(B: F)g^{-1}$. A subgroup G of $G_0(B: A)$ is a normal subgroup if and only if the fixed algebra F of G is mapped onto itself by every element of $G_0(B: A)$. In this case, F is a Galois extension of A with Galois group $G_0(B: A)/G$.*

3. Our goal in this section is to characterize the G_0-strong separable A-subalgebras of B in terms of our notion of a covering space. This characterization will be established under the assumptions that A is semi-simple and $G(B: A)$ contains an n^{th} order subgroup $G_0(B: A)$ such that $G_0(B: A)^*$ is simply transitive on the fibers of Φ_B. We first present a few facts concerning subalgebras of B.

Assume F is a closed subalgebra of B which contains A. We define the restriction mapping V_F of Φ_B into Φ_F by $V_F(h, \lambda) = (h, \lambda) \mid F$ for $(h, \lambda) \in \Phi_B$. Since V_F is the dual of the injection mapping of F into B, V_F is continuous [14, p. 116]. We now show V_F maps Φ_B onto Φ_F. That is, we show that any maximal ideal M in F extends to a maximal ideal in B. By [16, p. 254], B is an integral extension of A (i.e., each element of B is a zero of a monic polynomial in $A[x]$). Thus B is an integral extension of F. Therefore, by [16, p. 257], each prime ideal in F can be extended to a prime ideal in B. Since each maximal ideal in F is a prime ideal, and since each prime ideal in B can be extended to a maximal ideal in B, each maximal ideal in F can be extended to a maximal ideal in B.

Next, let π_F denote the restriction mapping of Φ_F onto Φ_A defined by $\pi_F(\mathscr{h}) = \mathscr{h} \mid A$ for each $\mathscr{h} \in \Phi_F$. Then, as with V_F, π_F is continuous. Any maximal ideal $h^{-1}(0)$ in A ($h \in \Phi_A$) can be extended to a maximal ideal in F since if $(h, \lambda) \in \pi^{-1}(h)$, then $(h, \lambda)^{-1}(0) \cap F$ is a maximal ideal in F. Therefore π_F maps Φ_F onto Φ_A. Finally, π_F is an open mapping since if U is an open subset of Φ_F, then $\pi(V_F^{-1}(U))$ is an open subset of Φ_A. The fact that $\pi_F V_F = \pi$ implies that $\pi_F(U) = \pi(V_F^{-1}(U))$.

In summary, we have the following:

PROPOSITION 3.1. *Assume F is a closed subalgebra of B which contains A. Then the restriction mapping V_F is a continuous mapping of Φ_B onto Φ_F, while the restriction mapping π_F is an open continuous mapping of Φ_F onto Φ_A.*

Now let G be a finite subgroup of $G(B: A)$, and let $F = \{b \in B: g(b) = b$ for all $g \in G\}$ be the fixed algebra of G. F is a subalgebra of B which contains A. Also, since each $g \in G$ is continuous, F is a closed subalgebra. Let E be the subgroup of $E(\Phi_B: \Phi_A)$ such that $G^* = E$. We denote by Φ_B/E the space of equivalence classes under the equivalence relation defined by means of the group E. (Thus $(h, \lambda) \sim (h, \lambda')$ if and only if for some $\phi \in E$, $\phi(h, \lambda) = (h, \lambda')$). Recally from §1 that Φ_B/E is a compact Hausdorff space and also recall that the projection mapping P is a continuous open mapping of Φ_B onto Φ_B/E. We will identify the carrier space Φ_F of F.

