PACIFIC JOURNAL OF MATHEMATICS
Vol. 31, No. 3, 1969

AFFINE COMPLEMENTS OF DIVISORS

MARIO BORELLI

Recently Goodman and Hartshorne have considered the question of characterizing those divisors in a complete linear equivalence class whose support has an affine complement. However their characterization is not clearly "linear", and in fact we have to resort to Serre's characterization of affine schemes to prove that, indeed, the condition "the support of an effective divisor has an affine complement" is, in the language of Italian geometry, expressed by linear conditions. In the language of Weil this means that the set of effective divisors, in a complete linear equivalence class, whose supports have affine complements is a linear system. This is our first result. Subsequently we study the intersection of all such affine-complement supports of effective divisors in the multiples of a given linear equivalence class, and prove the following: if the ambient scheme is a surface or a threefold, or if the characteristic of the groundfield is 0, (or assuming that we can resolve singularities!) then a minimal intersection cannot have zero-dimensional components, nor irreducible components of codimension 1, whose associated sheaf of ideals is invertible.

In particular we obtain anew Zariski's result (see [11]) that every complete nonsingular surface is projective, and that the examples of nonsingular, nonprojective threefolds given by Nagata and Hironaka (see [9] and [5]) are optimal, in the sense that no examples can be given of nonsingular, nonprojective threefolds in which the "bad" subsets are either closed points or two-dimensional subschemes.

The notation and terminology we use are, unless otherwise specifically stated, those of [4]. We consider only algebraic schemes, with an arbitrary, algebraically closed ground field k. For the sake of convenience we drop the adjective "algebraic", and speak simply of schemes.

When we refer to, say, Lemma 2.3 without further identification, we mean Lemma 2.3 of the present work, to be found as the third statement of § 2.

1. Let X be a scheme, $\mathscr{L}$ an invertible sheaf over X. A regular section $s \in \Gamma(X, \mathscr{L})$ identifies an exact sequence

$$0 \longrightarrow \mathscr{L}^{-1} \xrightarrow{\theta(s)} \mathscr{O}_X \longrightarrow \mathscr{K} \longrightarrow 0$$

with $\operatorname{Supp}(\mathscr{K}) = \operatorname{Supp}(s) = \{x \in X \mid s(x) \in \mathscr{M}_x\}$, $\mathscr{M}_x$ denoting the unique maximal submodule of the stalk $\mathscr{L}_x$ of $\mathscr{L}$ at x.

With these notations we quote a result of Goodman and Hartshorne (see [3], Proposition 3):

$X_s = X - \mathrm{Supp}(s)$ is affine if, and only if, X_s contains no complete curves and the following condition holds:

$$\dim_k \varinjlim_n H^1(X, \mathscr{L}^{\otimes n} \otimes \mathscr{F}) < \infty \tag{1}$$

for every coherent sheaf $\mathscr{F}$ over X, where the maps in the inductive system above are those induced by the injection $\mathscr{L}^{-1} \xhookrightarrow{\theta(s)} \mathscr{O}_X$.

In this part we shall prove the following

THEOREM 1.1. *Let $\mathscr{L}$ be an invertible sheaf over a scheme X. Let $A(\mathscr{L}) = \{s \in \Gamma(X, \mathscr{L})_{\mathrm{reg}} \mid X_s$ is affine$\}$. Then $A(\mathscr{L}) \cup \{0\}$ is a vector space over k.*

REMARK. While it is quite easy to show that the set of sections $A_c(\mathscr{L}) = \{s \in \Gamma(X, \mathscr{L}) \mid X_s$ contains no complete curve$\}$ is a vector space over k, we were not able to show that those elements of $\Gamma(X, \mathscr{L})$ which obey (1) also form a vector space. Leaving this open question aside, to prove Theorem 1.1 we make use instead of Serre's well known characterization of affine schemes, (see [10], and following lemma of Goodman and Hartshorne (see [3], Lemma 4).

LEMMA 1.2. *Let $\mathscr{L}$ be an invertible sheaf over a scheme X, and let $s \in \Gamma(X, \mathscr{L})$. Then, for every coherent sheaf over X, and every $i \geqq 0$,*

$$\varinjlim_n H^i(X, \mathscr{L}^{\otimes n} \otimes \mathscr{F}) = H^i(X_s, \mathscr{F} \mid X_s) .$$

The following lemma is needed in the proof of Theorem 1.1.

LEMMA 1.3. *Let $\{V_n\}_{n \geqq 0}$ be vector spaces over k, and let*

$$\theta_n \colon V_0 \longrightarrow \mathrm{Hom}_k(V_n, V_{n+1}) \qquad n > 0$$

be linear transformations such that, for all $v, w \in V_0$,

$$\theta_{n+1}(w) \circ \theta_n(v) = \theta_{n+1}(v) \circ \theta_n(w) .$$

For all $p > 0, q > 0$, let $\theta_{p+q,p}(v) = \theta_{p+q}(v) \circ \cdots \circ \theta_p(v)$. Then, for all $p > 0, q > 0$, and for all $v, w \in V_0$, and for all $\lambda, \mu \in k$

$$\theta_{p+q,p}(\lambda v + \mu w) = \sum_{i=0}^{q+1} \binom{q+1}{i} \lambda^i \mu^{q+1-i} \theta_{p+q,p+q-i+1}(v) \circ \theta_{p+q-i,p}(w) .$$

