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INJECTIVE HULLS OF SEMI-SIMPLE MODULES OVER REGULAR RINGS

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The object of this paper is to provide an explicit construction of the injective hull of a semi-simple module over a commutative regular ring.

The existence of injective hulls of an arbitrary module M and their uniqueness upto isomorphism over M was shown by B. Eckmann and A. Schopf in 1953 [6]. But only in few cases these hulls have been described explicitly [1, 2].

In the special case when the ring is regular as well as Noetherian, the problem is already solved since over such a ring every module is known to be semi-simple [9] and hence is its own injective hull [11, 10]. To begin with we show that every monotypic component of the module is injective and then prove a topological lemma about T_1-spaces. The Zariski topology of the maximal ideal space of the basic ring being T_1, we make use of the lemma to obtain the desired construction of an injective hull of the module. We show by an example that a semi-simple module over a regular ring need not always be injective and obtain finally a necessary and sufficient condition for the injectivity of the module.

DEFINITION 1. A ring R is called (von Neumann) *regular* if for every $a \in R$, there exists an element $x \in R$ such that $axa = a$. This condition reduces to $a^2x = a$ if R is commutative. A Boolean ring is an example of a commutative regular ring. It is well known that a commutative ring R with unit is regular if and only if every simple R-module is injective [11].

Throughout this paper we shall consider R to be a commutative regular ring with unit 1. Let Ω denote the set of maximal ideals of R. For each $a \in R$ define Ω_a by $\Omega_a = \{P \in \Omega \mid a \notin P\}$. It follows that $\Omega_a \cap \Omega_b = \Omega_{ab}$. Thus Ω can be made into a topological space with $\{\Omega_a \mid a \in R\}$ as the system of basic open sets. This topology of Ω is known as the Zariski topology. Ω is clearly a T_1-space since if P and Q are any two distinct points in Ω, there exists $a \in P - Q$ which implies that Ω_a is a neighbourhood of Q not containing P.

DEFINITION 2. Let M be a semi-simple R-module. For any simple submodule S of M, there exists exactly one $P \in \Omega$ with $S \cong R/P$. The sum of all those simple submodules of M which are isomorphic to R/P, will be denoted by M_P and will be called the *R/P-monotypic componet* of M. The *support of M*, to be denoted by Supp (M) is the set of all those maximal ideals P in Ω for which M_P is nonzero.

In our discussion M will always denote a semi-simple R-module with supp $(M) = S$. As usual for any function f, the symbol supp (f) will mean the set of all those elements in domain (f) for which $f(x) \neq 0$. We shall write $E = H(M)$ to express the fact that E is an injective hull of M. Where no ambiguity can arise, we let $H(M)$ stand for an arbitrary injective hull of M. If α is any cardinal number and L any module, the sombol $\alpha \odot L$ will stand for the external sum of α copies of L.

THEOREM 1. *For any $P \in S$, the associated monotypic component M_P is an injective module.*

Proof. Let α be the length of M_P and T a set with $|T| = \alpha$. Then $M_P \cong \alpha \odot R/P = E$. Let π be the set of all functions from T into R/P. Now each factor R/P of π being injective **[11]**, π is injective; hence there exists an $H(E) \subseteqq \pi$. Without loss of generality we can take α to be an infinite cardinal. Assume E is not injective. Then $E \subset H(E) \subseteqq \pi$. Take any element $f \in H(E) - E$. Since $H(E)$ is an essential extension of E, one has $Rf \cap E \neq 0$ which implies $0 \neq rf \in E$ for some $r \in R - P$. As R/P is a field and $f(t) \neq 0$ for infinitely many $t \in T$, we have $0 \neq (r + P)f(t) = rf(t)$ for infinitely many $t \in T$. But this contradicts the fact that $rf \in E$. Hence E is injective.

REMARK 1. $\prod_{P \in S} M_p$ is injective since each factor M_P is injective.

DEFINITION 3. Let X be any topological space and A any subset of X. An element $x \in A$ is called an *isolated point* of A if there exists a neighbourhood U of x such that $U \cap A = \{x\}$, i.e., if $\{x\}$ is an open set in the relative topology of A. A subset A of X is said to be *discrete* if every element x in A is an isolated point of A.

LEMMA 1. *Let $f \in \prod_{P \in S} M_P$ and $a \in R$ such that $0 \neq af \in \bigoplus_{P \in S} M_P$, then every element in* supp (af) *is an isolated point of* supp (f).