THEOREM 3.2. *Let G be a finite subgroup of $G(B: A)$, F the fixed* *algebra of G, and E the subgroup of $E(\Phi_B: \Phi_A)$ such that $G^* = E$. Then there exists a homeomorphism γ of Φ_F onto Φ_B/E such that $\gamma V_F = P$, where P is the projection mapping of Φ_B onto Φ_B/E, and V_F is the restriction mapping of Φ_B onto Φ_F.*

Proof. Define a mapping γ of Φ_F onto Φ_B/E by $\gamma(V_F(h, \lambda)) = P(h, \lambda)$ for $(h, \lambda) \in \Phi_B$. We must show γ is well-defined. Assume $G = \{g_1, g_2, \cdots, g_r\}$. If, for $h \in \Phi_A$ and for any two points (h, λ) and (h, λ') in $\pi^{-1}(h)$, $P(h, \lambda) \neq P(h, \lambda')$, then for each $j = 1, 2, \cdots, r$, $g_j^*(h, \lambda) \neq (h, \lambda')$. By [14, Lemma 2.6.9] there is an element $b \in B$ such that $\hat{b}(h, \lambda') = 0$ and $\hat{b}(g_j^*(h, \lambda)) = 1$ for each j. Let $c = \Pi_{j=1}^r g_j(b)$. Then $c \in F$, $\hat{c}(h, \lambda') = 0$ and $\hat{c}(h, \lambda) = \Pi_{j=1}^r g_j(b)\hat{}(h, \lambda) = \Pi_{j=1}^r \hat{b}(g_j^*(h, \lambda)) = 1$. Therefore, $(h, \lambda) \mid F \neq (h, \lambda') \mid F$. This means $V_F(h, \lambda) \neq V_F(h, \lambda')$. Consequently, γ is well-defined. It now follows that γ is a continuous, one-to-one mapping of Φ_F onto Φ_B/E. This completes the proof.

Throughout the remainder of this section we assume $G_0(B: A)$ is an n^{th} order subgroup of $G(B: A)$ such that $G_0(B: A)^* = E_0(\Phi_B: \Phi_A)$ is simply transitive on the fibers of Φ_B.

LEMMA 3.3. *Let F be a G_0-strong separable A-subalgebra of B. Then (Φ_B, V_F) is a covering space of Φ_F.*

Proof. By Theorem 2.7, F is the fixed algebra of $G_0(B: F) = \{g \in G_0(B: A): g(c) = c \text{ for all } c \in F\}$. We let E_0 be the subgroup of $E_0(\Phi_B: \Phi_A)$ such that $G_0(B: F)^* = E_0$. By Theorem 3.2, there exists a homeomorphism γ of Φ_F onto Φ_B/E_0 such that $\gamma V_F = P$. The fact that E_0 is a subgroup of $E_0(\Phi_B: \Phi_A)$ implies that no two elements of E_0 agree at any point of Φ_B. Consequently, by Lemma 1.13, (Φ_B, P) is a covering space of Φ_B/E_0. It follows that (Φ_B, V_F) is a covering space of Φ_F.

LEMMA 3.4. *Assume A is semi-simple and let F be a closed subalgebra of B which contains A. Also suppose (Φ_B, V_F) is a covering space of Φ_F. Then $G_0(B: F)^* = E_0(\Phi_B: V_F: \Phi_F) = \{\phi \in E_0(\Phi_B: \Phi_A): V_F\phi = V_F\}$. Furthermore, if F is G_0-strong, then the order of $E_0(\Phi_B: V_F: \Phi_F) =$ card $(V_F^{-1}(\not{k})) = k$, and the latter group is simply transitive on the fibers of (Φ_B, V_F).*

Proof. Assume $g \in G_0(B: F)$. Let $c \in F$, $\not{k} \in \Phi_F$ and $(h, \lambda) \in V_F^{-1}(\not{k})$. Then $\hat{c}(\not{k}) = \hat{c}(V_F(h, \lambda)) = \hat{c}(h, \lambda) = g(c)\hat{}(h, \lambda) = \hat{c}(g^*(h, \lambda)) = \hat{c}(V_F(g^*(h, \lambda)))$. (We use $\hat{c}$ to denote the element $\hat{c} \in \hat{B}$ as well as the element $\hat{c} \in \hat{F}$). Since $\hat{F}$ separates the points of Φ_F, $V_F(h, \lambda) = V_F(g^*(h, \lambda))$. Therefore $V_F = V_F g^*$. This means $G_0(B: F)^* \subset E_0(\Phi_B: V_F: \Phi_F)$.

