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ON AN INITIAL VALUE PROBLEM IN THE THEORY OF TWO-DIMENSIONAL TRANSONIC FLOW PATTERNS

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In the case of the differential equation

$$\mathrm{L}(\psi) = \frac{\partial^2\psi}{\partial\lambda^2} + \frac{\partial^2\psi}{\partial\theta^2} + N\frac{\partial\psi}{\partial\lambda} = 0, N \equiv N(\lambda, \theta) ,$$

where N is an analytic function, the integral operator of the first kind

$$\mathrm{P}(f) \stackrel{\mathrm{def}}{=} \int_{t=-1}^{1} E(\lambda, \theta, t)f(\zeta(1-t^2)/2)dt/\sqrt{1-t^2}$$

transforms analytic functions of a complex variable $\zeta = \lambda + i\theta$ into solutions of $\mathrm{L}(\psi) = 0$. Here E is a fixed function which depends only on L, while $f(\zeta)$ is an arbitrary analytic function of the complex variable ζ; f is assumed to be regular at $\zeta = 0$. Using this operator, one shows that many theorems valid for analytic functions of the complex variable can be generalized for the solutions ψ of $\mathrm{L}(\psi) = 0$. Continuing $\psi(\lambda, \theta)$ to complex values $U = \lambda + i\Lambda$ and setting $\lambda = 0$, one shows that many theorems in the theorems in the theory of functions of a real variable can be generalized to the case of solutions of

$$\mathrm{H}(\psi) \equiv -\frac{\partial^2\psi}{\partial\Lambda^2} + \frac{\partial^2\psi}{\partial\theta^2} - iN\frac{\partial\psi}{\partial\Lambda} = 0 .$$

By change of the variables,

$$\mathrm{M}(\psi) \equiv \frac{\partial^2\psi}{\partial x^2} + l(x)\frac{\partial^2\psi}{\partial y^2} = 0 ,$$

$l(x) > 0$ for $x < 0$, $l(x) < 0$ for $x > 0$, $l(0) = 0$, when considered for $x < 0$ can be reduced to the equation $\mathrm{L}(\psi) = 0$. The variables can be chosen so that $U = 0$ corresponds to $x = 0$. However, in this case the function $N(\lambda)$ becomes singular at $\lambda = 0$. Nevertheless, one can apply the theory of the so-called integral operators of the second kind. If $\psi(0, \theta) = \chi_1(\theta)$ and

$$\lim_{M \to 1^-} \psi_M(M, \theta) = \chi_2(\theta)$$

are given, one can determine the function f. Here M is the Mach number. In this way one can determine from χ_1 and χ_2 the location and character of singularities of ψ in the subsonic region. When considering ψ in the supersonic region, one can show that some theorems on functions of one real variable can be generalized to the case of certain sets of particular solutions $\psi_\nu(\Lambda, \theta)$, $\nu = 1, 2, \cdots$, of $\mathrm{H}(\psi) = 0$.

Suppose the streamfunction ψ of a transonic two-dimensional compressible fluid flow is given by the values of ψ and of ψ_M on a segment of the sonic line. Here ψ_M is the derivative with respect to the Mach number M.

One of the problems which arises is to determine the regularity domain, say $\mathscr{R}$, and the location and properties of the singularities of ψ in the subsonic region. Finally, it is of interest to determine ψ in a given domain $\mathscr{D}$, $\mathscr{D} \subset \mathscr{R}$. This problem complex will be called *the initial value problem in the large.*

ψ, when considered in the physical plane is a solution of a nonlinear partial differential equation. However, by introducing conveniently chosen new variables (instead of the coordinates x, y of the physical plane), we obtain for ψ a linear partial differential equation (see Chaplygin [12] and Molenbroek [23]).

The linear equation which we obtain in this way, see (1.4), is of mixed type. However, it is possible to use the theory of integral operators in the study of the behavior of ψ in the *subsonic* region.

The theory of integral operators investigates the solutions of linear partial differentiation equations of the form

$$\Delta\psi + \sum_{\nu=1}^{n} a_\nu \frac{\partial\psi}{\partial x_\nu} + a_{n+1}\psi = 0 \; . \tag{1.1}$$

$\Delta = \sum_{\nu=1}^{n} \partial^2/\partial x_\nu^2$ is the Laplace differential operator and a_ν are analytic functions of $x_1, \cdots, x_n$ regular in a sufficiently large domain.[1] Suppose the solution $\psi(x_1, \cdots, x_n)$ is given in the small, say in the neighborhood of the origin in the form of a series development

$$\psi(x_1, \cdots, x_n) = \sum_{\nu_1, \nu_2, \cdots, \nu_n = 0}^{\infty} a_{\nu_1 \cdots \nu_n} x_1^{\nu_1} \cdots x_n^{\nu_n} \; . \tag{1.2}$$

Then this approach reduces the study whether ψ is regular in a domain $\mathscr{D}$ to the investigation whether or not an analytic function $f(Z_1, \cdots, Z_m)$ of m complex variables $Z_k = x_k + iy_k$, $k = 1, 2, \cdots, m$, given by its power series development

$$f(Z_1, \cdots, Z_m) = \sum_{\nu_1, \nu_2, \cdots, \nu_m = 0}^{\infty} A_{\nu_1 \cdots \nu_m} Z_1^{\nu_1} \cdots Z_m^{\nu_m} \tag{1.3}$$

is regular in a domain $\mathscr{D}$. (See [1], [2], [9], [13], [20], [15], [16], [17].)

In the case of one variable, i.e., if $f(Z) = \sum A_\nu Z^\nu$ is given, two methods can be used to determine the regularity domain and the

[1] In the case of differential equations of the form (1.1) and for $n = 2$ one uses integral operators of the *first* kind, see **[9]**, pp. 9–27. If the coefficient $4F$ of the equation (see (1.9)) has the singularity indicated in (1.9a), we use the integral operator of the *second* kind (see **[5]**, p. 869, and **[6]**, p. 452 ff.).

location and character of the singularities of f from given A_ν, $\nu = 1, 2, \cdots$. (I) the Hadamard-Polya-Mandelbrojt approach, (II) the theory of Hilbert spaces possessing a kernel function. Two possibilities should be mentioned proceeding along the lines of (II): (a) the use of functions which are simultaneously orthogonal in two domains $\mathscr{B}$ and $\mathscr{D}$, $\bar{\mathscr{B}} \subset \mathscr{D}$, see [8], (b) some results by Schiffer, Siciak and the author which give conditions for the coefficients A_ν in order that an analytic function given by its series development $\sum_{\nu=1}^{\infty} A_\nu Z^\nu$ is regular and square integrable in a given domain $\mathscr{D}$ (possessing a kernel function), see [11], [30].

