PACIFIC JOURNAL OF MATHEMATICS
Vol, 33, No. 2, 1970

COMPARISON THEOREMS FOR ELLIPTIC DIFFERENTIAL SYSTEMS

C. A. SWANSON

Comparison theorems of Sturm's type are established for systems of quasilinear elliptic partial differential equations. Specialization to ordinary linear systems and to single partial differential inequalities yields sharper results than those previously available.

The classical Sturm-Picone comparison theorem was generalized to systems of ordinary linear second order equations by M. Morse in 1930 [11]. Refinements and extensions of Morse's theorem have been developed by Atkinson [2], Birkhoff and Hestenes [3], Coppel [4], Diaz and McLaughlin [5], Jakubovič [7], and Lidskiĭ [10].

A Sturmian comparison theorem for systems of linear elliptic partial differential equations was obtained by Kuks [9] in 1962, generalizing results of Picone [12] and Hartman and Wintner [6] for a single elliptic equation. The purpose here is to extend Kuks' result to quasilinear elliptic systems and to sharpen the original theorem. In particular, a "strong" comparison theorem of Kreith's type [8] is proved for arbitrary regular bounded domains by means of a device used by Allegretto [1] for single equations. An alternative procedure depending on Hopf's maximum principle, as employed by Kreith [8] and the author [13], could have been used to obtain the same conclusion under stronger regularity assumptions on the boundary.

The hypothesis used by Kuks, Morse, and others that the coefficients of the differential operators satisfy certain pointwise inequalities is replaced by a weaker integral inequality. In particular, specialization of our result to one dimension yields a refinement of Morse's theorem.

Let G be a nonempty regular bounded domain in n-dimensional Euclidean space R^n and let H be a domain in R^m containing the origin, $m, n = 1, 2, \cdots$. Points in R^n will be denoted by $x = (x_1, x_2, \cdots, x_n)$ and differentiation with respect to x_i by D_i, $i = 1, 2, \cdots, n$. The quasilinear elliptic partial differential operator l defined by

$$lu = \sum_{i,j=1}^{n} D_i[a_{ij}(x, u)D_j u] + b(x, u)u, \quad a_{ij} = a_{ji}$$

will be considered for $x \in G$, $u \in H$, where b and each a_{ij} are real symmetric $m \times m$ matrix functions of class $C^1(\bar{G} \times H)$, and the $mn \times mn$ matrix $(a_{ij}(x, u))(i, j = 1, 2, \cdots, n)$ is positive definite in $G \times H$. The domain $\mathfrak{D}$ of l is defined as the set of all vector functions $u \in C^2(G) \cap C^1(\bar{G})$ with range in H.

The notation $\mathfrak{D}^m$ will be used for the set of all $m \times m$ matrix functions whose column vectors $v_i \in \mathfrak{D}$, $i = 1, 2, \cdots, m$. The conclusion of the comparison theorems below concerns matrices $V \in \mathfrak{D}^m$ with the property that $V^T L V$ is negative semidefinite throughout G, where L is the partial differential operator defined by

$$LV = \sum_{i,j=1}^{n} D_i[A_{ij}(x, V)D_j V] + B(x, V)V, \quad A_{ij} = A_{ji} .$$

It is assumed that B and each A_{ij} are real symmetric $m \times m$ matrix functions of class $C^1(\bar{G} \times H^m)$ and that the $mn \times mn$ matrix $(A_{ij}(x, V))$ is positive definite in $G \times H^m$.

Let $f[u]$, $F[u, v]$ be the functionals defined by

$$(1) \qquad f[u] = \int_G \Big[\sum_{i,j} D_i u^T a_{ij}(x, u(x)) D_j u - u^T b(x, u(x)) u\Big] dx$$

$$(2) \qquad F[u, V] = \int_G \Big[\sum_{i,j} D_i u^T A_{ij}(x, V(x)) D_j u - u^T B(x, V(x)) u\Big] dx$$

with domains $\mathfrak{D}_f$, $\mathfrak{D}_f \times \mathfrak{D}^m$, respectively, where $\mathfrak{D}_f$ denotes the set of all vector functions $u \in C^1(\bar{G})$ with range in H such that u vanishes identically on ∂G.

In analogy with Morse's definition of a conjugate basis for an ordinary linear system [11, p. 56], a matrix V is said to be *conjugate* relative to L if and only if $Y_i(x, V) = 0$ identically for $i = 1, 2, \cdots, n$, where

$$(3) \qquad Y_i(x, V) = \sum_{j=1}^{n} [V^T A_{ij}(x, V) D_j V - (D_j V^T) A_{ij}(x, V) V] .$$

Similarly to the well known fact for ordinary linear systems, it follows easily from the symmetry of the matrices involved that

$$\sum_{i=1}^{n} D_i Y_i(x, V) = 0$$

identically in G for any solution $V \in \mathfrak{D}^m$ of $LV = 0$. As in the ordinary case, this motivates the definition of a conjugate matrix.