Conversely, if $\phi \in E_0(\Phi_B: V_F: \Phi_F)$, then $V_F\phi = V_F$. Since $\phi \in E_0(\Phi_B: \Phi_A)$, there exists an automorphism $g \in G_0(B: A)$ such that $g^* = \phi$. We must show $g \in G_0(B: F)$. That is, we must show $g(c) = c$ for all $c \in F$. For any $c \in F$ and $(h, \lambda) \in \Phi_B$, we have $g(c)\hat{}(h, \lambda) = \hat{c}(g^*(h, \lambda)) = \hat{c}(V_F(\phi(h, \lambda))) = \hat{c}(V_F(h, \lambda)) = \hat{c}(h, \lambda)$. Thus $g(c)\hat{} = \hat{c}$ on Φ_B. Since A is semi-simple and $\alpha(x) \in I$, B is semi-simple [2, Th. 4.3]. Therefore $g(c) = c$.

To prove the second assertion in the lemma, since (Φ_B, V_F) is a covering space of Φ_F and since no two elements of $E_0(\Phi_B: V_F: \Phi_F)$ agree at any point of (Φ_B, V_F) (i.e., on Φ_B viewed as the covering space (Φ_B, V_F) of Φ_F), $k = \text{card}\,(V_F^{-1}(\mathscr{k})) \geqq$ order of $E_0(\Phi_B: V_F: \Phi_F)$. We will show that the latter group is transitive on the fibers of (Φ_B, V_F). (This will imply that it is simply transitive on the fibers of (Φ_B, V_F)).

Let $\mathscr{k} \in \Phi_F$ and let (h, λ) and (h, λ') be any two points in $V_F^{-1}(\mathscr{k})$. Since $\pi(h, \lambda) = \pi(h, \lambda')$, there exists a homeomorphism ϕ in $E_0(\Phi_B: \Phi_A)$ such that $\phi(h, \lambda) = (h, \lambda')$. Thus $V_F\phi(h, \lambda) = V_F(h, \lambda)$. By Lemma 1.3, there exists an open and closed set Q in Φ_B such that $V_F\phi \mid Q = V_F \mid Q$. Let u be the idempotent in B such that $Q = \{(h, \lambda) \in \Phi_B: \hat{u}(h, \lambda) = 1\}$. Let $g \in G_0(B: A)$ such that $g^* = \phi$. We will show that $g \in G_0(B: F)$. For any $c \in F$ and any $(h, \lambda) \in Q$, $\hat{c}(h, \lambda) = g(c)\hat{}(h, \lambda)$. Since B is semi-simple, $g(c)u = cu$ for any $c \in F$. Consequently, for any $g_0 \in G_0(B: F)$, $g(c)u = cu = g_0(c)u$ for any $c \in F$. The fact that F is G_0-strong implies that g must be an element of $G_0(B: F)$. Therefore $\phi \in E_0(\Phi_B: V_F: \Phi_F)$. Thus the latter group is transitive on the fibers of (Φ_B, V_F). This completes the proof.

THEOREM 3.5. *Assume A is semi-simple. Let F be a G_0-strong closed subalgebra of B which contains A and is such that (Φ_B, V_F) is a covering space of Φ_F. Then F is the fixed algebra of $G_0(B: F)$.*