The streamfunction ψ of a two-dimensional compressible fluid flow satisfies an equation of mixed type, namely

$$\mathbf{M}(\psi) \equiv \frac{\partial^2 \psi}{\partial H^2} + l(H)\frac{\partial^2 \psi}{\partial \theta^2} = 0 , \qquad l(0) = 0 , \tag{1.4}$$

where $l(H)$ is an analytic function of H, which is real for real H and such that

$$l(H) > 0 \qquad \text{for } H < 0 , \tag{1.5a}$$

$$l(H) < 0 \qquad \text{for } H > 0 . \tag{1.5b}$$

$l(H)$ is supposed to be regular in a sufficiently large domain including $H = 0$. Further we assume that, if we reduce (1.4) to the normal form (1.7), $l(H)$ is chosen in such a way that N considered as a function of λ, see (1.6), has a development of the type indicated in (1.7a). The study of (1.4) can be reduced to the study of the equation (1.1) with *singular* coefficient a_{n+1}. By the transformation

$$-\lambda = \int_{\tau=0}^{-H} [l(-\tau)]^{1/2} d\tau , \tag{1.6}$$

the equation (1.4) in the region $H < 0$ is transformed into

$$\mathbf{L}(\psi) \equiv \psi_{\lambda\lambda} + \psi_{\theta\theta} + 4N\psi_\lambda = 0 , \tag{1.7}$$

$$N = -\frac{1}{8} l^{-3/2} l_H = \frac{1}{12\lambda}[1 + \beta_1(-\lambda)^{2/3} + \cdots], \beta_1 > 0, \lambda < 0 , \tag{1.7a}$$

(See (2.4), p. 860 of [5].)

Introducing

$$\psi^* = \frac{\psi}{H^*} , \tag{1.8}$$

$$\begin{aligned} H^* &= \exp\left[-\int_{-\infty}^{2\lambda} 2N(\tau)d\tau\right] \\ &= S_0(-2\lambda)^{-1/6}[1 + S_1(-2\lambda)^{2/3} + S_2(-2\lambda)^{4/3} + \cdots] , \end{aligned} \tag{1.8a}$$

(1.7) becomes

$$\mathbf{L}^*(\psi^*) \equiv \psi^*_{\lambda\lambda} + \psi^*_{\theta\theta} + 4F\psi^* = 0 \ , \tag{1.9}$$

$$F = \frac{5}{144}(-\lambda)^{-2} + \tilde{A}_{-2}(-\lambda)^{-2/3} + \tilde{A}_0 + \tilde{A}_2(-\lambda)^{2/3} + \cdots \ , \tag{1.9a}$$

see [4], [5], [6], [9, p. 106 ff.]

In the next section we shall discuss an integral representation for the solution ψ of (1.7) in terms of a function of one variable.

In the previous papers [5], [6] the conditions for the associate $f(Z) = Z^{1/6} \sum c_\nu Z^\nu$, $Z = \lambda + i\theta$, in order that ψ satisfies the relations (4.2), (4.3) on the segment of the sonic line, have been determined. However, the proof which shows that the relation (4.1′) is a sufficient condition that ψ satisfies (4.2) and (4.3) can be simplified. (see §§ 3 and 4 of this paper.)

REMARK. Formula (7.15) of [6] has been obtained in replacing c_ν by (4.1a) of the present paper and applying some further transformations. It should be noted that in formula (7.15) of [6] (as well as in (4) of [9], p. 121) $J_\nu^{(\kappa)}$ should be replaced by

$$J_\nu^{(\kappa)} \exp\left[-\frac{\pi(6-2\kappa)i}{3}\right], \ \kappa = 1, 2 \ .$$

A representation of ψ in the supersonic region is derived in § 7 by the use of integral operators.

2. An integral representation for the analytic solution of (1.4) in terms of functions of several complex variables. In this section we shall derive an integral representation for the solution ψ of equation (1.4). This representation is valid in a subdomain of the subsonic region.

DEFINITION. $\mathscr{E}(\mathscr{B}) = \bigcup_{Z \in \mathscr{B}} \mathscr{K}(Z)$ where $\mathscr{K}(Z) = \{\zeta \mid |\zeta - Z| \leq \left|\frac{Z}{2}\right|$.

In the following we assume that $\mathscr{B}$ is a stardomain with respect to the origin.

THEOREM 2.1. *Let $E(Z, Z^*, t)$ be a function of three complex variables Z, Z^*, t, $Z = \lambda + i\theta$, $Z^* = \lambda - i\theta$, which is defined for $t \in N(\mathscr{C})$ and $(Z, Z^*) \in \mathscr{G}$. Here $N(\mathscr{C})$ is a domain which includes the rectifiable (oriented) curve $\mathscr{C}$, with initial point $t = 1$ and end point $t = -1$, and $\mathscr{G}$ denotes a sufficiently small neighborhood of the origin $O = [Z = Z^* = 0]$. $\mathscr{C}$ is a curve of the complex t-plane, namely*

$$\mathscr{C} = \{|t| = 1\} \ . \tag{2.1}$$

We assume that E satisfies the following conditions:

(1) *E possesses continuous partial derivatives with respect to all three of its arguments up the second order for $(Z, Z^*, t) \in \mathscr{G} \times N(\mathscr{C})$.*

(2) *E satisfies the partial differential equation*

$$(2.2) \qquad (1 - t^2)(E_{Z^*t} + NE_t) - t^{-1}E_{Z^*} + 2tZ\mathbf{L}(E) = 0 \ ,$$

concerning $\mathbf{L}$ *see* (1.7).

If $f(\zeta) = (\zeta)^{1/6}p(\zeta/2)$, where $p(\zeta/2)$ is an analytic function of ζ which is defined in a simply connected domain $\mathscr{P}$, $\mathscr{P} \supset \mathscr{E}(\mathscr{B})$, then

$$(2.3) \qquad \psi(\lambda, \theta) = \mathbf{P}_2(f) \equiv \operatorname{Im} \int_{\mathscr{C}} E(Z, Z^*, t) f\Big(\frac{1}{2}Z(1 - t^2)\Big) \frac{dt}{\sqrt{1 - t^2}}$$

$$\operatorname{Im} = \textit{imaginary part} \ , \qquad Z = \lambda + i\theta, Z^* = \lambda - i\theta \ ,$$

is a solution of $\mathbf{L}(\psi) = 0$.

The function ψ is defined in $\mathscr{W} \cap \mathscr{B}$,

$$(2.4) \qquad \mathscr{W} = \{(\lambda, \theta) \mid 3\lambda^2 < \theta^2, \theta > 0, -s_0^{3/2} < \lambda < 0\} \ .$$

The proof of the above theorem is given in [5, p. 878 ff.] and [6] see also [1] and [9], Chapters I and V].

DEFINITION. $E(Z, Z^*, t)$ and f are denoted as *the generating and associate functions*, respectively, of the integral operator $\mathbf{P}_2$.

After certain auxiliary lemmas are obtained in § 3, we shall prove Theorem 4.1. The latter theorem will enable us to solve the problem mentioned in the introduction.

3. Auxiliary lemma . In this section we shall at first evaluate certain integrals which we shall need in § 4.

LEMMA 3.1.[2]

$$(3.1) \qquad \begin{aligned} I_\nu^{(1)} &\equiv \int_{\mathscr{C}} t^{-1/3}(1 - t^2)^{\nu+1/6} \frac{dt}{\sqrt{1 - t^2}} \\ &= -\frac{1}{2}(1 - e^{2\pi i/3}) \frac{\Gamma(1/3)\Gamma(\nu + 2/3)}{\Gamma(\nu + 1)} \ , \end{aligned}$$

$$(3.2) \qquad \begin{aligned} I_\nu^{(2)} &\equiv \int_{\mathscr{C}} t^{-5/3}(1 - t^2)^{\nu+5/6} \frac{dt}{\sqrt{1 - t^2}} \\ &= \frac{1}{2}(1 - e^{-2\pi i/3}) \frac{\Gamma(-1/3)\Gamma(\nu + 4/3)}{\Gamma(\nu + 1)} \ . \end{aligned}$$

[2] The counterclockwise orientation of $\mathscr{C}$ yields the negative sign in (3.1).

