PACIFIC JOURNAL OF MATHEMATICS
Vol. 31, No. 2, 1969

INFINITE SELF-INTERCHANGE GRAPHS

B. L. SCHWARTZ

Let G be an unoriented graph. Let $I(G)$ denote the interchange graph of G. If $G = I(G)$, we shall say G is a *self-interchange graph* (SIG). If for some positive integer $m \geqq 1$, we have $I^m(G) = G$, we shall say G is *eventually self interchange* (ESIG). This paper extends previous results to characterize all finite degree SIG's and ESIG's, (loops and parallel edges permitted), finite or infinite, connected or disconnected. It will be seen that when infinite graphs are considered, several earlier results change. For example, there are ESIG's which are not SIG's; and loop-free SIG's which are not regular.

1. **Terminology.** In this paper, we shall use the unmodified term *graph* to mean locally finite s-graph with loops permitted. The case of parallel edges forbidden will be denoted *restricted graph.* An elementary chain will be called a *line.* The interchange operation is so defined that the interchange of a loop is again a loop[1]. A loop is considered a complete 1-graph[1]. A loop contributes 1 to the degree of its vertex[1]. If a graph G has two parallel edges, the corresponding two vertices of $I(G)$ are likewise joined by two parallel edges[1].

DEFINITION. Let G be a graph. Suppose the components of G are $\{G_i \mid i \in A\}$ where A is some index set. We shall say that H is *component-subgraph* (hyphinated), or *C-subgraph* for short, if and only if components of H are $\{G_i \mid i \in B\}$ where B is some subset of A.

Similarly, if G' and G'' are disjoint graphs whose components are $\{G_i \mid i \in A\}$ and $\{G_i \mid i \in B\}$ respectively, where A and B are disjoint index sets, we say the graph consisting of the components of G' and the components of G'', $\{G_i \mid i \in A \cup B\}$ is the C-union of G' and G''. Where context makes it clear, we shall sometimes write this using the ordinary union symbol $\bigcup$, e.g. $G' \cup G''$.

2. **Preliminaries.** It has sometimes been asserted that G is a SIG if and only if G is regular of degree 2 [8], [12]. This assertion is valid only if the hypothesis include that G is loop-free and finite. Nonregular SIG's with loops have been known for some time [5]. The author has elsewhere [17] characterized all finite connected s-graph SIG's (loops permitted). We restate the result here for later use: all

[1] For the present purpose, these conventions appear to be the most appropriate, although we recognize that other conventions for these concepts are sometimes used.

finite, connected ESIG's are SIG's; and the only finite connected SIG's are graphs of the form of Figures (1c), (2b) or (2c) below.

The extension to infinite graphs is given in the following sections. The passage from finite to infinite graphs requires that certain existing tools be sharpened, since much of the current literature on interchange graphs applies only for finite ones, and sometimes to restricted and/or loop-free graphs [1], [7], [9], [11], [16].

The main result we use is an extension of a result of Krausz, originally given for finite restricted loop-free graphs [10]. Alternate approaches would involve extending various other theorems from the finite to the infinite case. One reviewer of this manuscript has suggested a result of van Rooij and Wilf [16] as one such possibility. While this would be quite possible, we believe our method is equally suitable.

The theorem of Krausz is stated for restricted finite loop-free graphs in [7] as follows:

KRAUSZ' THEOREM. *A graph H is an interchange graph of (another) graph G if and only if there exists an edge-disjoint partition of the edges of H into complete subgraphs, such that no vertex of G lies in more than two of those subgraphs.*

We now state and prove the following extension to locally finite s-graphs with loops.

EXTENDED KRAUSZ' THEOREM. *A locally finite s-graph G is an interchange graph if and only if it has no vertex with two or more loops, and there is an edge-disjoint partition of its edges into a set of complete graphs such that no vertex of G is in more than two of these subgraphs.*

Proof. It is obviously sufficient to consider only connected graphs of degree > 1.

The proof of Ore in [14] applies word for word to establish the following:

If G is a graph, then $I(G)$ has an edge-disjoint partition into complete graphs in which no vertex is in more than two of these subgraphs. (It remains undetermined whether $I(G)$ can have multiple loops.) Conversely, if H is a graph in which such an edge-disjoint partition exists, and no vertex of H has two loops, then there is a graph G such that $I(G) = H$.