Proof. Let $B_1 = F[z]/(\alpha(z))$. Since $\alpha(z) \in I(F)$, B_1 is a semi-simple Banach algebra [2, Th. 4.3] and its carrier space is $\Phi_{B1} = \{(\mathscr{k}, \lambda) \in \Phi_F \times \mathscr{C}: \alpha_{\mathscr{k}}(\lambda) = 0\}$. Let π_1 be the usual projection of Φ_{B1} onto Φ_F. Define a mapping w of (Φ_B, V_F) into Φ_{B1} by $w(h, \lambda) = (V_F(h, \lambda), \lambda)$ for $(h, \lambda) \in \Phi_B$. (Throughout this proof we view Φ_B as the covering space (Φ_B, V_F) of Φ_F and will thus write (Φ_B, V_F) in place of Φ_B). w is a continuous, one-to-one mapping of (Φ_B, V_F) onto a compact subset K of Φ_{B_1}. Since w is an open mapping (Lemma 1.2), K is an open subset of Φ_{B_1}. Furthermore, $\pi_1(K) = \pi_1 w(\Phi_B) = V_F(\Phi_B) = \Phi_F$.

By [12, Th. 2.1], there exists mutually orthogonal idempotents $u_1, u_2, \cdots, u_s$ in F, and polynomials $\beta_i(z), Q_i(z)$ in $F[z]$ for $i = 1, 2, \cdots, s$ which have the following properties: (i) $e = \sum_{i=1}^s u_i$, (ii) each $u_i\beta_i(z)$ is monic in $u_iF[z]$, (iii) $K = \bigcup_{i=1}^s \Phi_{D_i}$ where $D_i = u_iF[z]/(u_i\beta_i(z))$ and (iv) $\alpha(z) = \sum_{i=1}^s u_i\beta_i(z)Q_i(z)$. The fact that $u_i\alpha(z) \in I(u_iF)$ implies that $u_i\beta_i(z) \in I(u_iF)$ for $i = 1, 2, \cdots, s$. Since $\text{card}\,(V_F^{-1}(\mathscr{k})) = k$ for each $\mathscr{k} \in \Phi_F$, and since w is a homeomorphism of (Φ_B, V_F) onto K such that $\pi_1 w = V_F$, $k = \text{card}(\pi_1^{-1}(\mathscr{k}) \cap K)$ for each $\mathscr{k} \in \Phi_F$. It follows from (iii) that the degree of each $u_i\beta_i(z)$ must be k. Let $\beta(z) = \sum_{i=1}^{s} u_i\beta_i(z)$. This polynomial is monic and of degree k over F. Furthermore, since the u_i's are mutually orthogonal, $\beta(z) \in I(F)$.

If we denote the canonical root of $u_i\beta_i(z) = 0$ in D_i by $\mathfrak{z}_i$ for $i = 1, 2, \cdots, s$, then $c = \sum_{i=1}^{s} u_i\mathfrak{z}_i$ is a root of $\beta(z) = 0$ in $\sum_{i=1}^{s} D_i$. Let $D = F[y]/(\beta(y))$. (We change the variable from z to y in order to distinguish between the canonical roots $\mathfrak{z}$ and $\mathfrak{y}$ of $\beta(x) = 0$ in B_1 and D, respectively). It follows that the mapping $\sum_{i=0}^{k-1} f_i\mathfrak{y}^i \to \sum_{j=1}^{s} u_j \sum_{i=0}^{k-1} f_i\mathfrak{z}_j^i = \sum_{i=0}^{k-1} f_i c^i$ for $f_i \in F$ is an isomorphism of D onto $\sum_{j=1}^{s} D_j$. Therefore $K = \bigcup_{j=1}^{s} \Phi_{D_j}$ can be identified topologically with Φ_D.

If $(\mathscr{k}, \lambda) \in K$ and if $(h, \lambda) \in V_F^{-1}(\mathscr{k})$, then $\lambda = \hat{\mathfrak{x}}(h, \lambda) = \hat{\mathfrak{z}}(V_F(h, \lambda), \lambda) = \hat{\mathfrak{y}}(w(h, \lambda))$. This means $\hat{\mathfrak{x}} = \hat{\mathfrak{y}}w$. Thus $\hat{\beta}(\hat{\mathfrak{x}}) = 0$. This implies, since B_1 is semi-simple, that $\beta(\mathfrak{x}) = 0$.