In accordance with our assumptions, $\mathscr{C}$ is a rectifiable (oriented) curve connecting the points 1 and -1 and lying in $[|t| \geqq 1]$.

Proof. Applying Cauchy's theorem to the integral of (3.1), we can reduce the curve $\mathscr{C}$ to the segment $(1, -1)$ of the real t-axis.[3] Thus

$$\begin{aligned} I_\nu^{(1)} &= I_\nu^{(11)} + I_\nu^{(12)} , \\ I_\nu^{(11)} &= \int_1^0 t^{-1/3}(1 - t^2)^{\nu+1/6} \frac{dt}{\sqrt{1-t^2}} , \\ I_\nu^{(12)} &= \int_0^{-1} t^{-1/3}(1 - t^2)^{\nu+1/6} \frac{dt}{\sqrt{1-t^2}} . \end{aligned} \tag{3.3}$$

Introducing $\tau = t^2$, we obtain

$$I_\nu^{(11)} = -\frac{1}{2}\int_0^1 \tau^{-2/3}(1-\tau)^{\nu-1/3} d\tau = -\frac{1}{2}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\nu+\frac{2}{3}\right)}{\Gamma(\nu+1)} . \tag{3.4}$$

When considering $I_\nu^{(12)}$, we note that for $-1 < t < 0$, $t = re^{i\pi}$, $r > 0$, and therefore

$$I_\nu^{(12)} = e^{-4\pi i/3}\int_0^1 r^{-1/3}(1-r^2)^{\nu-1/3} dr = \frac{1}{2} e^{-4\pi i/3} \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\nu+\frac{2}{3}\right)}{\Gamma(\nu+1)} . \tag{3.5}$$

Thus (3.1) is obtained. (Γ = Gamma function.)

When evaluating (3.2), we assume at first that $t = 0$ does not belong to the integration curve denoted by $\mathscr{C}$. Integrating by parts yields

$$\begin{aligned} I_\nu^{(2)} &= -\frac{3}{2}\int_{\mathscr{C}} (1 - t^2)^{\nu+1/3} d(t^{-2/3}) \\ &= -\left(\frac{3}{2}\right)[t^{-2/3}(1-t^2)^{\nu+1/3}]_{t=1}^{t=-1} - 3\left(\nu+\frac{1}{3}\right)\int_{\mathscr{C}} t^{1/3}(1-t^2)^{\nu-2/3} dt . \end{aligned} \tag{3.6}$$

The first term on the right-hand side of (3.6) vanishes. In the second term we replace $\mathscr{C}$ by the segment $(1, -1)$ of the real t-axis. Introducing $\tau = t^2$, we obtain

$$\begin{aligned} -3\left(\nu+\frac{1}{3}\right)\int_1^0 t^{-1/3}(1-t^2)^{\nu+1/3} dt &= \frac{3}{2}\left(\nu+\frac{1}{3}\right)\int_0^1 \tau^{-2/3}(1-\tau)^{\nu+1/3} d\tau \\ &= \frac{\Gamma\left(-\frac{1}{3}\right)\Gamma\left(\nu+\frac{4}{3}\right)}{\Gamma(\nu+1)} . \end{aligned} \tag{3.7}$$

[3] We replace $\mathscr{C}$ at first by the sum of segments $[1 > \operatorname{Re} t > \varepsilon]$, $[t = \varepsilon e^{i\theta}, 0 < \theta < \pi]$, $[-\varepsilon > \operatorname{Re} t > -1]$, $\varepsilon > 0$. Then we consider the limit of the integrals for $\varepsilon \to 0$.

When integrating from 0 to -1, we introduce $t = re^{i\pi}$ and thus obtain

$$I_\nu^{(2)} = \frac{1}{2}(1 - e^{-2\pi i/3})\frac{\Gamma\left(-\frac{1}{3}\right)\Gamma\left(\nu + \frac{4}{3}\right)}{\Gamma(\nu + 1)} . \tag{3.8}$$

The generating function E yielding the representation (2.3) has been determined in [5], [6], [7], [10]. In particular, it has been shown that two functions

$$E^{(k)} = H^* E^{*(k)},\ E^{*(k)} = \sum_{n=0}^{\infty} \frac{q^{(n,k)}(\lambda)}{(-t^2 Z)^{n-(1/2)+(2/3)k}},\ k = 1, 2 , \tag{3.9}$$

$$q^{(n,k)}(\lambda) = \sum_{\nu=0}^{\infty} C_\nu^{(n,k)}(-\lambda)^{n-(1/2)+(2/3)(k+\nu)},\ C_\nu^{(n,k)} = \text{const.}, \quad C_0^{(01)} = 2^{1/6},\ C_0^{(02)} = 2^{5/6}, \tag{3.10}$$

(see (1.8a) and [6], p. 453) are solutions of (2.2) for $(\lambda, \theta) \in \mathscr{W}^{-}$ (see (2.4)). Let ψ be equal to the right-hand side of (2.3) where

$$E = \tilde{E}(\lambda, \theta, t) \overset{\text{def}}{\equiv} A_1 E^{(1)} + A_2\left[Z\frac{(1 - t^2)}{2}\right]^{2/3} A_2 E^{(2)} , \tag{3.11}$$

then obviously $\mathbf{L}(\psi) = 0$. Here A_1 and A_2 are two complex numbers such that

$$\operatorname{Im}(\bar{A}_1 A_2) \neq 0 . \tag{3.12}$$

In the following considerations we need Lemma 3.2 yielding the limit relations for the generating function $\tilde{E}$ introduced in (3.11).

LEMMA 3.2.

$$\lim_{\lambda\to 0^-} \tilde{E}(\lambda, \theta, t) = -d_2 t^{-1/3}\theta^{-1/6} , \tag{3.13}$$

$$\lim_{\lambda\to 0^-} (-\lambda)^{1/3}\tilde{E}_\lambda(\lambda, \theta, t) = [d_1 t^{-1/3} + d_0 t^{-5/3}(1 - t^2)^{2/3}]\theta^{-1/6} \tag{3.14}$$

where

$$d_0 = -\frac{2}{3} i^{3/2} S_0 A_2,\ d_1 = -\frac{2^{5/3}}{3} i^{1/6} S_0 S_1 A_1,\ d_2 = -i^{1/6} S_0 A_1 , \tag{3.15}$$