Proof. We proceed by induction on q. For $q = 1$ we have

$$\begin{aligned}&\theta_{p+1}(\lambda v + \mu w) \circ \theta_p(\lambda v + \mu w)\\ &\qquad = [\lambda\theta_{p+1}(v) + \mu\theta_{p+1}(w)][\lambda\theta_p(v) + \mu\theta_p(w)] ,\end{aligned}$$

which gives us our assertion. Now

$$\begin{aligned}&\theta_{p+q,p}(\lambda v + \mu w)\\ &= \theta_{p+q}(\lambda v + \mu w)\Big[\sum_{j=0}^{q}\binom{q}{j}\lambda^j\mu^{q-j}\theta_{p+q-1,p+q-j}(v)\circ\theta_{p+q-j-1,p}(w)\Big]\\ &= \sum_{i=0}^{q+1}\lambda^i\mu^{q-i+1}\Big[\binom{q}{i-1}\theta_{p+q,p+q-i+1}(v)\circ\theta_{p+q-i,p}(w)\\ &\quad + \binom{q}{i}\theta_{p+q}(w)\circ\theta_{p+q-1,p+q-i}(v)\circ\theta_{p+q-i-1,p}(w)\Big]\\ &= \sum_{i=0}^{q+1}\lambda^i\mu^{q-i+1}\Big[\binom{q}{i-1}\theta_{p+q,p+q-i+1}(v)\circ\theta_{p+q-i,p}(w)\\ &\quad + \binom{q}{i}\theta_{p+q,p+q-i+1}(v)\circ\theta_{p+q-i,p}(w)\Big]\end{aligned}$$

and the lemma is proved.

Proof of Theorem 1.1. For any $s \in \Gamma(X, \mathscr{L})_{\text{reg}}$ and any coherent sheaf $\mathscr{F}$ let

$$\theta_n(s)\colon H^1(X, \mathscr{L}^{\otimes n} \otimes \mathscr{F}) \longrightarrow H^1(X, \mathscr{L}^{\otimes(n+1)} \otimes \mathscr{F})$$

denote the homomorphism corresponding to the injection

$$0 \longrightarrow \mathscr{L}^{-1} \xrightarrow{\theta(s)} \mathscr{O}_X .$$

By Lemma 1.2, and Theorem 1 of [10], it clearly suffices to prove the following statement: If $s_i \in \Gamma(X, \mathscr{L})$, $i = 1, 2$, are such that

$$\varinjlim_n\, [H^1(X, \mathscr{L}^{\otimes n} \otimes \mathscr{J}), \theta_n(s_i)] = 0, i = 1, 2$$

for all coherent sheaves of ideals $\mathscr{J}$ over X, then, for all $\lambda, \mu \in k$,

$$\varinjlim_n\, [H^1(X, \mathscr{L}^{\otimes n} \otimes \mathscr{J}), \theta_n(\lambda s_1 + \mu s_2)] = 0 .$$

Now the homomorphisms $\theta_n(s)$ define a homomorphism

$$\theta_n\colon H^\circ(X, \mathscr{L}) \longrightarrow \operatorname{Hom}_k[H^1(X, \mathscr{L}^{\otimes n} \otimes \mathscr{J}), H^1(X, \mathscr{L}^{\otimes(n+1)} \otimes \mathscr{J})]$$

with the additional property that the following diagram commutes:

$$
\begin{tikzcd}
H^1(X, \mathscr{L}^{\otimes n} \otimes \mathscr{F}) \arrow[r, "\theta_n(s_1)"] \arrow[d, "\theta_n(s_2)"'] & H^1(X, \mathscr{L}^{\otimes(n+1)} \otimes \mathscr{F}) \arrow[d, "\theta_{n+1}(s_2)"] \\
H^1(X, \mathscr{L}^{\otimes(n+1)} \otimes \mathscr{F}) \arrow[r, "\theta_{n+1}(s_1)"'] & H^1(X, \mathscr{L}^{\otimes(n+2)} \otimes \mathscr{F}) .
\end{tikzcd}
\tag{1.3.1}
$$

We can therefore apply Lemma 1.3, and obtain, with the same notations as in the lemma:

$$\theta_{p+i,p}(\lambda s_1 + \mu s_2) = \sum_{j=0}^{i+1} \binom{i+1}{j} \lambda^j \mu^{i+1-j} \theta_{p+i,p+i-j+1}(s_1) \circ \theta_{p+i-j,p}(s_2) ,$$

for all $p \geqq 1$, and all $i \geqq 1$.

The theorem now follows from the above equation, the commutativity of diagram (1.3.1), and the fact that

$$\varinjlim_n [H^1(X, \mathscr{L}^{\otimes n} \otimes \mathscr{F}), \theta_n(s_i)] = 0 \qquad i = 1, 2$$

implies that, for all $z \in H^1(X, \mathscr{L}^{\otimes p} \otimes \mathscr{F})$, and all $p > 0$,

$$\theta_{p+n,p}(s_i)(z) = 0$$

for $n \gg 0$ (depending possibly on z).