The only thing remaining to prove is that an interchange graph cannot have any vertex with two or more loops. For this, we invoke the following obvious:

LEMMA. *For any finite graph G, the number of loops in $I(G)$ is the same as the number of loops in G.*

Now suppose $H = I(G)$ is an interchange graph with some vertex α incident to two loops. Since H can be decomposed into complete graphs K_i (by Ore's argument) and α can be included in only two of the K_i, it follows that no other edge of H is incident to α. Hence one component H_0 of H, must consist of just these two loops, Figure (2e). But by the lemma, the corresponding component G_0 of G must have two loops, and therefore at least two edges. Hence by definition of the I operation $I(G_0) = H_0$ has at least two vertices. But Figure (2e) has only one vertex. This contradiction completes the proof.

3. Graphs of degree 2. For reference, we catalog here certain graphs referred to later. These include all possible connected graphs of degree less than 3. In Figure 1 are the graphs of degree 0, and

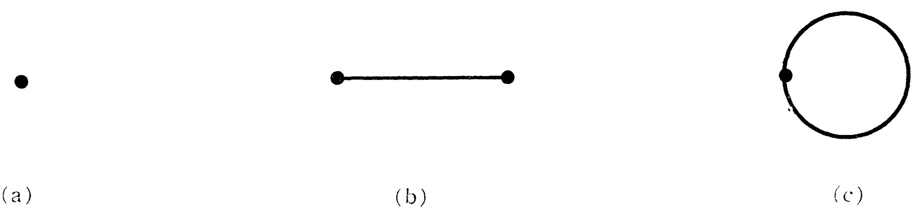


Figure 1.

1. The only graph of degree 0 is Figure (1a), the isolated vertex. The two graphs of degree 1 are in Figure (1b) and (1c): a line of length 1 and a loop, respectively. Figure 2 shows all the finite graphs of degree 2. These include a line of arbitrary length $\geqq 2$ (Figure (2a));

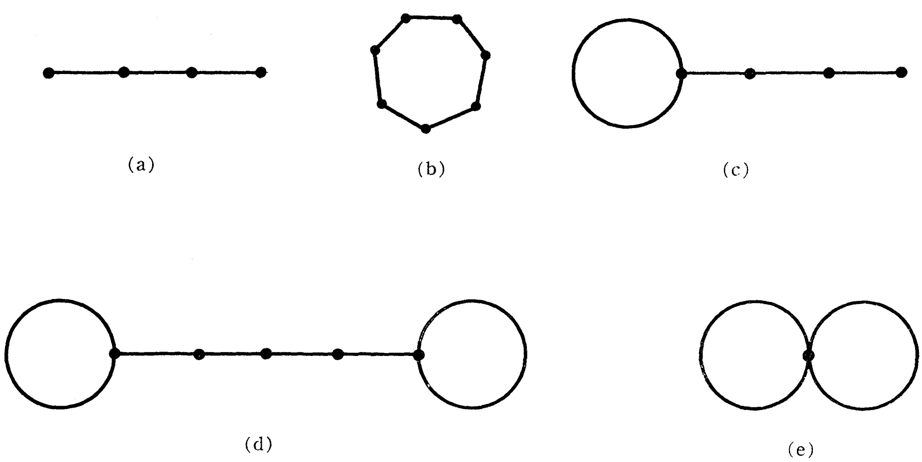


Figure 2.

a cycle of arbitrary length $\geqq 2$ (Figure (2b)); a line of arbitrary length $\geqq 1$ with a loop adjoined to one end (Figure (2c)); a line of arbitrary length > 1 with a loop adjoined to each end (Figure (2d)); and two loops on the same vertex (Figure (2e)). Finally, Figure 3 shows the