Proof. Let supp $(af) = \{P_1, P_2, \cdots, P_n\}$ where $P_i \neq P_j$ if $i \neq j$. This implies that there exist elements $a_i \in P_i - P_1$ $(i = 2, 3, \cdots, n)$. Put $b = aa_2a_3 \cdots a_n$. Then $b \notin P_1$ and $b \in P$ for each $P \in \text{supp}(f)$ with $P \neq P_1$. Hence $\Omega_b \cap \text{supp}(f) = \{P_1\}$ showing that P_1 is an isolated point of supp(f). Similar argument will prove that $P_2, \cdots, P_n$ are also isolated points of supp(f).

REMARK 2. It follows from the lemma that the support of any nonzero element in an essential extension of $\bigoplus_{P \in S} M_P$ contains an isolated point.

LEMMA 2. *Let E be a proper essential extension of $\bigoplus_{P \in S} M_P$. Then for any $f \in E - \bigoplus_{P \in S} M_P$, $\operatorname{supp}(f)$ contains infinitely many isolated points.*

Proof. Since E is an essential extension of $\bigoplus_{P \in S} M_P$ and $0 \neq f \in E$, we can find an element $a \in R$ such that $0 \neq af \in \bigoplus_{P \in S} M_P$. Let $\operatorname{supp}(af) = \{P_1, P_2, \cdots, P_n\}$. By Lemma 1, each P_1 is an isolated point of $\operatorname{supp}(f)$. Choose an element $Q \in \operatorname{supp}(f) - \operatorname{supp}(af)$. As $P_i \not\subseteq Q$, there exist elements $r_i \in P_i - Q (i = 1, 2, \cdots, n)$. Then

$$r = r_1 r_2 \cdots r_n \in (P_1 \cap P_2 \cap \cdots \cap P_n) - Q .$$

It follows that $0 \neq rf \in E$. Since for some $s \in R$, $0 \neq srf \in \bigoplus_{P \in S} M_P$, we can apply Lemma 1 to show that the elements in $\operatorname{supp}(srf)$ are isolated points of $\operatorname{supp}(f)$ and they are all distinct from $P_1, P_2, \cdots, P_n$. Now $\operatorname{supp}(f)$ being infinite, we can find an element in

$$\operatorname{supp}(f) - (\operatorname{supp}(af) \cup \operatorname{supp}(srf))$$

which will give rise to another set of finitely many elements isolated points of $\operatorname{supp}(f)$ each being different from the ones obtained before. Proceeding thus we get infinitely many isolated points of $\operatorname{supp}(f)$. This proves the lemma.

We now prove the following topological fact about T_1-spaces:

LEMMA 3. *In any T_1-space X, if A and B are nonvoid subsets such that A as well as every nonvoid subset of B has an isolated point, then there exists an isolated point in $A \cup B$.*

Proof. Let the complement of a subset C of X be denoted by C'. Since A is given to have an isolated point p, there exists an open neighbourhood U of p such that $U \cap A = \{p\}$. From

$$U \cap (A \cup (B \cap U')) = U \cap A$$

we conclude that p is also an isolated point of $A \cup (B \cap U')$. If $B \cap U$ is empty, then p is an isolated point of $A \cup B$ and so the lemma holds. We have therefore to consider only the case when $B \cap U$ is nonvoid.

By hypothesis $B \cap U$ contains an isolated point q which can be assumed to be distinct from p without any loss in generality. This assumption, together with the fact that X is T_1 implies that $\{p\}'$ is an open set containing q. Now q being an isolated point of $B \cap U$, we have $V \cap B \cap U = \{q\}$ for some neighbourhood V of q. Thus we obtain

$$U \cap V \cap \{p\}' \cap (A \cup B) = U \cap V \cap \{p\}' \cap B = \{q\} \cap \{p\}' = \{q\} .$$

Since $U \cap V \cap \{p\}'$ is a neighbourhood of q, the above relation implies that q is an isolated point of $A \cup B$.

REMARK 3. From Lemma 3 we immediately have the following

(i) Let B be a discrete subset of a T_1-space X and A any subset of X with an isolated point, then $A \cup B$ has an isolated point.

(ii) If A and B are nonvoid subsets of a T_1-space X with the property that each of their nonvoid subsets has an isolated point then $A \cup B$ has the same property.