The first comparison theorem is "weak" in the sense that the conclusion applies to $\bar{G}$ rather than G. The rather simple proof of the weak theorem suggests the proof of the strong Theorem 2. For the weak theorem, ∂G is required only to be piecewise C^1.

THEOREM 1. *If*

(i) *There exists a nontrivial vector function* $u \in \mathfrak{D}_f$ *such that* $f[u] \leqq 0$;

(ii) *$V \in \mathfrak{D}^m$ is a conjugate matrix such that $V^T L V$ is negative semidefinite throughout G; and*

(iii) $f[u] \geqq F[u, V]$,

then det $V(x)$ *must vanish at some point in* $\bar{G}$.

Proof. Suppose to the contrary that $V(x)$ is nonsingular for all $x \in \bar{G}$. Then there exists a unique $w \in \mathfrak{D}_f$ satisfying $u(x) = V(x)w(x)$ identically in $\bar{G}$. An easy calculation similar to that given in [14, p. 188] yields the following identity:

$$\begin{aligned} &\sum_{i,j} (VD_i w)^T A_{ij}(x, V) VD_j w + \sum_i D_i\Big[(Vw)^T \sum_j A_{ij}(x, V)(D_j V)w\Big] \\ (4) \quad &= \sum_{i,j} D_i(Vw)^T A_{ij}(x, V) D_j(Vw) - (Vw)^T B(x, V) Vw \\ &\quad + (Vw)^T(LV)w + \sum_i w^T Y_i(x, V) D_i w \end{aligned}$$

where $Y_i(x, V)$ is given by (3). Since $Y_i(x, V) = 0$ identically for $i = 1, 2, \cdots, n$, $V^T L V \leqq 0$ in G, and $w = 0$ on ∂G, integration of (4) over G and use of Green's identity gives the inequality

$$(5) \qquad F[u, V] \geqq 0 ,$$

where F is given by (2), equality if and only if $D_i w = 0$ identically in G for each $i = 1, 2, \cdots, n$ and $LV \equiv 0$, i.e., $u(x) = V(x)w(x) = V(x)c$ for some constant vector c and $LV \equiv 0$. However, $u = 0$ on ∂G and $c \neq 0$ since $u(x)$ is nontrivial by hypothesis, and hence equality in (5) implies that $V(x)$ is singular on ∂G. Thus the assumption that $V(x)$ is nonsingular throughout $\bar{G}$ leads to the contradiction

$$f[u] \geqq F[u, V] > 0 .$$

Theorem 2 (*strong comparison theorem*). *Under the hypotheses of Theorem* 1 (*where* $\partial G \in C^1$) *either det* $V(x)$ *vanishes at some point in* G *or there exists a constant vector* $c \neq 0$ *such that* $u(x) = V(x)c$ *throughout* $\bar{G}$.

Proof. If det $V(x) \neq 0$ in G, there exists a unique $w \in C^1(G)$ such that $u(x) = V(x)w(x)$ for all $x \in G$. Since ∂G is of class C^1, it is well known that u belongs to the Sobolev space $H_0^1(G)$ (the closure in the norm $||\cdot||_1$ defined by

$$(6) \qquad ||u||_1^2 = \int_G |u|^2 dx + \sum_{i=1}^n \int_G |D_i u|^2 dx$$

of the class $C_0^\infty(G)$ of infinitely differentiable vector functions with compact support in G.)

Let $\{u_n\}$ denote a sequence of $C_0^\infty(G)$ functions converging to u in the norm (6). It follows analogously to (5) that

$$F[u_n, V] \geqq \int_G \sum_{i,j} (VD_i w_n)^T A_{ij}(x, V) VD_j w_n dx \geqq 0 \tag{7}$$

where w_n is the unique solution of $u_n(x) = V(x)w_n(x)$, $x \in G$. Since $A_{ij}(x, V(x))(i, j = 1, 2, \cdots, n)$ and $B(x, V(x))$ are uniformly bounded in G, use of (2) shows that there is a constant $K > 0$ such that

$$\begin{aligned} |F[u_n, V] - F[u, V]| & \\ \leqq K\int_G \Big| \sum_{i,j} D_i u_n^T D_j(u_n - u) + D_i(u_n^T - u^T)D_j u \Big| dx & \\ + K\int_G |u_n^T(u_n - u) + (u_n^T - u^T)u| dx \, . & \end{aligned}$$

Application of the Schwarz inequality then yields the estimate

$$|F[u_n, V] - F[u, V]| \leqq K(n^2 + 1)(\|u_n\|_1 + \|u\|_1)\|u_n - u\|_1 \, . \tag{8}$$

Since $\lim \|u_n - u\|_1 = 0$ $(n \to \infty)$, $F[u, V] \geqq 0$ in view of (7). If $F[u, V] > 0$, we obtain the contradiction $f[u] > 0$ as in Theorem 1, and hence $F[u, V] = 0$.