REMARKS. Clearly $A(\mathscr{L}) \subset A_c(\mathscr{L})$, but the question "does $A(\mathscr{L}) \neq \emptyset$ imply $A_c(\mathscr{L}) = A(\mathscr{L})$?" has a negative answer. In fact, if $A(\mathscr{L}) \neq \emptyset$, and $s \in A_c(\mathscr{L})$, X_s need not even be quasi-affine. A counterexample can be found in the birational blow-up of a point on the exceptional divisor of the blow-up of a point of the projective plane. To the author's knowledge, however, there are no counterexamples to the affirmative answer with Supp(s) irreducible. One might therefore conjecture with Goodman that $A(\mathscr{L}) \neq \emptyset$, $s \in A_c(\mathscr{L})$, Supp(s) irreducible imply $s \in A(\mathscr{L})$.

2. We continue with the notations introduced in § 1.

DEFINITION 2.1. Let X be a scheme, $\mathscr{L}$ an invertible sheaf over X, U an open subset of X. We define, for all $n > 0$,

(a) $$X_n(\mathscr{L}) = \bigcup_{s \in A(\mathscr{L}^{\otimes n})} X_s$$

(b) $$X(\mathscr{L}) = \bigcup_{n>0} X_n(\mathscr{L}) .$$

Furthermore, we say that U is $\mathscr{L}$-projective if $U \subset X(\mathscr{L})$.

REMARK. If U is $\mathscr{L}$-projective, then $\mathscr{L} \mid U$ is ample, but the converse need not be true. In fact $H^\circ(X, \mathscr{L}^{\otimes n})$ may have base points for all $n > 0$, while choosing U sufficiently small will always result in $\mathscr{L} \mid U$ being ample. Of course, to say that X is $\mathscr{L}$-projective is to say that $\mathscr{L}$ is ample.

We proceed to study $\mathscr{L}$-projective open subsets. If $(F, \mathscr{O}_F)$ is a closed subscheme of the scheme X, we say that the invertible sheaf $\mathscr{L}$ *is ample on* F when the invertible sheaf $\mathscr{L} \otimes \mathscr{O}_F$ over F is ample.

LEMMA 2.2. *Let $\mathscr{L}$ be an invertible sheaf over the scheme X, let U be an $\mathscr{L}$-projective open subset, let $x_1, \cdots, x_m$ be a finite subset of U. Then there exists a suitable $s \in A(\mathscr{L}^{\otimes n})$ such that*

$$x_1, \cdots, x_m \in X_s \subset U .$$

Proof. It clearly suffices to prove the lemma taking $U = X(\mathscr{L})$. Using the quasi-compacity of $X(\mathscr{L})$ and a well known argument, we see that, for a sufficiently high integer n, and a suitable finite number of elements $s_0, s_1, \cdots, s_t \in A(\mathscr{L}^{\otimes n})$, there exists an injection

$$X(\mathscr{L}) \hookrightarrow \operatorname{Proj}(k[s_0, s_1, \cdots, s_t]) .$$

By Theorem 1.1, a homogeneous element of degree d of the ring $k[s_0, s_1, \cdots, s_t]$ is an element of $A(\mathscr{L}^{\otimes nd})$, and the statement of the lemma is trivially true for $\operatorname{Proj}(k[s_0, s_1, \cdots, s_t])$. The lemma is proved.

PROPOSITION 2.3. *Let X be a scheme proper over k, $\mathscr{L}$ an invertible sheaf over X, $s \in A_c(\mathscr{L}^{\otimes n})$. If $\mathscr{L}$ is ample on $F = X - X_s$, then $\mathscr{L}$ is ample.*

Proof. We shall apply the Nakai-Moishezon-Kleiman criterion for ampleness. (See [6] or [7]). We may clearly assume that X is integral (see [4], Ch. III, 2.6.2), and proceed by induction on $r = \dim X$. The case $r = 1$ is trivial. Now let Y be an integral closed subscheme of X, and let $\dim Y = t$. If $t = r$, i.e., if $Y = X$, then $(\mathscr{L}^r \cdot X) = n(\mathscr{L}^{r-1} \cdot F) > 0$, since $\mathscr{L}$ is ample on F. Here $(\cdot)$ denotes the intersection pairing, as defined in [6]. If $t < r$, then either $Y \subset F$ or $Y \cap X_s \neq \varnothing$. In the latter case the canonical image of s in

$$H^\circ(Y, \mathscr{L}^{\otimes n} \otimes \mathscr{O}_Y)$$

is an element of $A_c(\mathscr{L}^{\otimes n} \otimes \mathscr{O}_Y)$, and therefore, by the induction assumption, $(\mathscr{L}^t \cdot Y) > 0$. In the former case, since $\mathscr{L}$ is ample on F by hypothesis, it is a fortiori ample on Y, and $(\mathscr{L}^t \cdot Y) > 0$ follows. The proposition is proved.

REMARK. The hypothesis $s \in A_c(\mathscr{L}^{\otimes n})$ is essential, in fact $Y \cap X_s \neq \varnothing$ does not imply $(\mathscr{L}^t \cdot Y) > 0$ in general. An easy counterexample to the proposition can be given, where $\mathscr{L}$ is not ample, but it is ample on $X - X_s$ for some section $s \in H^\circ(X, \mathscr{L})$. Take, for instance, $X =$ the blow-up of $\boldsymbol{P}_2(k)$ at a point, $f: X \to \boldsymbol{P}_2(k)$ the associated surjection, $\mathscr{L} = f^*[\mathscr{O}_{\boldsymbol{P}_2(k)}(1)]$.