We now show that B (viewed as a Banach algebra extension of F) can be identified algebraically with D. Define the mapping T of D into B by $T(\sum_{i=0}^{k=1} f_i\mathfrak{y}^i) = \sum_{i=0}^{k=1} f_i\mathfrak{x}^i$ for $f_i \in F$. This is a well-defined homomorphism which, since $F[\mathfrak{x}] = B$, maps D onto B. If $(h, \lambda) \in (\Phi_B, V_F)$, then $T(\mathfrak{y})\hat{}(h, \lambda) = \hat{\mathfrak{x}}(h, \lambda) = \hat{\mathfrak{y}}(w(h, \lambda))$. Consequently, the dual mapping T^* of T must be equal to w. By [14, Th. 3.1.17], T^* maps (Φ_B, V_F) homeomorphically onto the hull of $T^{-1}(0)$ in Φ_D. But $T^* = w$ maps (Φ_B, V_F) onto $K = \Phi_D$. Thus the hull of $T^{-1}(0)$ in Φ_D must be equal to Φ_D. Since D is semi-simple, $T^{-1}(0) = (0)$. Therefore B is isomorphic to D.

By (2.4), if $G_0(B\colon A) = \{g_1, g_2, \cdots, g_n\}$, then $\alpha(x) = \Pi_{i=1}^{n}(x - g_i(\mathfrak{x})) \in B[x]$. For each $i = 1, 2, \cdots, n$, let $c_i \in D$ such that $T(c_i) = g_i(\mathfrak{x})$. Then for each i, $\beta(c_i) = \beta(T^{-1}g_iT(\mathfrak{y})) = 0$ since $T^{-1}g_iT$ is an automorphism on D. We relable the g_i's if need be and assume $G_0(B\colon F) = \{g_1, \cdots, g_k\}$. (Lemma 3.4 shows that the order of $G_0(B\colon F)$ is indeed k.) Since the mapping $g \to T^{-1}gT$ for $g \in G(B\colon F)$ is a group isomorphism of $G(B\colon F)$ onto $G(D\colon F)$, the latter group possesses a k^{th} order subgroup $G_0(D\colon F)$ isomorphic to $G_0(B\colon F)$. Since $G_0(B\colon F)^*$ is simply transitive on the fibers of Φ_F (Lemma 3.4), and since w is a homeomorphism of Φ_B onto Φ_D such that $\pi_1 w = V_F$, $G_0(D\colon F)^*$ must be simply transitive on the fibers of Φ_D. Now, by Lemma 2.5, F is the fixed algebra of $G_0(D\colon F)$. Therefore F must also be the fixed algebra of $G_0(B\colon F)$. This completes the proof of Theorem 3.5.

We note that the above theorem need not be true if A is not semi-simple, since there may exist two distinct closed subalgebras of B which contain A and which have the same carrier space.

COROLLARY 3.6. *Under the same assumptions as in Theorem* 3.5, *there exists a polynomial* $\beta(y) \in I(F)$ *of degree* $k = \text{card}(V_F^{-1}(\mathscr{k}))$ *with* *the property that B (viewed as an extension of F) can be algebraically and topologically identified with $F[y]/(\beta(y))$.*

Proof. From the above proof, the mapping T defined by $T(\sum_{i=0}^{k-1} f_i\mathfrak{y}^i) = \sum_{i=0}^{k-1} f_i\mathfrak{x}^i$ for $f_i \in F$ is an isomorphism of D onto B. We show that T is also a homeomorphism.