S_0, S_1 *positive.*

Proof. By (1.8), (3.9),

$$\begin{aligned} A_1 E^{(1)} &= A_1[S_0(-2\lambda)^{-1/6} + S_0 S_1(-2\lambda)^{1/2} + \cdots]\left[\frac{q^{(0,1)}(\lambda)}{(-t^2 Z)^{1/6}} + \cdots\right] \\ &= \frac{A_1 S_0 C_0^{(01)} 2^{-1/6} + A_1 S_0 S_1 C_0^{(01)} 2^{1/2}(-\lambda)^{2/3} + \cdots}{(-t^2(\lambda + i\theta))^{1/6}} , \end{aligned} \tag{3.16}$$

$$A_2E^{(2)} = A_2(S_0(-2\lambda)^{-1/6} + S_0S_1(-2\lambda)^{1/2} + \cdots]\left[\frac{q^{(0,2)}(\lambda)}{(-t^2Z)^{5/6}} + \cdots\right]$$

$$(3.17) \quad = \frac{A_2C_0^{(02)}S_0 2^{-1/6}(-\lambda)^{2/3} + A_2S_0S_1C_0^{(02)}2^{1/2}(-\lambda)^{8/6} + \cdots}{[-t^2(\lambda + i\theta)]^{5/6}} + \cdots,$$

$$(3.18) \quad \left[\frac{1}{2}Z(1 - t^2)\right]^{2/3} A_2E^{(2)} = \frac{A_2C_0^{(02)}S_0 2^{-5/6}(-\lambda)^{2/3}(1-t^2)^{2/3}(\lambda+i\theta)^{2/3}+\cdots}{[-t^2(\lambda + i\theta)]^{5/6}}.$$

Thus

$$\tilde{E} = \frac{A_1S_0C_0^{(01)}2^{-1/6} + A_1S_0S_1C_0^{(01)}2^{1/2}(-\lambda)^{2/3} + \cdots}{[-t^2(\lambda + i\theta)]^{1/6}}$$

$$(3.19) \quad + \frac{\left[\frac{1}{2}Z(1 - t^2)\right]^{2/3} A_2C_0^{(02)}S_0 2^{-1/6}(-\lambda)^{2/3} + \cdots}{[-t^2(\lambda + i\theta)]^{5/6}} + \cdots,$$

$C_\nu^{(0,k)}$ have been introduced in (3.10). (see also [6] p. 453.) From (3.19) the limit relations (3.13) and (3.14) follow. The justifications of the above operations follow from considerations in [5], p. 882, and [10], p. 336. To derive (3.14), we note that after differentiation of (3.18) with respect to λ, the only terms of $\tilde{E}_\lambda$ which contribute to $\lim_{\lambda\to 0^-}(-\lambda)^{1/3}\tilde{E}_\lambda$ are the coefficients of $(-\lambda)^{-1/3}$. Hence

$$\lim_{\lambda\to 0^-}(-\lambda)^{1/3}\tilde{E}_\lambda = -\frac{2}{3}i^{3/2}S_0A_2t^{-5/3}(1 - t^2)^{2/3}\theta^{-1/6}$$

$$(3.20) \quad - \frac{2^{5/3}}{3}i^{1/6}S_0S_1A_1t^{-1/3}\theta^{-1/6}$$

which implies (3.14).

4. The determination of the associate f in (2.3) from given values of ψ and ψ_M on the sonic line.

THEOREM 4.1. *Let* $\chi_k(\theta) = \sum_{\nu=0}^{\infty} a_\nu^{(k)}\theta^\nu$, $k = 1, 2$, ($a_\nu^{(k)}$ real) *be two power series which converge uniformly for* $0 \leqq \theta \leqq \theta_1$, $\theta_1 > 0$. *Suppose further that* $\breve{E}(Z, Z^*, t)$ *is given by* (3.11), *and let*

$$(4.1) \quad f(Z) = Z^{1/6}\sum_{\nu=1}^{\infty} c_\nu Z^\nu,$$

$$(4.1') \quad c_\nu = (-2i)^{\nu+1/6}\frac{(\bar{d}_0\bar{I}_\nu^{(2)} + \bar{d}_1\bar{I}_\nu^{(1)})a_\nu^{(1)} + \bar{d}_2\bar{I}_\nu^{(1)}a_\nu^{(2)}}{\operatorname{Im}[d_0\bar{d}_2\bar{I}_\nu^{(1)}I_\nu^{(2)}]},$$

where $f(Z)$ *is the associate function of the integral operator* $\mathbf{P}_2(f)$ *in* (2.3). *Then* ψ *given by* (2.3) *is a solution of* (1.7) *satisfying the conditions*

$$(4.2) \quad \lim_{\lambda\to 0^-}\psi(\lambda, \theta) = \sum_{\nu=0}^{\infty} a_\nu^{(1)}\theta^\nu,$$

$$\lim_{\lambda \to 0^-} (-\lambda)^{1/3} \psi_\lambda(\lambda, \theta) = \sum_{\nu=0}^{\infty} a_\nu^{(2)} \theta^\nu . \tag{4.3}$$

Proof. In order to prove the theorem, it is sufficient to show that

$$\psi_\nu(\lambda, \theta) = \operatorname{Im} \int_{\mathscr{C}} E(Z, Z^*, t) \left(\frac{1}{2} Z(1 - t^2) \right)^{\nu + 1/6} c_\nu \frac{dt}{\sqrt{1 - t^2}} \tag{4.4}$$

satisfies the relations (4.2), (4.3) with the right-hand side replaced by $a_\nu^{(k)} \theta^\nu$, $k = 1, 2$, respectively. Using Lemma 3.2, we obtain the general term

$$\frac{-\{\bar{d}_0 d_2 I_\nu^{(1)} \bar{I}_\nu^{(2)} + \bar{d}_1 d_2 I_\nu^{(1)} \bar{I}_\nu^{(1)}\} a_\nu^{(1)} - d_2 \bar{d}_2 I_\nu^{(1)} \bar{I}_\nu^{(1)} a_\nu^{(2)}}{\operatorname{Im} [d_0 \bar{d}_2 \bar{I}_\nu^{(1)} I_\nu^{(2)}]} \theta^\nu . \tag{4.5}$$

Since $\bar{d}_1 d_2$ is real, from (4.4), (3.12) and (4.5) we infer (4.2). It is easy to see that $\operatorname{Im} [d_0 \bar{d}_1 \bar{I}_\nu^{(1)} I_\nu^{(2)}] \neq 0$ in the case under consideration.

We now consider the second condition. We note that

$$\lim_{\lambda \to 0^-} (-\lambda)^{1/3} \frac{\partial \psi}{\partial \lambda} = \operatorname{Im} \int_{\mathscr{C}} \lim_{\lambda \to 0^-} (-\lambda)^{1/3} \tilde{E}_\lambda f \frac{dt}{\sqrt{1 - t^2}} \tag{4.6}$$

since

$$\lim_{\lambda \to 0^-} (-\lambda)^{1/3} \tilde{E} f_\lambda = 0 . \tag{4.7}$$

Concerning the interchange of $\lim_{\lambda \to 0^-}$ and $\int_{\mathscr{C}}$ compare [10, pp. 336, 339], and [5, (5.28), p. 882 ff]. Using Lemmas 3.1 and 3.2, we obtain the general term

$$\begin{aligned} &\frac{[\bar{d}_0 d_1 I_\nu^{(1)} \bar{I}_\nu^{(2)} + |d_0^2| \, |I_\nu^{(2)}|^2 + |d_1|^2 \, |I_\nu^{(1)}|^2 + d_0 \bar{d}_1 \bar{I}_\nu^{(1)} I_\nu^{(2)}] a_\nu^{(1)}}{\operatorname{Im} [d_0 \bar{d}_2 \bar{I}_\nu^{(1)} I_\nu^{(2)}]} \theta^\nu \\ &\qquad + \frac{[d_1 \bar{d}_2 \, |I_\nu^{(1)}|^2 + d_0 \bar{d}_2 \bar{I}_\nu^{(1)} I_\nu^{(2)}] a_\nu^{(2)}}{\operatorname{Im} [d_0 \bar{d}_2 \bar{I}_\nu^{(1)} I_\nu^{(2)}]} \theta^\nu . \end{aligned} \tag{4.8}$$

Noting (4.6), we infer (4.3) from (4.8). This completes the proof of Theorem 4.1.