COROLLARY 2.4. *If U is an open, $\mathscr{L}$-projective subset of the scheme X, and $\mathscr{L}$ is ample on $F = X - U$, then $\mathscr{L}$ is ample.*

Proof. We may assume that X is integral, and proceed by induction on $r = \dim X$. The case $r = 1$ is trivial. Let now $s \in H^\circ(X, \mathscr{L}^{\otimes n})$ be such that $X_s \subset U$ is affine. Such s exists by Lemma 2.2. Let

$$0 \longrightarrow \mathscr{L}^{\otimes -n} \xrightarrow{\theta(s)} \mathscr{O}_X \longrightarrow \mathscr{O}_D \longrightarrow 0$$

be the exact sequence associated to s. Let $D_1, D_2, \cdots, D_t$ be the irreducible components of the subscheme D. If $D_i \cap U$ is empty, then $\mathscr{L}$ is ample on D_i by hypothesis. If $D_i \cap U \neq \varnothing$, then $\mathscr{L}$ is ample on D_i by the induction assumption. In fact $D_i \cap U$ is clearly $\mathscr{L} \otimes \mathscr{O}_{D_i}$-projective. In either case $\mathscr{L}$ is ample on D_i, and therefore, by 2.6.2 of Ch. III of [4], $\mathscr{L}$ is ample on D. Since X_s is affine we can apply Proposition 2.3. The corollary is proved.

COROLLARY 2.5. *If U is an open, $\mathscr{L}$-projective subset of the scheme X, and* $\dim(X - U) = 0$, *then $\mathscr{L}$ is ample.*

Proof. $\mathscr{L}$ is locally free of rank 1, hence trivially ample on $X - U$. Apply the previous corollary.

We proceed to study the behavior of $\mathscr{L}$-projective open subsets under certain types of morphisms.

PROPOSITION 2.6. *Let $f: X \to Y$ be a proper surjective morphism of integral schemes, $U \subset Y$ an open subscheme, and assume that:*

(a) *$f_*(\mathscr{O}_X)$ is locally free over Y.*

(b) *$f \mid f^{-1}(U): f^{-1}(U) \to U$ is a finite morphism.*

Let $\mathscr{L}$ be an invertible sheaf over Y. Then U is $\mathscr{L}$-projective if, and only if, $f^{-1}(U)$ is $f^*(\mathscr{L})$-projective.

Proof. The necessity follows from the fact that $f \mid f^{-1}(U)$ is an affine morphism, and 1.3.2 of Ch. II of [4].

To prove the sufficiency, let $y \in U$ be a given closed point. Since $f^{-1}(U)$ is $f^*(\mathscr{L})$-projective, and $f^{-1}(y)$ is a finite set of closed points by hypothesis (b) above, for some $n > 0$ there exists, by Lemma 2.2, a section $s \in H^\circ(X, f^*(\mathscr{L}^{\otimes n}))$ such that X_s is affine and $f^{-1}(y) \subset$

$X_s \subset f^{-1}(U)$.

Let now $N: f_*(\mathcal{O}_X) \to \mathcal{O}_Y$ denote the norm mapping, as defined in [4], Ch. II, 6.5.1 and ff. By 5.4.10 of Ch. 0_I of [4] we see that s corresponds to a section $t' \in H^\circ(Y, f_*(\mathcal{O}_X) \otimes \mathcal{L}^{\otimes n})$. Let

$$t = (N \otimes I)(t') \in H^\circ(Y, \mathcal{L}^{\otimes n}) .$$

By 6.5.7 of Ch. II of [4], t is such that $y \in Y_t$ and $f^{-1}(Y_t) \subset X_s$. Since $f \mid f^{-1}(U)$ is a finite morphism, it follows from a theorem of Chevalley's (see [4], Ch. II, 6.7.1) that Y_t is affine. Clearly $Y_t \subset U$. Therefore the proposition is proved.

The following proposition, which will enable us to obtain our main results, is a generalization of Theorem 1 of [2]. We shall say that a morphism of integral schemes $f: X \to Y$ is dominating (and also say, less precisely, that X dominates Y) if the morphism is proper, birational and surjective.

PROPOSITION 2.7. *Let Y be a normal, integral scheme, $\mathcal{L}$ an invertible sheaf over Y, U an open subscheme of Y. Then U is $\mathcal{L}$-projective if, and only if, the following condition holds:*

There exists an integral scheme X dominating Y, an open subscheme $V \subset X$ with $V \approx U$, and invertible sheaves of ideals $\mathcal{T}$, $\mathcal{J}$ of $\mathcal{O}_X$ with support off V such that, for $t \gg 0$, $f^*(\mathcal{L}^{\otimes nt}) \otimes \mathcal{J}^{\otimes t} \otimes \mathcal{T}$ is ample on X, f being the morphism $f: X \to Y$.
Furthermore the scheme X can be obtained from Y by blowing up a suitable sheaf of ideals of $\mathcal{O}_Y$ with support off U.

Proof. We essentially follow the argument given by Goodman in [2]. The sufficiency is obvious from Proposition 2.6. In fact, since the support of $\mathcal{J}^{\otimes t} \otimes \mathcal{T}$ is off V, the fact that $f^*(\mathcal{L}^{\otimes nt}) \otimes \mathcal{J}^{\otimes t} \otimes \mathcal{T}$ is ample clearly implies that V is $f^*(\mathcal{L}^{\otimes n})$-projective. Now, $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, since Y is normal, and Proposition 2.6 applies.