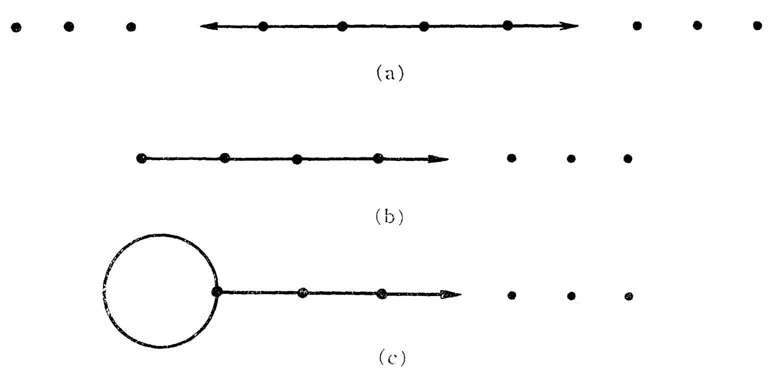


Figure 3.

infinite graphs of degree 2. These are: an infinite line whose edges can be consecutively numbered from $-\infty$ to $+\infty$ (Figure (3a)); a semi-infinite line, with edges numbered from 1 to ∞ (Figure (3b)); and a loop with a semi-infinite line adjoined (Figure (3c)).

4. Finite disconnected SIG's. *Theorem* 1. *Let G be a finite graph with p distinct components $G_1, \cdots, G_i, \cdots, G_p$.*

If G is an ESIG, *then G_i is a* SIG *for $i = 1, \cdots, p$. Hence, each G_i is of either the form of Figure* (1c), (2b), *or* (2c); *and G is a* SIG.

Proof. The I mapping clearly preserves connectivity. And since for a known fixed $m \geqq 1$, we have $I^m(G) = G$ each component G_i is mapped into (another) component of G by the function I^m, i.e., the mapping on the set of components $\{G_i\}$ induced by I^m is a permutation of order p. From group theory, we know that this permutation is the product of disjoint (algebraic) cycles [3]. Let r be the l.c.m. of the orders of the algebraic cycles. Then $(I^m)^r = I^{m \cdot r}$ induces the identity mapping on the components G_i. Hence, for any component G_i, we have $I^{mr}(G_i) = G_i$; that is, G_i is an ESIG.

But since G_i is connected and finite, we can use the previous results cited in § 2 to conclude that G_i is a SIG, and of the form of Figure (1c), (2b), or (2c).

5. Infinite SIG's first result. Now consider a general G. We require neither finiteness nor connectedness of G.

THEOREM 2. *Let G be an* ESIG. *There exists a* SIG, *H such* *that G is a C-subgraph of H; and, if G is of finite degree, so is H.*

Proof. Let the components of G be $\{G_i \mid i \in A\}$ for some index set A. The components of H consist of the components of G, plus the components of $I^j(G)$ for $j = 1, 2, \cdots, m-1$: viz

$$H = \bigcup_{j=0}^{m-1} \{I^j(G)\} .$$

The result is then immediate.

6. Infinite SIG's of finite degree.

THEOREM 3. *If C is a* SIG *of finite degree, then G is of degree* $\leqq 2$. (This theorem is of interest only for infinite SIG's, since all finite SIG's have already been characterized in Theorem 1.)

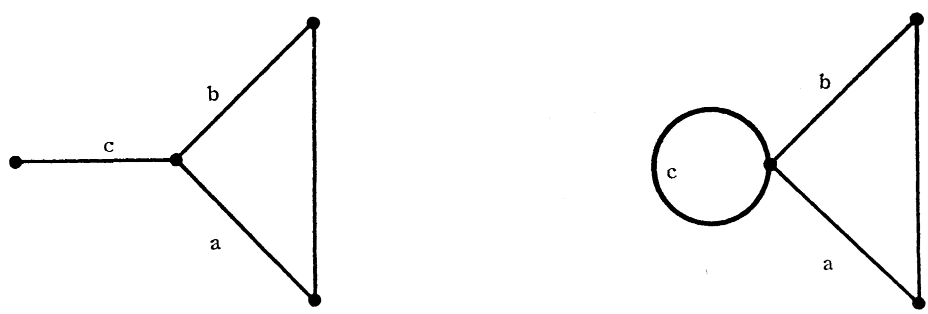
Proof. If G is a SIG of finite degree, clearly $I^k(G)$ is of the same degree for all k. Hence it suffices to prove that if degree $G > 2$, then for some k, degree $I^k(G)$ is arbitrarily large. Menon [12] has proved that if G is of degree > 3, then $I^k(G)$ is of arbitrary high degree for sufficiently large k.