5. The conditions imposed on the coefficients $a_\nu^{(k)}$ in order that ψ has singularities of specific types. In § 4 we expresed the associate function $f(Z)$ in terms of values $a_\nu^{(k)}$, $k = 1, 2$, $\nu = 0, 1, 2, \cdots$ (see (4.2) and (4.3)), which appear in the initial value problem considered here. Suppose $f(Z)$ is regular, say in some simply connected domain $\mathscr{D}$, $\mathscr{D} \subset \mathscr{W}$, the stream function ψ will be regular there. As mentioned in the introduction, there exist various procedures for the determination of the location of singularities of a function $Z^{1/6} f(Z)$ given by its series development $Z^{1/6} \sum c_\nu Z^\nu$ at the origin.

In the following we shall discuss a procedure using the approach indicated by (I), see [18], [21].

THEOREM 5.1. *Suppose that the solution* $\psi(\lambda, \theta)$ *is defined in a sufficiently small neighborhood of the origin and on a segment* $(\lambda = 0, -\theta_1 \leqq \theta \leqq \theta_1), \theta_1 > 0$, *and satisfies the conditions* (4.2) and (4.3).

Here $\sum_{\nu=0}^{\infty} a_\nu^{(k)} \theta^\nu$ *are power series converging absolutely and uniformly for* $|\theta| \leqq \theta_1$. *Let*

$$(5.1) \qquad \frac{1}{\rho} = 2 \overline{\lim_{\nu \to \infty}} \left[\left| \frac{a_\nu^{(1)} \sum_{k=1}^{2} \bar{d}_{k-1} \bar{I}_\nu^{(3-k)} + a_\nu^{(2)} \bar{d}_2 I_\nu^{(1)}}{\operatorname{Im}[d_0 \bar{d}_2 \bar{I}_\nu^{(1)} I_\nu^{(2)}]} \right| \right]^{1/\nu} < \infty ,$$

where d_k, $I_\nu^{(k)}$, $k = 1, 2$, *have been introduced in* (3.15), (3.1), (3.2), *and let*

$$(5.2) \qquad \cos \varphi = \lim_{h \to 0^+} \frac{\sigma(h) - 1}{h}, \; \sigma(h) = \overline{\lim_{n \to \infty}} [|\boldsymbol{d}_n(h)|]^{1/n} ,$$

$$(5.3) \qquad \boldsymbol{d}_n(h) = \sum_{N=0}^{n} \frac{2C_n^N h^{n-N}(-2i)^{N+1/6} \left[a_N^{(1)} \sum_{k=1}^{2} \bar{I}_N^{(3-k)} \bar{d}_{k-1} + a_N^{(2)} I_N^{(2)} \right]}{\operatorname{Im}[d_0 \bar{d}_2 \bar{I}_N^{(1)} I_N^{(2)}]} \frac{1}{\rho} .$$

(C_n^N are binomial coefficients.) *Suppose*

$$(5.4) \qquad -\frac{1}{2} < \lim_{h \to 0^+} \frac{\sigma(h) - 1}{h} < 0 .$$

Let us denote by $\mathscr{D}^2$ *the domain*

$$(5.5) \qquad \{(\lambda, \theta) \mid 3^{1/2} |\lambda| < \theta, \; -s_0^{3/2} < \lambda \leqq 0, \; \lambda^2 + \theta^2 < \rho^2\} .$$

and let

$$(5.6) \qquad s^1 = \left\{ (\lambda, \theta) \mid \lambda = \rho \cos \varphi, \theta = \rho \sin \varphi, \right. \\ \left. \lim_{h \to 0^+} \frac{\sigma(h) - 1}{h} < \cos \varphi < 0, \; \lambda^2 + \theta^2 = \rho^2 \right\} .$$

Then $\psi(\lambda, \theta)$ *is regular in* $\mathscr{D}^2 \cup s^1$. *Concerning* s_0 *see* [5], p. 878.

Proof. Since $E(Z, Z^*, t)$ is regular in $\mathscr{W}$, see (2.4), the solution ψ is regular in every subdomain of $\mathscr{W}$ which does not include $\theta = 0$ in which $f(Z)/Z^{1/6}$ is regular. Here $f(Z) = Z^{1/6} \sum_{\nu=0}^{\infty} c_\nu Z^\nu$. By the theorems of Hadamard and Mandelbrojt, the function $g(Z) = \sum_{\nu=0}^{\infty} c_\nu Z^\nu$ is regular in the circle $|Z| < \rho, \rho = 1/\overline{\lim}_{\nu \to \infty} |c_\nu|^{1/\nu} > 0$ and on the arc $[(0, \varphi), |Z| = \rho$ where

$$(5.7) \qquad \cos \varphi = \lim_{\sigma \to 0^+} \left\{ \frac{\sigma(h) - 1}{h} \right\}, \; \sigma(h) = \overline{\lim_{n \to \infty}} [|d_n(\lambda)|]^{1/n} .$$

6. The representation of ψ in a simply connected domain $\mathscr{D}$, $\mathscr{D} \subset \mathscr{W}$. As indicated in [2], [9], the integral operators enable us to translate many theorems in the theory of analytic functions of complex variables into theorems on functions ψ satisfying a linear partial differential equation of elliptic type. As an example of an application of this method, we shall determine for the domain $\mathscr{D}$ systems $\{\psi_\nu(\lambda, \theta)\}$ of solutions of (1.7) such that every solution ψ regular in $\overline{\mathscr{D}}_1$, $\overline{\mathscr{D}}_1 = \mathscr{E}(\overline{\mathscr{D}})$, can be represented in $\mathscr{D}$ in the form

$$\psi(\lambda, \theta) = \sum_{\nu=0}^{\infty} A_\nu \psi_\nu(\lambda, \theta) . \tag{6.1}$$

Given a simply connected domain $\mathscr{D}_1$, there exist various systems $\{\varphi_\nu(Z)\}$ of analytic functions of one complex variable such that a function $g(Z)$ regular in $\overline{\mathscr{D}}_1$ can be developed in $\mathscr{D}_1$ in the form

$$g(Z) = \sum_{\nu=1}^{\infty} a_\nu \varphi_\nu(Z) . \tag{6.2}$$

For instance, one can choose for $\{\varphi_\nu(Z)\}$, $\nu = 1, 2, \cdots$, the system of functions which are orthogonal in $\mathscr{D}_1$, or functions $\{[\tilde{g}(Z)]^\nu\}$, $\nu = 0, 1, 2, 3, \cdots$, $[(\tilde{g}(Z))^0 \overset{\text{def}}{=} \text{const}]$, where $\tilde{g}(Z)$ maps $\mathscr{D}_1$ onto the unit circle. Suppose now that $\mathscr{D}$ is a star domain with respect to $Z = O$, $\overline{\mathscr{D}} \subset \mathscr{W}$ and $g(Z/2)$ is regular in $\overline{\mathscr{D}}_1 = \mathscr{E}(\overline{\mathscr{D}})$, every solution ψ regular for $\lambda, \theta \in \overline{\mathscr{D}}_1$ can be represented in $\mathscr{D}$ in the form (6.1), where

$$\psi_{2\nu}(\lambda, \theta) = \operatorname{Im} \int_{\mathscr{C}} E(Z, Z^*, t) Z^{1/6} \varphi_\nu\Big(\frac{1}{2} Z(1 - t^2)\Big) \frac{dt}{(1 - t^2)^{1/3}} , \tag{6.3}$$

and $\psi_{2\nu-1}(\lambda, \theta)$ are the real parts of the above integral.