To prove the necessity, choose $t \gg 0$ so that the scheme

$$Z = \operatorname{Proj}\Big[\bigoplus_{n \geqq 0} H^\circ(Y, \mathcal{L}^{\otimes nt})\Big]$$

is birational to Y and contains an open subscheme $W \approx U$. Note that, if $\mathcal{H} = \mathcal{O}_Z(1)$, $\mathcal{H}$ is very ample for Z. As in [2], we can obtain a scheme X which dominates both Z and Y by successively blowing up sheaves of ideals of $\mathcal{O}_Z$ with support off W and taking joins (see theorems 3.2, 3.3 of [8]).

Let $Z_1 \xrightarrow{g} Z$ be obtained by blowing up a sheaf of ideals $\mathcal{J} \subset \mathcal{O}_Z$ with support off W. Then the following three statements hold:

(i) Z_1 contains an open subscheme $W_1 \approx W \approx U$.

(ii) $\mathscr{I}\mathscr{O}_{Z_1}$ is an invertible sheaf of ideals of $\mathscr{O}_{Z_1}$ with support off W_1.

(iii) for $n \gg 0$ the invertible sheaf $g^*(\mathscr{H}^{\otimes n}) \otimes \mathscr{I}\mathscr{O}_{Z_1}$ is ample on Z_1 (see [4], Ch. II, 4.6.13).

The join X of a finite number of such blow-ups has therefore the following three properties:

(a) X dominates Z. Let $h: X \to Z$ denote the corresponding surjective, birational, proper morphism.

(b) X contains an open subscheme $V \approx W \approx U$.

(c) There exists an invertible sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X$ with support off V such that, for $n \gg 0$, the invertible sheaf $h^*(\mathscr{H}^{\otimes n}) \otimes \mathscr{J}$ is ample on X.

Let now $s \in H^\circ(Y, \mathscr{L}^{\otimes nt})$, and let

$$0 \longrightarrow \mathscr{L}^{\otimes -nt} \xrightarrow{\theta(s)} \mathscr{O}_Y \longrightarrow \mathscr{O}_D \longrightarrow 0$$

be the corresponding sequence of sheaves. We have $s \in H^\circ(Z, \mathscr{H}^{\otimes n})$, and therefore we have an exact sequence of sheaves

$$0 \longrightarrow \mathscr{H}^{\otimes -n} \xrightarrow{\theta'(s)} \mathscr{O}_Z \longrightarrow \mathscr{O}_H \longrightarrow 0 .$$

Let $f: X \to Y$ denote the surjective, birational, proper morphism which has been constructed. Since the equations of D, H at corresponding points P, Q differ by an element of $O_{P,Y'}$ for some invertible sheaf of ideals $\mathscr{J}$ with support off $f^{-1}(U)$

$$f^*(\mathscr{L}^{\otimes n}) \otimes \mathscr{J} = h^*(\mathscr{H}) .$$

The above proves the first statement of the proposition. To prove the second we observe that, once more as in [2], we can obtain an integral scheme X_1, which dominates X, by blowing up a suitable sheaf of $\mathscr{I}$ of ideals $\mathscr{O}_Y$ with support off U. That X_1 has the desired properties follows from 8.1.7 and 4.6.13 of Ch. II of [4]. The proposition is proved.

We are now in the position of proving our main result, namely:

THEOREM 2.8. *Let Y be a normal, integral scheme, proper over k. Let $\mathscr{L}$ be an invertible sheaf over Y, let U be an $\mathscr{L}$-projective open subscheme of Y, and let P be the generic point of an irreducible component of $Y - U$.*

(I) *If P is a closed point, then $P \in Y(\mathscr{L})$.*

(II) *Let $\bar{P} = D$, and assume that either* $\dim Y \leqq 3$ *or that* char $k = 0$. *If* $\operatorname{codim} D = 1$, *and if the sheaf of ideals $\mathscr{G}$ which defines* *the reduced scheme structure on D is invertible, then, for some $r \geqq 0$ and $n \gg 0$, $P \in Y(\mathscr{L}^{\otimes n} \otimes \mathscr{G}^{\otimes r})$.*

Proof of (I). Let $Y - U = F \cup \{P\}$, where $P \notin F$, and let $f: X \to Y$ be the blow-up morphism constructed in Proposition 2.7. Now, $f^{-1}(P)$ is the "antiregular total transform" of the closed point P (see [9] for the definition of antiregular total transforms, and for the existence of the scheme X' below), therefore we can construct a scheme X' with the following properties:

(a) The diagram
$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & X' \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$
commutes, and the morphisms g and h are surjections.

(b) The morphism $g \mid g^{-1}(Y - F)$ is an isomorphism.

(c) The morphism $h \mid [X - f^{-1}(P)]$ is an isomorphism.

Let $U_1 = Y - F$, $U_2 = X - f^{-1}(P)$, $U_1' = g^{-1}(U_1)$, and let $\mathscr{I}$ be the sheaf of ideals of $\mathscr{O}_Y$, constructed in Proposition 2.7, such that f is the blow-up morphism of Y at $\mathscr{I}$.