Assume the norm on D is given by $\| \sum_{i=0}^{k-1} f_i\mathfrak{y}^i \| = \sum_{i=0}^{k-1} \| f_i \| t^i$ for some positive number t. Let $L = \min\{t^i : i = 0, 1, \cdots, k-1\}$. Then

$$\left\| T\left(\sum_{i=0}^{k-1} f_i\mathfrak{y}^i\right)\right\| = \left\| \sum_{i=0}^{k-1} f_i\mathfrak{x}^i \right\| \leqq \sum_{i=0}^{k-1} \| f_i \|$$
$$\leqq L^{-1} \sum_{i=0}^{k-1} \| f_i \| t^i = L^{-1} \left\| \sum_{i=0}^{k-1} f_i\mathfrak{y}^i \right\| .$$

Thus T is a continuous mapping of D onto B. By the inverse mapping theorem, T^{-1} is a continuous mapping of B onto D. This completes the proof.

Combining Lemma 3.3 with Theorem 3.5, we have the following characterization of the fixed algebras of subgroup of $G_0(B: A)$.

THEOREM 3.7. *Assume A is semi-simple, B is the Arens-Hoffman extension of A with respect to the n^{th} degree polynomial $\alpha(x) \in I$, and $G(B: A)$ contains an n^{th} order subgroup $G_0(B: A)$ such that $G_0(B: A)^*$ is simply transitive on the fibers Φ_B. Then a G_0-strong closed subalgebra F of B which contains A is the fixed algebra of a subgroup G of $G_0(B: A)$ if and only if (Φ_B, V_F) is a covering space of Φ_F.*

4. In this section, we will prove a theorem similar to Theorem 3.7, but the condition that a subgroup of $E(\Phi_B: \Phi_A)$ exists which is simply transitive on the fibers of Φ_B will be weakened.

Our assumptions throughout this portion of the paper are as follows: A is semi-simple, and $\alpha(x)$ factors into monic linear factors over B. (Or equivalently by Theorem 2.3, $E(\Phi_B: \Phi_A)$ is transitive on the fibers of Φ_B).

By Corollary 1.11, there exists a finite covering of Φ_A by mutually disjoint open and closed sets $Q_1, Q_2, \cdots, Q_s$ with the property that corresponding to each Q_j, there exists an n^{th} order subgroup E_j of $E(\pi^{-1}(Q_j): Q_j)$ which is simply transitive on the fibers of $\pi^{-1}(Q_j)$. Furthermore, each homeomorphism in E_j is the identity homeomorphism off of $\pi^{-1}(Q_j)$. We denote by $E_0(\Phi_B: \Phi_A)$ the subgroup $E_1 \times E_2 \times \cdots \times E_s$ of $E(\Phi_B: \Phi_A)$. The order of this subgroup is $n \cdot s$.

For each j, let u_j be the idempotent in A corresponding to the open and closed set Q_j in Φ_A. (i.e., $Q_j = \{h \in \Phi_A: \hat{u}_j(h) = 1\}$.) Then $\Phi_{u_jA} = Q_j$ and $\Phi_{u_jB} = \pi^{-1}(Q_j)$ for each j. We note that $e = \sum_{j=1}^{s} u_j$. For $j = 1, 2, \cdots, s$, let G_j be the n^{th} order subgroup of $G(u_jB: u_jA)$ such that $G_j^* = E_j$. We denote by $G_0(B\colon A)$ the subgroup $G_1 \times G_2 \times \cdots \times G_s$ of $G(B\colon A)$. It follows that the order of $G_0(B\colon A)$ is $n \cdot s$ and $G_0(B\colon A)^* = E_0(\Phi_B\colon \Phi_A)$. (In the event $s = 1$, then these two subgroups coincide with the subgroups $G_0(B\colon A)$ and $E_0(\Phi_B\colon \Phi_A)$ used in §2 and §3). If $g \in G_0(B\colon A)$, then $g = (g_1, g_2, \cdots, g_s)$ where $g_j \in G_j$ for each j. If $b = \sum_{j=1}^s u_j b \in B$, then $u_j g(b) = g(u_j b) = g_j(u_j b)$. Also, $g(b) = \sum_{j=1}^s g(u_j b) = \sum_{j=1}^s u_j g(u_j b) = \sum_{j=1}^s g_j(u_j b)$. We relate fixed algebras of subgroups of $G_0(B\colon A)$ to those of subgroups of G_j.