Proof. Since we assumed that $Z^{-1/6} f(Z/2)$ is regular in $\overline{\mathscr{D}}_1$, the representation

$$Z^{-1/6} f\Big(\frac{Z}{2}\Big) = \sum a_\nu \varphi_\nu\Big(\frac{Z}{2}\Big) \tag{6.4}$$

converges for $(Z/2) \in \mathscr{D}_1$ uniformly and absolutely.

In the development

$$\begin{aligned} \psi(\lambda, \theta) = \operatorname{Im} \int_{\mathscr{C}} & E(Z, Z^*, t)(Z(1 - t^2))^{1/6} \times \\ & \sum_{\nu=1}^{\infty} a_\nu \varphi_\nu\Big(\frac{Z}{2}(1 - t^2)\Big) \frac{dt}{\sqrt{1 - t^2}} \end{aligned} \tag{6.5}$$

we can interchange the order of summation and integration, and we obtain the development (6.1), where $\psi_\nu(\lambda, \theta)$ are given by the real and imaginary parts of $\int_{\mathscr{C}} \cdots$ in (6.3).

7. A representation of a stream function ψ in the supersonic region. As indicated in [10], the approach of the present paper can be generalized to the case where λ is replaced by the complex variable

$$U = \lambda + i\Lambda , \tag{7.1}$$

and under some assumptions about

$$\chi_1(\theta) = \lim_{U \to 0} \psi(U, \theta) \quad \text{and} \quad \chi_2(\theta) = \lim_{U \to 0} U^{1/3} \frac{\partial \psi(U, \theta)}{\partial U}$$

one can determine the associate f in terms of χ_1 and χ_2. Consequently, the method of integral operators can also be used to consider the initial value problem in the supersonic region. Replacing λ by U, see (7.1), and setting $\lambda = 0$, we obtain

$$\begin{gathered} \Lambda = h^{-1} \arctan [h(M^2 - 1)^{1/2}] - \arctan [(M^2 - 1)^{1/2}] , \\ h = \left(\frac{k-1}{k+1}\right)^{1/2}, k > 1 . \end{gathered} \tag{7.2}$$

Here M is the Mach number and $p = c\rho^k$ is the pressure density relation, k, c are constants. (See [22] and [5], p. 861). Equation (1.7) assumes the form

$$\begin{gathered} \mathbf{H}(\psi) \equiv \psi_{\Lambda\Lambda} - \psi_{\theta\theta} + 4N_1\psi_\Lambda = 0 , \\ N_1 = \frac{k+1}{8} \frac{M^4}{(M^2 - 1)^{3/2}}, M > 1 . \end{gathered} \tag{7.3}$$

In the supersonic case it is convenient to introduce the variable

$$\tilde{\tilde{\tilde{T}}}{}^2 = M^2 - 1 . \tag{7.3a}$$

If we write in analogy to (1.8)

$$\psi(\Lambda, \theta) = \tilde{\tilde{\tilde{H}}}_1(\Lambda)\tilde{\psi}^*(\Lambda, \theta) , \qquad \tilde{\psi}^*(\Lambda, \theta) \equiv \psi^*(i\Lambda, \theta) , \tag{7.3b}$$

where

$$\tilde{\tilde{\tilde{H}}}_1(\Lambda) = \exp\left[-2\int_{\Lambda_0}^{\Lambda} N_1(\tau) d\tau\right] \tag{7.3c}$$

and

$$\Lambda = \Lambda(M) , \qquad \Lambda_0 = \Lambda(2^{1/2}) ,$$

then ψ^* satisfies

$$\mathbf{H}^*(\psi^*) \equiv \psi^*_{\Lambda\Lambda} - \psi^*_{\theta\theta} - 4F_1\psi^* = 0 , \tag{7.4}$$

$$F_1 = \frac{(k+1)M^4}{64}\left[\frac{-(3k-1)M^4 - 4(3-2k)M^2 + 16}{(1-M^2)^3}\right] \tag{7.4'}$$

(see [5], p. 861). A formal computation yields that

$$(7.4'') \qquad \tilde{\tilde{H}}(\tilde{\tilde{T}}) = \frac{1}{\tilde{\tilde{T}}^{1/2}}\left[\frac{2k}{(k+1)+(k-1)\tilde{\tilde{T}}^2}\right]^{1/2(k-1)} = \tilde{\tilde{H}}_1(\Lambda(M)) .$$

In general, we use in the following the same notations as in [5] and [6]. As a rule we write $\tilde{p}(\Lambda) \equiv p(i\Lambda)$, e.g., $\tilde{q}^{(nk)}(\Lambda) \equiv q^{(n,\kappa)}(i\Lambda)$. Further, instead of $H(T)$ (see [5] (4.3), p. 870) we introduced here $\tilde{\tilde{H}}(\tilde{\tilde{T}})$ (see (7.4″) and (7.3a)). Consequently the generating function $\tilde{\tilde{E}}$ *differs* from $E(i\Lambda, i\Lambda, \theta)$.

In defining the operation[4]

$$(7.5) \qquad \begin{aligned} \mathbf{P}(f) &\overset{\text{def}}{=} \operatorname{Im}\left[\tilde{\tilde{H}}_1(\Lambda)\int_{\mathscr{C}} \tilde{E}^*_{22}(\Lambda, \theta, t)\tilde{f}\Big(\frac{1}{2}(\Lambda+\theta)(1-t^2)\Big)\frac{dt}{\sqrt{1-t^2}}\right], \\ &\tilde{\tilde{H}}_1(\Lambda) \equiv \tilde{\tilde{H}}(\Lambda(T)),\ \Lambda > 0,\ 2\Lambda < |\Lambda+\theta| , \\ &\tilde{E}^*_{22}(\lambda + i\Lambda, \theta, t) = E^*_{22}(Z, Z^*, t) , \end{aligned}$$

it was assumed that $\mathscr{C}$ is a curve connecting $t = 1$ with $t = -1$ and lying in $|t| \geqq 1$ (see [5], p. 872). As in the subsonic case (see [6] (7.12) p. 468), we set

$$(7.6) \qquad \begin{aligned} \tilde{E}^*_{22}(\Lambda, \theta, t) &= A_1\tilde{E}^{*(1)}(i\Lambda, \theta, t) \\ &\quad + [(i\Lambda + i\theta)/(1-t^2)/2]^{2/3}A_2\tilde{E}^{*(2)}(i\Lambda, \theta, t) , \\ &\operatorname{Im}(A_2\bar{A}_1) \neq 0 . \end{aligned}$$

$$(7.6') \qquad \tilde{E}_{22}(\Lambda, \theta, t) = \tilde{\tilde{H}}_1(\Lambda)\tilde{E}^*_{22}(\Lambda, \theta, t) .$$