LEMMA 4. *Let $A = \bigcup_{i \in I} A_i$ where each A_i is without an isolated point. Then A has no isolated point.*

Proof. Suppose A has an isolated point p. Then $p \in A_i$ for some $i \in I$ and $\{p\} = U \cap A$ for some neighbourhood U of p. Hence $\{p\} = U \cap A_i$ contrary to the hypothesis that A_i is without an isolated point. Thus A has no isolated point.

LEMMA 5. *If A has no isolated point, then $\bar{A}$, the closure of A also has no isolated point.*

Proof. Assume p is an isolated point in $\bar{A}$ with $V \cap \bar{A} = \{p\}$ for some neighbourhood V of p, then $p \in \bar{A} \cap A'$ implies the existence of an element $q \in V \cap A \subseteqq V \cap \bar{A}$ with q distinct from p, a contradiction. Hence A has no isolated point.

REMARK 4. We know that the semi-simple module $M = \sum_{P \in S} M_P$ (direct) hence $M \cong \bigoplus_{P \in S} M_P$. Since the injective module $\prod_{P \in S} M_P$ contains $\bigoplus_{P \in S} M_P$ as a submodule, it also contains an $H(\bigoplus_{P \in S} M_P)$. Thus to find an injective hull of M, it is sufficient to obtain one of $\bigoplus_{P \in S} M_P$ inside $\prod_{P \in S} M_P$. This is done in the following:

THEOREM 2. *Let $H = \{f \in \prod_{P \in S} M_P \mid$ Every nonvoid subset of $\operatorname{supp}(f)$ has an isolated point$\}$. Then H is an injective hull of $\bigoplus_{P \in S} M_P$.*

Proof. Let f, g be any two elements in H, then since

$\text{supp}\,(f+g) \subseteqq \text{Supp}\,(f) \cup \text{supp}\,(g)$, we have $f+g \in H$ by Remark 3 (ii) following Lemma 3. Now if $a \in R$, $f \in H$, then $\text{supp}\,(af) = \Omega_a \cap \text{supp}\,(f)$ implies that $af \in H$. Hence H is an R-submodule of $\prod_{P \in S} M_P$ and it contains $\bigoplus_{P \in S} M_P$ since every nonvoid subset of a finite set is discrete. Now let $0 \neq f \in H$, then $\text{supp}\,(f)$ is nonempty and hence contains an isolated point P so that for some

$$a \in R,\ \text{supp}\,(af) = \Omega_a \cap \text{supp}\,(f) = \{p\}\ .$$

Thus $0 \neq af \in \bigoplus_{P \in S} M_P$. Hence H is an essential extension of $\bigoplus_{P \in S} M_P$.

As to the injectivity of H assume by way of contradiction that H has a proper essential extension E. Then $H \subset E \subseteqq \prod_{P \in S} M_P$. Take $f \in E, f \notin H$. Then there exists a nonvoid subset of $\text{supp}\,(f)$ without isolated points. Denote by X, the union of all those subsets of $\text{supp}\,(f)$ which have no isolated points. By Lemma 4, X has no isolated point. Let $Y = \text{supp}\,(f) \cap X'$ where X' is the complement of X in S. Then Y is nonvoid since by Remark 2, Lemma 1, $\text{supp}\,(f)$ contains an isolated point which cannot belong to X. Thus $\text{supp}\,(f) = X \cup Y$ is a decomposition of $\text{supp}\,(f)$ into disjoint nonempty subsets X and Y. Moreover every nonvoid subset of Y contains an isolated point for otherwise it will have to be contained in X which is not possible. Now for any subset $A \subseteqq \text{supp}\,(f)$, define f_A to be the function such that

$$f_A(P) = \begin{cases} f(P) & \text{if } P \in A \\ 0 & \text{if } P \in S - A \end{cases}$$

we can then write $f = F_X + f_Y$. Since $\text{supp}\,(f_Y) = Y$, one has $f_Y \in H$ and hence from $f_X = f - f_Y$, it follows that $f_X \in E$. The fact that f_X is a nonzero element in an essential extension E of $\bigoplus_{P \in S} M_P$, then implies that $X = \text{supp}\,(f_X)$ has an isolated point. We thus arrive at a contradiction. Hence H is injective. This completes the proof.