The case of degree 3 is more delicate.

Let α be a vertex of G of degree 3. Let the three incident edges to α be denoted a, b, and c. By the extended Krausz' theorem, α is part of one or two complete graphs. Since there are three edges incident to α, there are just two possible cases.

Case 1. All three edges are part of a complete graph. In this case, the complete graph clearly must be a complete 4-graph. Then G must contain this complete 4-graph, K_4.

Case 2. Two edges, say a and b, are part of a complete graph; and one edge, say c, is part of another complete graph. Then a and b are two edges of a complete 3-graph. And either c is a loop, and



(a) (b)

Figure 4.

thus a complete 1-graph, or c is a complete 2-graph of one edge. These two possible cases are illustrated in Figure 4.

By direct calculation, in each case, it is verified that $I(K_4)$, $I^2(4a)$, and $I^2(4b)$ all contain vertices of degree $\geqq 4$. By Menon's result, we have a contradiction. This completes the proof.

COROLLARY. *If G is an* ESIG *of finite degree, then G is of degree $\leqq 2$.*

Proof. By Theorem 2, G is a C-subgraph of a SIG, which fulfills the hypotheses of Theorem 3.

7. Connected infinite SIG's.

THEOREM 4. *Let G be a connected infinite* ESIG *of finite degree. Then G is of one of the three forms of Figure* 3, *and therefore is a* SIG.

Proof. By the corollary to Theorem 3, G is of degree $\leqq 2$. Hence, the only candidates are the connected infinite graphs of degree 2; these are completely listed in Figure 3. By direct calculation, each of these is shown to be a SIG.

8. Disconnected infinite SIG's. We have shown that for connected graphs of finite degree, all ESIG's are SIG's. If the connectivity condition is dropped, the result no longer holds. However, in view of Theorem 2, it suffices to consider SIG's in the infinite disconnected case, since all ESIG's are C-subgraphs.

We shall use the notation L_i to denote a line of length i; $i = 0, 1, 2, \cdots$. (Hence L_0 and L_1 are graphs of the forms Figure (1a) and (1b), respectively.) We shall denote graphs of the forms Figure (2d) and (2e) by M_i, where i is the length of the line between the two loops.

The graph $\mathscr{L} = \bigcup_{i=0}^{\infty} L_i$ is obviously a SIG, as is immediately verified by direct calculation. We now show that this is essentially the only SIG of finite degree beyond those we have already described.

LEMMA 1. *For $i \geqq 0$, L_i is a component of $I(G)$ if and only if L_{i+1} is a component of G* [18].

LEMMA 2. *For $i > 0$, M_i is a component of $I(G)$ if and only if M_{i-1} is a component of G.*

The proofs of these lemmata are easy, and are omitted.

THEOREM 5. *Let G be a* SIG *of finite degree which does not contain as a C-subgraph any of the following forms:*
Figure (2b), *a cycle;*
Figure (2c), *a finite line with attached loop;*
Figure (3a), *an infinite line;*
Figure (3b), *a semi-infinite line;*
Figure (3c), *a semi-infinite line with attached loop; or*

$$\mathscr{L} = \bigcup_{i=0}^{\infty} L_i \, .$$

Then G is the empty graph.

Proof. If L_i, $i > 0$, is a component of G, then $\mathscr{L}$ is a C-subgraph of G. This follows by infinite induction, using Lemma 1.

Since by hypothesis G does not contain $\mathscr{L}$, we conclude that G does not contain any L_i, $i = 0, 1, 2, \cdots$, as a component.

All other components of degree less than 3 have been explicitly excluded except M_i.

If for some $i > 0$, M_i is a component of G, then M_{i-1} is a component. This follows as above from Lemma 2.

Hence by finite induction, if M_i is a component of G, so is M_0, (Figure (2e)). By the extended Krausz theorem, however, M_0 is not an interchange graph. But this is a contradiction.

It follows that G contains no component M_i. Hence G contains no component at all, and hence is empty. This completes the proof.