We shall show that by imposing some additional restrictions on the domain of definition of $\tilde{E}(\Lambda, \theta, t)$ we can use for $\mathscr{C}$ the curve

$$(7.7) \qquad \mathscr{C}^* = \mathscr{C}_{21} \cup \mathscr{C}_{22} \cup \mathscr{C}_{23} ,$$

$$(7.7') \qquad \begin{aligned} &\mathscr{C}_{21} = (-1 \leqq t \leqq -t_0) , \\ &\mathscr{C}_{22} = (t = t_0e^{i\varphi},\ t_0 > 0,\ -\pi \leqq \varphi \leqq 0) , \\ &\mathscr{C}_{23} = (t_0 \leqq t \leqq 1) . \end{aligned}$$

Analogously to [5, (5.5), p. 878], or [6, (4.7), p. 453],

$$(7.8) \qquad \begin{aligned} &\tilde{\tilde{E}}^{(\kappa)}(U, \theta, t) = \tilde{\tilde{H}}_1(U)\tilde{E}^{*(\kappa)}(U, \theta, t) , \\ &\tilde{E}^{*(\kappa)}(U, \theta, t) = \sum_{n=0}^{\infty}\frac{q^{(n,\kappa)}(U)}{(-t^2Z)^{n-1/2+2\kappa/3}} , \end{aligned}$$

where $q^{(n,\kappa)}(U)$, $\kappa = 1, 2$, $n = 0, 1, 2, \cdots$, are solutions of the equations

---

[4] Both the real and imaginary parts of $\tilde{\tilde{H}}_1(\Lambda)\int_{\mathscr{C}}\cdots$ are solutions of (7.3). In accordance with the previous definition we choose here "Im". In view of definition (2.4) it is sufficient to assume that $|t| \geqq 1$. If $0 < t_0 \leqq |t|$, where $t_0 < 1$, then $\mathscr{W}$ has to be replaced by $\{2|\lambda| < |Z|t_0^2\}$.

$$q_{UU}^{(0,\kappa)} + 4F(U)q^{(0,\kappa)} = 0 \,, \tag{7.8$'$}$$

$$2\left[n + \left(\frac{2\kappa}{3}\right)\right]q_U^{(n,\kappa)} + q_{UU}^{(n+1,\kappa)} + 4F(U)q^{(n+1,\kappa)} = 0 \tag{7.8$''$}$$

(see [6, p. 453, (4.4)])[5]. These solutions can be written in the form

$$q^{(n,\kappa)} = \sum_{\nu=0}^{\infty} C_\nu^{(n,\kappa)}(-U)^{n-1/2+2/3(\kappa+\nu)} \tag{7.9}$$

where

$$C_0^{(01)} = 2^{1/6},\ C_0^{(n1)}\frac{\left(\frac{1}{6}\right)_n\left(\frac{2}{3}\right)_n 2^{n+1/6}}{n!\left(\frac{1}{3}\right)_n},\ n \geqq 1,\ C_1^{(n1)} = 0,\ n \geqq 0 \,, \tag{7.10}$$

$$\begin{gathered} C_0^{(0,2)} = 2^{5/6},\ C_0^{(n,2)} = \frac{\left(\frac{5}{6}\right)_n\left(\frac{4}{3}\right)_n 2^{n+5/6}}{n!\left(\frac{5}{3}\right)_n},\ n \geqq 1 \,, \\ (a)_n \equiv a(a+1)\cdots(a+n-1) \end{gathered} \tag{7.11}$$

(see [6, (4.4), (4.5), (4.6a), (4.6b), (4.6c), (3.11), (3.12)] or [9, p. 113, (1a), (1b), (2), (3a), (3b), 3c)].)

REMARK. *One* initial value condition determines uniquely $q^{(n,2)}(U)$. Indeed, the general solution of (7.8′) and (7.8″) can be written in the form

$$C_{-1}^{(n,2)}(-U)^{n+1/6} + \sum_{\nu=0}^{\infty} C_\nu^{(n,2)}(-U)^{n-1/2+2/3(n+1)} \,. \tag{7.12}$$

From (7.9) follows that the second initial value condition used for (7.12) is $C_{-1}^{(n,2)} = 0$.

In [6, p. 459, (4.43)], it has been shown that

$$|\tilde{E}_{2\kappa}^*(U, \theta, t)| \leqq \sum_{n=0}^{\infty}\left|\frac{w^{(\kappa)}(U)u^{(n\kappa)}(U)}{(-t^2Z)^{n-1/2+2\kappa/3}}\right| \,. \tag{7.13}$$

and that

$$|u^{(n\kappa)}(U)| \leqq C\frac{2^n\Gamma\left(n + \frac{2\kappa}{3}\right)}{\Gamma\left(\frac{2\kappa}{3}\right)\Gamma(n+1)}(1+\varepsilon)^n\,|U|^n \,, \qquad \varepsilon > 0 \,. \tag{7.14}$$

LEMMA 7.1. *Let* $U = i\Lambda$, $0 < \Lambda < t_1^2\theta/(1-t_1^2)$, *where* $(2+\varepsilon)t_1^2 < t_0^2$,

[5] In the first equation of [6, (4.4)], q is missing after $4F(\lambda)$.

$0 < t_0 < 1$, $\varepsilon > 0$ $[0 < t_0 \leqq |t|, t \in \mathscr{C}^*]$, *see* (7.6). *Then the series on the right-hand side of* (7.13) *converges absolutely and uniformly for* $t \in \mathscr{C}^*$ *and* $2|U| < t_0^2 |Z|$.

Proof. Since $\{\Gamma(n + 4/3)/\Gamma(n + 1)\} \leqq n + 1$, it is sufficient to show that $2\Lambda/|t^2(\Lambda + \theta)| < 1$. From our assumptions follows that $\theta/\Lambda > (1 - t_1^2)/t_1^2$, therefore, for $t \in \mathscr{C}^*$ and $\Lambda > 0$,

$$
\begin{aligned}
\frac{\Lambda}{|t^2(\Lambda + \theta)|} &= \frac{1}{\left|t^2\left(1 + \dfrac{\theta}{\Lambda}\right)\right|} \leqq \frac{1}{|t|^2 \left|1 + \dfrac{1 - t_1^2}{t_1^2}\right|} \\
&= \frac{|t_1|^2}{|t|^2} \leqq \frac{t_0^2}{(2 + \varepsilon)|t|^2} \leqq \frac{1}{2 + \varepsilon}\,.
\end{aligned} \tag{7.15}
$$

Analogously to Theorem 4.1 we can express the associate $\tilde{f}(\Lambda + \theta)$ in terms of

$$
\lim_{\Lambda \to 0^+} \tilde{\psi}(\Lambda, \theta) \quad \text{and} \quad \lim_{\Lambda \to 0^+} [\Lambda^{1/3} \tilde{\psi}_\Lambda(\Lambda, \theta)]\,, \tag{7.16}
$$

see also [10], [28] and [29]. In formula (7.5) with $U = i\Lambda$ the associate $f \equiv f((1/2)i(\Lambda + \theta)(1 - t^2)) \equiv \tilde{f}((1/2)(\Lambda + \theta)(1 - t^2))$ is a function of a *real* variable $(\Lambda + \theta)$. In this case we obtain some modifications of our results.