We then have that, for a suitable invertible sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X$, for all $n \gg 0$, and all $t \gg 0$, the invertible sheaf

$$f^*(\mathscr{L}^{\otimes nt}) \otimes \mathscr{J}^{\otimes t} \otimes \mathscr{I}\mathscr{O}_X$$

is ample. We proceed in steps.

Case 1. $P \notin \operatorname{Supp}(\mathscr{I})$. Then $X - f^{-1}(F) \approx U_1$, and $\operatorname{Supp}(\mathscr{J}) \subset f^{-1}(F)$. (We omit here the case $\dim Y = 1$, the theorem being trivially true in this case). Therefore $\operatorname{Supp}(\mathscr{I}\mathscr{O}_X \otimes \mathscr{J}) \subset f^{-1}(F)$, and hence $X - f^{-1}(F)$ is $f^*(\mathscr{L}^{\otimes n})$-projective. By Proposition 2.6 U_1 is $\mathscr{L}$-projective, and we are done in this case.

Case 2. $P \in \operatorname{Supp}(\mathscr{I})$. We then have

$$\mathscr{I}\mathscr{O}_X = \mathscr{I}_1 \otimes \mathscr{I}_2$$
$$\mathscr{J} = \mathscr{J}_1 \otimes \mathscr{J}_2$$

where $\mathscr{I}_i$, $\mathscr{J}_i$ are invertible sheaves of ideals of $\mathscr{O}_X$, with

$$\operatorname{Supp}(\mathscr{I}_1) \cup \operatorname{Supp}(\mathscr{J}_1) \subset f^{-1}(F),$$

and $\operatorname{Supp}(\mathscr{I}_2) \cup \operatorname{Supp}(\mathscr{J}_2) \subset f^{-1}(P)$.

Since $f^*(\mathscr{L}^{\otimes nt}) \otimes \mathscr{I}\mathscr{O}_X \otimes \mathscr{J}^{\otimes t}$ is ample, we see that U_2 is $f^*(\mathscr{L}^{\otimes nt}) \otimes \mathscr{I}_1 \otimes \mathscr{J}_1^{\otimes t}$-projective. Now $U_2 \approx X' - g^{-1}(P)$, and $g^{-1}(P)$ is a closed point. Furthermore the morphism $h \mid X - f^{-1}(P)$ is an isomorphism of $X - f^{-1}(P)$ onto $X' - g^{-1}(P)$; this, together with the fact that Y is normal, shows that $h_*(\mathscr{O}_X) = \mathscr{O}_{X'}$. Therefore we see that $h_*(\mathscr{I}_1)$ and $h_*(\mathscr{J}_1)$ are invertible sheaves of ideals of $\mathscr{O}_{X'}$ with supports on $g^{-1}(F)$. Also, we can now apply Proposition 2.6, and obtain that $X' - g^{-1}(P)$ is $g^*(\mathscr{L}^{\otimes nt}) \otimes h_*(\mathscr{I}_1) \otimes h_*(\mathscr{J}_1^{\otimes t})$-projective. Therefore, by Corollary 2.5, the invertible sheaf $g^*(\mathscr{L}^{\otimes nt}) \otimes h_*(\mathscr{I}_1) \otimes h_*(\mathscr{J}_1^{\otimes t})$ is ample on X'. Now $X' - g^{-1}(F)$ is isomorphic to $Y - F$, and $X' - g^{-1}(F)$ is clearly $g^*(\mathscr{L}^{\otimes nt})$-projective, since $\operatorname{Supp}(h_*(\mathscr{I}_1))$ and $\operatorname{Supp}(h_*(\mathscr{J}_1))$ are both contained in $g^{-1}(F)$. Therefore, by Proposition 2.6, applied to the morphism g, we see that $Y - F$ is $\mathscr{L}$-projective, i.e., $P \in Y(\mathscr{L})$. Statement (I) of the theorem is proved.

Proof of (II). We let $X, \mathscr{I}, \mathscr{J}, f$ be as in the previous proof. First of all, we observe that we may assume that $P \notin \operatorname{Supp}(\mathscr{I})$. In fact, if $P \in \operatorname{Supp}(\mathscr{I})$, then a simple application of Theorem 14, p. 154, and Corollary, p. 277 of [12] shows that, for some $r > 0$, the sheaf $\mathscr{G}' = \mathscr{I}\mathscr{G}^{\otimes -r}$ is a sheaf of ideals of $\mathscr{O}_Y$ with $P \notin \operatorname{Supp}(\mathscr{G}')$. Now, by 8.1.3 of Ch. II of [4], the blow-ups of Y at $\mathscr{G}'$ and $\mathscr{I}$ respectively are isomorphic. So we can indeed assume $P \notin \operatorname{Supp}(\mathscr{I})$.

Note that we now have that $f^{-1}(P)$ is a point of X, which we denote by Q, and we have that $\mathscr{O}_{P,Y} \approx \mathscr{O}_{Q,X}$. Suppose first that $Q \notin \operatorname{Supp}(\mathscr{J})$ either. (Clearly $Q \notin \operatorname{Supp}(\mathscr{I}\mathscr{O}_X)$). For $t \gg 0$ take a section $\rho \in H^\circ[X, f^*(\mathscr{L}^{\otimes nt}) \otimes \mathscr{I}\mathscr{O}_X \otimes \mathscr{J}^{\otimes t}]$ which has the following properties:

(a) $Q \in X_\rho$ and X_ρ is affine.