COROLLARY 1. *$\prod_{P \in S} M_P$ is an injective hull of $\bigoplus_{P \in S} M_P$ if and only if every nonvoid subset of S has an isolated point. In particular if S is discrete in Ω, then $\prod_{P \in S} M_P \cong H(M)$.*

Proof. If S has the property that each of its nonvoid subsets has an isolated point, then for every $f \in \prod_{P \in S} M_P$, $\text{supp}\,(f)$ has the same property. Hence by Theorem 2, $\prod_{P \in S} M_P = H(\bigoplus_{P \in S} M_P)$. On the other hand let $\prod_{P \in S} M_P = H(\bigoplus_{P \in S} M_P)$. Suppose that some non-empty subset A of S has no isolated point. Then A must be an infinite set. We can find a function $f \in \prod_{P \in S} M_P$ with $\text{supp}\,(f) = A$. Then $f \notin \bigoplus_{P \in S} M_P$ and hence $f \neq 0$. Since $\prod_{P \in S} M_P$ is an essential extension of $\bigoplus_{P \in S} M_P$, by

Remark 2, $\operatorname{supp}(f)$ has an isolated point contrary to the assumption that A has no isolated point. Hence every nonvoid subset of S has an isolated point. The last part of the corollary follows immediately from the fact that every element in a discrete set is an isolated point.

COROLLARY 2. *If S contains only principal ideals, then*

$$\prod_{P \in S} M_P = H(\bigoplus_{P \in S} M_P) .$$

Proof. Let Ra be any maximal ideal in S. If P in S is different from Ra, then $a \notin P$ since $a \in P$ would mean $Ra \subsetneqq P$, hence $Ra = P$, a contradiction. Regularity of R implies that $a = a^2x$ for some $x \in R$. Since $0 = a(1 - ax)$ belongs to every P in S, $1 - ax$ belongs to every element in S different from Ra. Also $1 - ax \notin Ra$ since other wise $1 \in Ra$. It follows that $\Omega_{1-ax} \cap S = \{Ra\}$. Thus every element in S is an isolated point. By Corollary 1, we have $\prod_{P \in S} M_P = H(\bigoplus_{P \in S} M_P)$.

REMARK 5. For any module M over a regular and Noetherian ring R, $\prod_{P \in S} M_P = H(\bigoplus_{P \in S} M_P) = \bigoplus_{P \in S} M_P$ since every ideal of R is a principal ideal [9] and every R-module is injective [10, 11].

COROLLARY 3. *There exist semi-simple modules over a regular ring which are not injective.*

Proof. Let R_0 be the two-element Boolean ring $\{0, e_0\}$, I an infinite index set and R, the set of all functions $f: I \rightarrow R_0$. Then R is a complete Boolean ring and hence a commutative regular ring. For each $\alpha \in I$, define P_α by $P_\alpha = \{f \in R \mid f(\alpha) = 0\}$. It is easily seen that P_α is a maximal ideal of R[7]. Let $M = \bigoplus_{\alpha \in I} R/P_\alpha$. Then M is a semi-simple module with $\operatorname{Supp}(M) = \{P_\alpha \mid \alpha \in I\}$. Take any $P_{\alpha_0} \in \operatorname{Supp}(M)$ and define f by

$$f(\alpha) = \begin{cases} e_0 & \text{if} \quad \alpha = \alpha_0 \\ 0 & \text{if} \quad \alpha \neq \alpha_0 \end{cases}$$

then $f \in R - P_{\alpha_0}$ and $f \in P_\beta$ for all $\beta \in I$ with $\beta \neq \alpha_0$. Thus

$$\Omega_f \cap \operatorname{Supp}(M) = \{P_{\alpha_0}\}$$

which implies that $\operatorname{Supp}(M)$ is discrete. Hence by Corollary 1, $\prod_{\alpha \in I} (R/P_\alpha) = H(\bigoplus_{\alpha \in I} (R/P_\alpha)$. The fact that I is infinite then shows that $\bigoplus_{\alpha \in I} (R/P_\alpha)$ is not injective.

COROLLARY 4. *If $S = A \cup D_1 \cup D_2 \cup \cdots \cup D_n$ where A has an* *isolated point and* $D_i (i = 1, 2, \cdots, n)$ *are discrete sets, then* $\prod_{P \in S} M_P \cong H(M)$.

Proof. It follows immediately from Lemma 3 and Corollary 1.