Operating with functions of one *real* variable, it is convenient for many purposes to represent them in form of trigonometric series. Analogously, in the case of solutions of (7.3) we introduce a set of solutions

$$
\begin{aligned}
C_n(\Lambda, \theta) = \operatorname{Re} \int_{\mathscr{C}^*} & \tilde{E}_{22}(\Lambda, \theta, t)(\Lambda + \theta)^{1/6} \\
& \times \cos [n(\Lambda + \theta)(1 - t^2)/2] \frac{dt}{(1 - t^2)^{1/3}}\,, \qquad n = 0, 1, 2, \cdots,
\end{aligned} \tag{7.17}
$$

$$
\begin{aligned}
S_n(\Lambda, \theta) = \operatorname{Re} \int_{\mathscr{C}^*} & \tilde{E}_{22}(\Lambda, \theta, t)(\Lambda + \theta)^{1/6} \\
& \times \sin [n(\Lambda + \theta)(1 - t^2)/2] \frac{dt}{(1 - t^2)^{1/3}}\,,
\end{aligned} \tag{7.18}
$$

of (7.3)

REMARK. We note that when introducing a set of particular solutions of the equation of elliptic type, we used $\{U^n\}$, $n = 0, 1, 2, \cdots$, as associates (see, e.g., [9], p. 22]); here we use

$$
\frac{1}{2}(e^{in(\Lambda+\theta)} + e^{-in(\Lambda+\theta)}) \quad \text{and} \quad \frac{1}{2i}(e^{in(\Lambda+\theta)} - e^{-in(\Lambda+\theta)})\,, \qquad n = 0, 1, 2, \cdots. \tag{7.19}
$$

Further, instead of using $i \cos [n(\Lambda+\theta)(1-t^2/2]$ and $i \sin [\Lambda+\theta)(1-t^2)/2]$ as associates, we take the *real* part of the integral in (7.17) (7.18). The integral operator (7.5) assumes the form

$$(7.20)\qquad \begin{aligned}\psi(\Lambda, \theta) = \operatorname{Re} \int_{\mathscr{C}^*} &\tilde{E}_{22}(\Lambda, \theta, t)(\Lambda + \theta)^{1/6} \\ &\times f\left[\frac{1}{2}(\Lambda + \theta)(1 - t^2)\right]\frac{dt}{(1 - t^2)^{1/3}} .\end{aligned}$$

When considering the integration along $\mathscr{C}_2^*$, it is useful to introduce the auxiliary variable $\tau = \tilde{t}^{-1}(t)$ given by

$$(7.21)\qquad \begin{aligned} &\tau = \operatorname{Re} t + 1 && \text{for } t \in \mathscr{C}_{21} = [-1 \leqq t \leqq t_0] \\ &\tau = \frac{1}{i}\left(\log \frac{t}{t_0}\right) && \text{for } t \in \mathscr{C}_{22} = [t = t_0 e^{i\varphi}, 0 \leqq \varphi \leqq \pi] \\ &\tau = \operatorname{Re} t - t_0 + \pi t_0 && \text{for } t \in \mathscr{C}_{23} = [-t_0 \leqq t \leqq 1] .\end{aligned}$$

LEMMA 7.2. *$d\tau/dt = 1$ for $t \in (\mathscr{C}_{21} - P_1) \cup (\mathscr{C}_{23} - P_2)$ and $dt/d\tau = ie^{i\tau} t_0$ for $t \in \mathscr{C}_{22} - P_1 - P_2$, $P_1 = \mathscr{C}_{21} \cap \mathscr{C}_{22}$, $P_2 = \mathscr{C}_{22} \cap \mathscr{C}_{23}$.*

THEOREM 7.1. *Suppose that the coefficients $\{a_n\}$, $\{b_n\}$, $n = 0, 1, 2, \cdots$, of the series*

$$(7.22)\qquad \sum_{n=9}^{\infty} (a_n \cos nx + b_n \sin nx) \sim g(x) .$$

are chosen in such a way that

$$(7.23)\qquad \int_{\mathscr{C}^*} \sum_{n=0}^{\infty} \left[\frac{\left|a_n \cos\left(\frac{n}{2}(\Lambda + \theta)(1 - \tilde{t}^2(\tau))\right)\right| + \left|b_n \sin\left(\frac{n}{2}(\Lambda + \theta)(1 - \tilde{t}^2(\tau))\right)\right|}{|\sqrt{1 - \tilde{t}^2(\tau)}|} \right] \times d\tau < \infty .$$

Then if

$$(7.24)\qquad \begin{aligned}\psi(\Lambda, \theta) = \operatorname{Re} \int_{\mathscr{C}^*} &\tilde{E}(\Lambda, \theta, t)(\Lambda + \theta)^{1/6} \\ &\times g\left(\frac{1}{2}(\Lambda + \theta)(1 - t^2)\right)\frac{dt}{(1 - t^2)^{1/3}} ,\end{aligned}$$

it holds

$$(7.25)\qquad \psi(\Lambda, \theta) = \sum_{n=0}^{\infty} (a_n C_n(\Lambda, \theta) + b_n S_n(\Lambda, \theta)) .$$

Proof. Since for $0 < 2\Lambda < |\Lambda + \theta| t_0^2$, $t \in \mathscr{C}^*$, $E(Z, Z^*, t) \equiv \tilde{E}(\Lambda, \theta, t)$

and $dt/d\tau$ are uniformly bounded on $\mathscr{C}^* - P_1 - P_2$ and $|d\tau/dt| \geqq A \geqq 0$ by the Lebesgue theorem (see, e.g., [31, p. 347]), it follows that we can interchange the order of summation and integration in (7.24). Using (7.17) and (7.18), we obtain (7.25).

The above investigations suggest that we write the solution[6] ψ as a sum of two operators, namely,

$$\psi(\Lambda, \theta) = \psi_\tau(\Lambda, \theta) + \Psi_\tau(\Lambda, \theta) , \tag{7.26}$$

$$\psi_\tau(\Lambda, \theta) = \int_{\mathscr{C}_{21} \cup \mathscr{C}_{23}} \tilde{E}(\Lambda, \theta, t)\zeta^{1/6}f(\zeta)\frac{dt}{(1-t^2)^{1/2}}, \ \zeta \equiv \frac{1}{2}(\Lambda + \theta)(1 - t^2) \tag{7.26a}$$

$$\Psi_\tau(\Lambda, \theta) = \int_{\mathscr{C}_{22}} \tilde{E}(\Lambda, \theta, t)\zeta^{1/6}f(\zeta)\frac{dt}{(1-t^2)^{1/2}} , \tag{7.26b}$$

$$\begin{gathered}\mathscr{C}_{21} \cup \mathscr{C}_{23} = [-1 \leqq t < -\tau] \cup [\tau \leqq t \leqq 1] , \\ \mathscr{C}_{22} = [t = \tau e^{i\varphi}, \ -\pi \leqq \varphi \leqq 0] , \\ 0 < \tau < 1 .\end{gathered}$$

When considering $\{\psi_\tau(\Lambda, \theta)\}$, one can apply various results in the theory of trigonometrical series, while considering $\{\Psi_\tau(\Lambda, \theta)\}$, we apply theorems on analytic functions of one complex variable.

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[6] Concerning "mixed" solutions see also [**24**], [**25**], [**26**], [**27**].

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