4.1. If F is the fixed algebra of a subgroup G' of $G_0(B\colon A)$, then for each j, $u_j F$ is the fixed algebra of $G'_j = G' \cap G_j$.

Proof. If $c \in F$, then since $g_j(u_j c) = u_j g(c) = u_j c$ for each $g_j \in G'_j$, $u_j F$ is contained in the fixed algebra of G'_j. Conversely if, for some $c \in B$, $g_j(u_j c) = u_j c$ for each $g_j \in G'_j$, then $g(u_j c) = g_j(u_j c) = u_j c$ for each $g \in G'$. Therefore $u_j F$ must be the fixed algebra of G'_j.

4.2. If for each j, F_j is a subalgebra of $u_j B$ which is the fixed algebra of a subgroup G'_j of G_j, then $F = \sum_{j=1}^s u_j F_j$ is the fixed algebra of $G' = G'_1 \times G'_2 \times \cdots \times G'_s$.

Proof. For $c \in F$, $u_j c \in F_j$ for each j. Therefore, for any automorphism $g \in G'$, we have $g(c) = \sum_{j=1}^s g(u_j c) = \sum_{j=1}^s g_j(u_j c) = \sum_{j=1}^s u_j c = c$. This means F is contained in the fixed algebra of G'. On the other hand, if $g(c) = c$ for some $c \in B$ and for all $g \in G'$, then $g_j(u_j c) = u_j g(c) = u_j c$. Therefore $u_j c \in u_j F_j$. Thus $c \in F$. Consequently, F is the fixed algebra of G'.

As a consequence of Lemma 2.5 and 4.2, we have

4.3. A is the fixed algebra of $G_0(B\colon A)$.

We also note that by Lemma 2.6, $u_j B$ is a Galois extension of $u_j A$ with Galois group G_j.

LEMMA 4.4. *If F is the fixed algebra of*

$$G_0(B\colon F) = \{g \in G_0(B\colon A)\colon g(c) = c$$

for all $c \in F\}$, then (Φ_B, V_F) is a local covering space of Φ_F.

Proof. By 4.1, $u_j F$ is the fixed algebra of

$$G_j(B\colon F) = \{g \in G_j\colon g(u_j c) = u_j c \text{ for all } c \in F\}\ .$$

Thus, (Theorem 2.7), $u_j F$ is a separable $u_j A$-subalgebra of $u_j B$ which is G_j-strong. Now by Lemma 3.3, $(\Phi_{u_j B}, V_j)$ is a covering space of

Φ_{u_jF}, where $V_j = V_F \mid \Phi_{u_jB}$. The proof is completed by applying Corollary 1.8.

LEMMA 4.5. *If F is a closed subalgebra of B which contains A and each u_jF is G_j-strong, and if (Φ_B, V_F) is a local covering space of Φ_F, then F is the fixed algebra of $G_0(B: F)$.*

Proof. By Corollary 1.8, there exists a finite covering of Φ_F by mutually disjoint, open and closed sets $R_1, R_2, \cdots, R_t$ such that for each i, $(V_F^{-1}(R_i), V_F \mid V_F^{-1}(R_i))$ is a covering space of R_i. Let $Z_{ij} = R_i \cap \Phi_{u_jF}$ for i $= 1, 2, \cdots, t$ and $j = 1, 2, \cdots, s$. Without loss of generality we assume that these sets are mutually disjoint. Let $X_{ij} = V_F^{-1}(Z_{ij})$ and $Y_{ij} = \pi(X_{ij}) = V_F(Z_{ij})$. These open and closed sets are such that (X_{ij}, V_{ij}) is a covering space of Z_{ij}, where $V_{ij} = V_F \mid X_{ij}$, and (X_{ij}, π_{ij}) is a covering space of Y_{ij} where $\pi_{ij} = \pi \mid X_{ij}$. Since $Y_{ij} \subset \Phi_{u_jA}$ and E_j is an n^{th} order subgroup of $E(u_jB: u_jA)$ which is simply transitive of the fibers of Φ_{u_jB}, it follows that $E_{ij} = E_j \mid X_{ij}$ is an n^{th} order subgroup of $E(X_{ij}: Y_{ij})$ which is simply transitive on the fibers of X_{ij}.