(b) $X_\rho \cap [\operatorname{Supp}(\mathscr{I}\mathscr{O}_X) \cap \operatorname{Supp}(\mathscr{J})] = \varnothing$.

The section ρ exists since the invertible sheaf $f^*(\mathscr{L}^{\otimes nt}) \otimes \mathscr{I}\mathscr{O}_X \otimes \mathscr{J}^{\otimes t}$ is ample. By 5.4.10 of Ch O_I of [4], the section ρ corresponds to a section τ' of the (not necessarily invertible) sheaf

$$\mathscr{L}^{\otimes nt} \otimes f_*(\mathscr{I}\mathscr{O}_X \otimes \mathscr{J}^{\otimes t}) .$$

Since Y is normal, the sheaf $f_*(\mathscr{I}\mathscr{O}_X \otimes \mathscr{J}^{\otimes t})$ is a sheaf of ideals of $\mathscr{O}_Y$, hence we have an injection

$$0 \longrightarrow \mathscr{L}^{\otimes nt} \otimes f_*(\mathscr{I}\mathscr{O}_X \otimes \mathscr{J}^{\otimes t}) \longrightarrow \mathscr{L}^{\otimes nt} .$$

Applying (b) we see that the section $\tau \in H^\circ(Y, \mathscr{L}^{\otimes nt})$, which corresponds to τ' under the above injection, has the property that $Y_\tau \approx X_\rho$. Clearly $P \in Y_\tau$, and statement (II) is proved in this case, simply by taking $r = 0$.

Let now $Q \in \operatorname{Supp}(\mathscr{J})$. Let $g: X' \to X$ be a desingularization of X, and let $h = f \circ g$. Since Y is normal and $P \notin \operatorname{Supp}(\mathscr{I})$ we see that Q is simple on X, and therefore $g^{-1}(Q)$ is a point Q' of X' such that $\mathscr{O}_{Q',X'} \approx \mathscr{O}_{Q,X}$. We denote by $\mathscr{H}$ the invertible sheaf of ideals of $\mathscr{O}_{X'}$ which defines the reduced scheme structure on $\bar{Q}'$. $\mathscr{H}$ is indeed invertible, since X' is nonsingular.

We observe that, first of all, for all $m > 0$

(2.8.1) $$h_*(\mathscr{H}^{\otimes m}) = \mathscr{G}^{\otimes m} .$$

To see the above it suffices to assume that Y is affine, and in this case it becomes an easy verification, using the facts that Y is normal, that $\mathscr{G}$ is principal and selfradical, and that $\mathscr{O}_{P,Y}$ is a discrete valuation ring.

Let S_Y and S_X denote the singular loci of Y and X respectively. Then $Q' \notin h^{-1}(S_Y) \cup g^{-1}(S_X)$, since both P and Q are simple on Y and X respectively. Since the invertible sheaf $f^*(\mathscr{L}^{\otimes nt}) \otimes \mathscr{I}\mathscr{O}_X \otimes \mathscr{J}^{\otimes t}$ is ample, the open subscheme $X' - g^{-1}(S_X)$ is $h^*(\mathscr{L}^{\otimes nt}) \otimes g^*(\mathscr{I}\mathscr{O}_X) \otimes g^*(\mathscr{J}^{\otimes t})$-projective, and therefore we can find an open subscheme V' of X', containing Q', and having the following two properties:

(i) $V' \cap [\mathrm{Supp}(g^*(\mathscr{I}\mathscr{O}_X) \cup h^{-1}(S_Y) \cup g^{-1}(S_X)] = \varnothing$

(ii) V' is $h^*(\mathscr{L}^{\otimes nt}) \otimes g^*(\mathscr{I}\mathscr{O}_X) \otimes g^*(\mathscr{J}^{\otimes t})$-projective.

From (i) we see that $h \mid V'$ is an isomorphism of V' onto an open subscheme V of Y, with the property that $P \in V$.

Since $g^*(\mathscr{J})$ is an invertible sheaf of ideals over the nonsingular scheme X', and since $Q' \in \mathrm{Supp}[g^*(\mathscr{J})]$, we see that we have $g^*(\mathscr{J}) = \mathscr{H}^{\otimes r} \otimes \mathscr{K}$, where r is some positive integer, and $\mathscr{K}$ is an invertible sheaf of ideals of $\mathscr{O}_{X'}$, with $Q' \notin \mathrm{Supp}(\mathscr{K})$.

By property (ii) of V' we have that the open subscheme

$$W = V - [\mathrm{Supp}(h_*(\mathscr{K})) \cup \mathrm{Supp}(h_*(g^*(\mathscr{I}\mathscr{O}_X)))]$$

is $\mathscr{L}^{\otimes nt} \otimes h_*(\mathscr{H}^{\otimes rt})$-projective. Since $P \in W$ we are done, by (2.8.1). The theorem is proved.

Let X be a scheme. We shall say that the open subscheme U of X is *divisorially quasi-projective in* X if, for some invertible sheaf $\mathscr{L}$ over X, U is $\mathscr{L}$-projective. With this terminology we state

COROLLARY 2.9. *Let Y be a normal, integral scheme, proper over k. Let U be a maximal divisorially quasi-projective open subscheme of Y. Then $Y - U$ has no irreducible components of codimension* 1 *whose associated sheaf of ideals is invertible.* (Under the assumption made in (II) of Theorem 2.8).