In Corollary 3 we have a concrete example showing that not every semi-simple R-module is injective. It is therefore worthwhile to ask under what conditions a semi-simple R-module is injective. The following theorem gives a characterisation for the injectivity of a semi-simple module.

THEOREM 3. *M is injective if and only if S has only finite discrete subsets.*

Proof. Let M be injective. Assume that $D \subseteqq S$ is an infinite discrete subset. We can find $f \in \prod_{P \in S} M_P$ with $\operatorname{supp}(f) = D$. Since D is infinite, $f \notin \bigoplus_{P \in S} M_P$. The fact that $\operatorname{supp}(f)$ is discrete implies by Theorem 2, that $f \in H(\bigoplus_{P \in S} M_P) = \bigoplus_{P \in S} M_P$ and so we get a contradiction. Hence S contains only finite discrete subsets.

Conversely suppose that S has only finite discrete subsets. Assume that M is not injective. Then $\bigoplus_{P \in S} M_P$ has a proper essential extension E inside $\prod_{P \in S} M_P$. Hence for any $f \in E - \bigoplus_{P \in S} M_P$, $\operatorname{supp}(f)$ contains an infinite discrete subset by Lemma 2. This contradiction then proves that M is injective.

Added in Proof.

REMARK 6. Under the assumptions of Theorem 3, S is a compact subset of Ω.

Proof. Let $S \subseteqq U_{i \in I} \Omega_{a_i}$ so that $S = U_{i \in I} (S \cap \Omega_{a_i})$ where we assume without loss of generality that each $S \cap \Omega_{a_i}$ is nonvoid. For each i in I, pick one P_i from $S \cap \Omega_{a_i}$ and let A be the set of all such P_i. Then $\Omega_{a_i} \cap A = \{P_i\}$ for each i in I. This implies that A is a discrete subset of S and hence by Theorem 3, A is finite. Consequently S is compact.

As a consequence of the above remark, we obtain as a corollary of Theorem 3, the following result of J. Levine, announced in an abstract in the Notices:

COROLLARY. (Levine) *If an injective module M over a commutative regular ring R is a direct sum of simple submodules, then there are only finitely many nonisomorphic simples in the sum.*

Proof. Let $M^* = \sum_P X_P$ be the sum of nonisomorphic simple submodules in the direct sum decomposition of M. Then for each X_P, there exists exactly one P in S with X_P isomorphic to R/P and hence the R/P-monotypic component of M^* is X_P. Moreover, M^* being a direct summand of M, is injective and, therefore, by Remark 6, its support S^* is compact. Any nonvoid subset of S^* also has this property since it is injective. We propose to show that S^* is discrete. Take any P in S^* and let $\{P\}'$ be the complement of $\{P\}$ in S^*. Then $\{P\}'$ being open and compact, we have $\{P\}' = U_{i\in 1}^{n} S_{c_i}$, where $S_{c_i} = \Omega_{c_i} \cap S^*$. Now, c_i in R implies that there exists x_i in R with $c_i = c_i^2 x_i$ $i = 1, 2, \cdots, n$. Put $d_i = 1 - c_i x_i$. Then from $c_i d_i = 0$, it follows that $d = d_1 d_2 \cdots d_n$ belongs to every Q in S^*, different from P and does not belong to P. Hence $\{P\} = S_d$. Thus every point in S^* is an isolated point as was required. By Theorem 3 we have S^* finite.

Remark 7. Theorem 1 is a special case of a more general Proposition of C. Faith [Proposition 3, Rings with ascending condition on annihilators, Nagoya Math. J. 27 (1966), 179–181]. Let a module M be called *Σ-injective* if it is injective and every direct sum of copies of M is also injective. Then Proposition 3 of Faith has the following corollaries:

Corollary 1. *Let R be any ring, and let M be any injective simple module. Then if M is finite dimensional over the field $K = End\, M_R$, then M is Σ-injective.*

Corollary 2. *If R is any commutative ring, and M is an injective simple module, then M is Σ-injective.*

Theorem 1 is a special case of Corollary 2 when R is a regular ring.

Remark 8. Corollary 3 of Theorem 2 provides an example of a semisimple module over a commutative regular ring which is not injective. C. Faith has sketched an example of a simple module over a noncommutative regular ring which is not injective [Chapter 15, "Lectures on Injective Modules and Quotient Rings" Springer Verlag, New York 1967].

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