For each i and j, let e_{ij} be the idempotent in A corresponding to the open and closed set Y_{ij}. Also, let G_{ij} be the n^{th} order subgroup of $G_j \mid e_{ij}B$ such that $G_{ij}^* = E_{ij}$. It follows that $G_{ij} = G_j \mid e_{ij}B$ and thus $G_j = G_{1j} \times G_{2j} \times \cdots \times G_{tj}$. Since $e_{ij}A$ is the fixed algebra of G_{ij} (Lemma 2.5), by Lemma 2.6, $e_{ij}B$ is a Galois extension of $e_{ij}A$ with Galois group G_{ij}.

We next show that each $e_{ij}F$ is G_{ij}-strong. By our assumptions on the u_jF's, for each j, for any two distinct automorphisms g_1 and g_2 in G_j, and for any idempotent u_0 in u_jB, there exists an element c in F such that $g_1(u_jc)u_0 \neq g_2(u_jc)u_0$. In particular, for g_1, g_2 in G_{ij} and for any idempotent e_0 in $e_{ij}B \subset u_jB$, there exists an element $c \in F$ such that

$$g_1(e_{ij}c)e_0 = g_1(u_jc)e_{ij}e_0 \neq g_2(u_jc)e_{ij}e_0 = g_2(e_{ij}c)e_0 \ .$$

Therefore each $e_{ij}F$ is G_{ij}-strong.

By Theorem 3.5, $e_{ij}F$ is the fixed algebra of $G_{ij}(B: F) = \{g \in G_{ij}: g(e_{ij}c) = e_{ij}c$ for all $c \in F\}$ for each i and j. Since the proof of 4.2 does not depend upon the particular decomposition of F, F is the fixed algebra of

$$G_{11}(B: F) \times \cdots \times G_{1s}(B: F) \times \cdots \times G_{t1}(B: F) \times \cdots \times G_{ts}(B: F)$$
$$= G_1(B: F) \times \cdots G_s(B: F) = G_0(B: F) \ .$$

This completes the proof.

In conclusion, we have the following charcterization of the fixed algebras of $G_0(B\colon A)$.

THEOREM 4.6. *Assume A is semi-simple, $B = A[x]/(\alpha(x))$ and $\alpha(x) \in I$ factors into monic linear factors over B. Let $G_0(B\colon A) = G_1 \times G_2 \times \cdots \times G_s$ be defined as in the opening remarks of this section. Then a closed subalgebra F of B which contains A with the property that $u_j F$ is G_j-strong for each j is the fixed algebra of a subgroup of $G_0(B\colon A)$ if and only if (Φ_B, V_F) is a local covering space of Φ_F.*

The author would like to express his appreciation to Professor John A. Lindberg, Jr. The author would also like to thank the referees for their helpful suggestions and especially for pointing out the error in the original version of Lemma 1.10.

The author is also indebted to Mr. Philip Downum for a careful reading of the preprint and for pointing out two minor mistakes, which have been corrected.

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Received April 5, 1967 and in revised form December 12, 1968. This paper is an outgrowth of the author's doctoral dissertation written at Syracuse University under the direction of Professor John A. Lindberg, Jr. While at Syracuse, the author was partially supported by a grant from the U. S. Army Research office in Durham and by a National Aeronautics and Space Administrative Fellowship. The writing of the original version of this paper was supported in part by a Faculty Summer Research grant from Hiram College.

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