Proof. By assumption U is $\mathscr{L}$-projective, for some invertible sheaf $\mathscr{L}$ over Y. Assume that P is the generic point of an irreducible component of $Y - U$, such that the associated sheaf of ideals $\mathscr{G}$ of $\bar{P}$ in $\mathscr{O}_Y$ is invertible. Let r be fixed as in the proof of (II) of Theorem 2.8. Then, for $n \gg 0$, $U = Y(\mathscr{L}) \subsetneqq Y(\mathscr{L}^{\otimes n} \otimes \mathscr{G}^{\otimes r})$. In fact, the first equality follows from the fact that U is maximal divisorially quasi-projective, and the inequality from the fact that $P \notin U$, while, by Theorem 2.8, P does belong to the open subscheme $Y(\mathscr{L}^{\otimes n} \otimes \mathscr{G}^{\otimes r})$.

This contradicts the maximality of U. The corollary is proved.

COROLLARY 2.10. *Let Y be a normal, integral scheme, proper over k. Let U be a maximal, quasi-projective open subscheme of Y which contains all the singularities of Y. Then* $\operatorname{codim}(Y - U) > 1$.

Proof. Under the basic assumption made in (II) of the statement of Theorem 2.8, this corollary is an immediate consequence of the previous one. In fact, it suffices to observe that, if $\mathscr{H}$ denotes an ample, invertible sheaf over U, then, since Y is nonsingular off U, there exists at least one invertible sheaf $\mathscr{L}$ over Y which extends $\mathscr{H}$. Therefore U is $\mathscr{L}$-projective, and hence maximal divisorially quasi-projective. Since Y is nonsingular off U, the corollary follows from Corollary 2.9.

However, as the Editor has pointed out to the author, the corollary is valid without the basic assumption made in (II) of the statement of Theorem 2.8. In fact, since, as before, U is $\mathscr{L}$-projective for some suitable invertible sheaf $\mathscr{L}$ over Y, by Proposition 2.7 there exists an integral scheme X which dominates Y, and such that the fundamental locus of the morphism $f: X \to Y$ in Y is, by the maximality of U, precisely $Y - U$. However, since Y is normal, such fundamental locus is of codimension >1. The corollary is proved.

In particular, every nonsingular, nonprojective threefold (with no assumptions on the field k, other than it be algebraically closed) must have quasi-projective open subschemes whose complements are of pure dimension 1. (See the examples of nonsingular, nonprojective threefolds given by Hironaka and Nagata in [5] and [9] respectively).

To say that the singularities of a normal surface Y are contained in an open affine subscheme is equivalent, by Proposition 1 of [2], to saying that the singularities of Y are contained in an open $\mathscr{L}$-projective subscheme, for a suitable invertible sheaf $\mathscr{L}$ over Y. This observation, combined with Corollaries 2.9 and 2.5, give us another proof of the well known result of Zariski (see [11]) that every normal surface, whose singularities are contained in an open affine subscheme is quasi-projective.

In [1] the author has studied divisorial schemes, i.e., schemes which admit a finite open cover of the form $\{X(\mathscr{L}_i)\}_{i=1,\cdots,n}$. Corollary 2.10 implies that, if Y is a normal, divisorial scheme, then the invertible sheaves $\mathscr{L}_i$ can be chosen so that no zero-dimensional subscheme of Y, nor integral subschemes of codimension 1, whose associated sheaves of ideals are invertible, appear as components of the closed subsets $Y - Y(\mathscr{L}_i)$.

BIBLIOGRAPHY

1. M. Borelli, *Divisorial varieties*, Pacific J. Math. **13** (1963), 375-388.

2. J. Goodmann, *Affine open subsets of algebraic varieties and ample divisors*, Ann. of Math. **89**, (1969), 160-183.

3. J. Goodman, and R. Hartshorne, *Schemes with finite dimensional cohomology groups*, (to appear in Amer. J. Math.)

4. A. Grothendieck, and J. Dieudonné, *Éléments de géometrie algébrique*, Publ. Math. de 1' I.H.E.S. **4**, **8**, **11**, Paris, 1960, 1961.

5. H. Hironaka, *An example of a non Kahlerian complex-analytic deformation of Kahlerian complex structures*, Ann. of Math. **75** (1962), 190-208.

6. S. Kleiman, *Towards a numerical theory of ampleness*, Ann. of Math. **84** (1966), 293-344.

7. ———, *A note on the Nakai-Moisezon test for ampleness of a divisor*, Amer. J. Math. (1965), 221-226.

8. M. Nagata, *Imbedding of an abstract variety in a complete variety*, J. Math. Kyoto Univ. (1962), 1-10.

9. ———, *Existence theorems for nonprojective complete algebraic varieties*, Illinois J. Math., **2**, (1958), 490-498.

10. J. P. Serre, *Sur la cohomologie des variétés algébriques*, J. Math. Pures et Appl. **36** (1957), 1-16.

11. O. Zariski, *Introduction to the problem of minimal models in the theory of algebraic surfaces*, Publ. Math. Soc. Japan (1958).

12. O. Zariski, and P. Samuel, *Commutative algebra*, Vol. I, van Nostrand, Princeton, 1958.

Received January 30, 1969.

UNIVERSITY OF NOTRE DOME