COROLLARY. *A graph G of finite degree is a* SIG *if and only if it is C-union of graphs of the form of* (1c), (2b), (3a), (3b), (3c), *and $\mathscr{L}$.*

9. ESIG's that are not SIG's. From Theorem 5, we can characterize all the ESIG's of finite degree which are not SIG's. They are all C-subgraphs of the graph $\mathscr{L}$ of the previous section. Let G be an ESIG. Clearly if L_i is a component of G, so is $I^m(L_i) = L_{i-m}$ for $m \leqq i$. It is therefore easy to see that any graph of the following form is an ESIG of this type.

$J_{m,n} = \bigcup_{i=0}^{\infty} L_{mi+n}$ where m, n are positive integers and $n < m$. Furthermore, we have:

THEOREM 6. *C-unions of these graphs $J_{m,n}$ are the only* ESIG*'s of finite degree which do not contain any* SIG *C-subgraphs.*

The proof follows the lines of the proof of Theorem 5. We omit the details.

COROLLARY. *A graph G of finite degree is an* ESIG *if and only if it is the C-union of graphs of the form of* (1c), (2b), (2c), (3a), (3b), (3c), $\mathscr{L}$, *and* $J_{m,n}$.

The corollaries of this and the previous section provide the characterization of finite degree SIG's and ESIG's promised in the introduction.

10. **Acknowledgement.** The author gratefully acknowledges the contribution of the referee, who pointed out the necessity for distinguishing between the concepts of subgraph and C-subgraph. This has facilitated an improvement in both the clarity and the rigor of the presentation.

BIBLIOGRAPHY

1. A. Andreatta, *On finite graphs which are line graphs* (Italian), 1st Lombardo Accad. Sci. Lett. Remd. A. **98** (1964), 133-156.
2. Claude Berge, *The theory of graphs and its applications*, translated by Alison Doig, John Wiley and Son, 1962.
3. Garrett Birkhoff and Saunders MacLane, *A Survey of modern algebra*, revised ed., Macmillan, 1953.
4. G. Chartrand, *The existence of complete cycles in repeated line graphs*, Bull. Amer. Math. Soc. **71** (1965), 668-670.
5. Capt. Bernard Clark, USMC; private communication with the author, 1964.
6. Anna Maria Ghirlanda, *Osservazioni sulle caratteristiche dei grafi o singrammi*, Ann. Univ. Ferrara Sez. (N.S.) **11** (1962-65), 93-106.
7. Frank Harary, "Combinatorial problems in graphical enumeration", Chapter 6 of *Applied combinatorial mathematics*, Edwin. F. Beckenbach, editor, John Wiley and Sons, 1964.
8. ———, Review of No. 16 below, Math. Review **33**, No. 3959.
9. A. Hoffman and D. Ray-Chaudhuri, *On the line graph of a finite affine plane*, Canad. J. Math. **17** (1965), 687-694.
10. J. Krausz, *Domenstration nouvelle d'une theoreme de Whitney sur les reseaux*, Mat. Fiz. Lapok **50** (1943), 75-85.
11. V. Menon, *The isomorphism between graphs and their adjoint graphs*, Canad. Math. Bull **8** (1965), 7-15.
12. ———, *On repeated interchange graphs*, Amer. Math. Monthly **13** (1966), 986-989.
13. L. Muracchini and A. Ghirlanda, *Sul grafo commutato e sul grafo opposto di un grafo orientato*, Atti. Sem. Mat. Fis. Univ. Modena **14** (1965), 87-97.
14. Oystein Ore, *Theory of graphs*, American Mathematical Society Colloquium Publications, Volume 38, Providence, R.I., 1962.
15. D. Ray-Chaudhuri, *Characterization of line graphs*, Math. Res. Ctr. Univ. Wis. Tech. Report 642, 1966.
16. A. van Rooij and H. Wilf, *The interchange graph of a finite graph*, Acta. Math. Acad. Sci. Hungar **16** (1965), 263-269.
17. B. L. Schwartz, *On interchange graphs*, Pacific J. Math. **27** (1968), 393-396.
18. H. Whitney *Congruent graphs and the connectivity of graphs*, Amer. J. Math. **54** (1932), 150-168.

Received September 23, 1968, and in revised form May 6, 1969.

MITRE CORPORATION