Let S denote a ball with $\bar{S} \subset G$ and define

$$H_S[u_n, V] = \int_S \sum_{i,j} (VD_i w_n)^T A_{ij}(x, V) VD_j w_n dx \, .$$

Then (7) implies that

$$0 \leqq H_S[u_n, V] \leqq F[u_n, V] \, , \tag{9}$$

and the following analogue of (8) is valid:

$$|H_S[u_n, V] - H_S[u, V]| \leqq M(\|w_n\|_{1,S} + \|w\|_{1,S})\|w_n - w\|_{1,S}$$

where M is a positive constant and the subscript S indicates that the integrals involved in the norm (6) are taken over S only. Since $V^{-1}(x) \in C^1(\bar{S})$ and $w = V^{-1}u$, $w_n = V^{-1}u_n$, it follows that

$$H_S[u_n, V] \to H_S[u, V] \text{ as } \|u_n - u\|_1 \to 0 \, .$$

Since $F[u_n, V] \to F[u, V] = 0$ from (8), we conclude from (9) that $H_S[u, V] = 0$, and hence that $D_i w = 0$ identically in S for $i = 1, 2, \cdots, n$. Since S is arbitrary, $w(x) = c$ or $u(x) = V(x)c$ throughout G, and hence throughout $\bar{G}$ by continuity, for some nonzero constant vector c. This completes the proof of Theorem 2.

It follows from Green's formula that hypothesis (i) of Theorem 1 or 2 is implied by the existence of a solution $u \in \mathfrak{D}$ of the differential inequality $u^T lu \geqq 0$ in G such that $u = 0$ on ∂G.

Also, hypothesis (iii) is implied by the conditions that the matrices

$$[a_{ij}(x, u(x)) - A_{ij}(x, V(x))],\ B(x, V(x)) - b(x, u(x))$$

of order mn and m, respectively, are positive semidefinite for all $x \in G$.

In the linear case, hypothesis (iii) reduces to $E[u] \geqq 0$, where

$$E[u] = \int_G \Big[\sum_{i,j} D_i u^T [a_{ij}(x) - A_{ij}(x)] D_j u + u^T [B(x) - b(x)] u \Big] dx ,$$

which is independent of V. The following special case of Theorem 2 is then immediate.

THEOREM 3. (*Linear case*). *If there exists a nontrivial solution* $u \in \mathfrak{D}$ of $u^T l u \geqq 0$ *in* G *such that* $u = 0$ *on* ∂G *and* $E[u] \geqq 0$, *then every conjugate matrix* V *for which* $V^T L V$ *is negative semidefinite in* G *is singular at some point in* G *unless* $u(x) = V(x)c$ *for a constant vector* $c \neq 0$.

This sharpens Kuks' theorem [9] in two directions: (i) Theorem 3 is "strong" in the sense described above; and (ii) The integral inequality $E[u] \geqq 0$ is weaker than Kuks' pointwise inequalities $[a_{ij}(x)] \geqq [A_{ij}(x)]$ and $B(x) \geqq b(x)$ (as forms) throughout G, as shown by an example in [14, p. 189] in the case $m = 1$. (Since the statement of Kuks' theorem does not include the alternative $u(x) = V(x)c$, it is false as stated for (open) domains.)

In the case $m = 1$, Theorem 3 extends results of Kreith [8] and Diaz and McLaughlin [5] to arbitrary regular bounded domains G and to differential inequalities. In the case that $n = 1$ and the hypotheses are strengthened to $lu = 0$ identically in an interval (x_1, x_2), $LV = 0$ in (x_1, x_2), and $a(x) - A(x)$ and $B(x) - b(x)$ are positive semidefinite at every point, Theorem 3 reduces to a result of Morse [11], also stated by Diaz and McLaughlin [5] in a different form.

REFERENCES

1. W. Allegretto, *A comparison theorem for nonlinear elliptic operators*, (submitted for publication)
2. F.V. Atkinson, *Discrete and continuous boundary problems*, Academic Press, New York, 1964.
3. G.D. Birkhoff and M.R. Hestenes, *Natural isoperimetric conditions in the calculus of variations*, Duke Math. J. **1** (1935), 198-286.
4. W.A. Coppel, *Comparison theorems for canonical systems of differential equations*, J. Math. Anal. Appl. **12** (1965), 306-315.
5. J.B. Diaz and J.R. McLaughlin, *Sturm comparison theorems for ordinary and partial differential equations*, Bull. Amer. Math. Soc. **75** (1969), 335-339.
6. Philip Hartman and Aurel Wintner, *On a comparison theorem for self-adjoint partial differential equations of elliptic type*, Proc. Amer. Math. Soc. **6** (1955), 862-865.
7. V.A. Jakubovič, *Oscillatory properties of solutions of canonical equations* (Russian), Mat. Sb. (98) **56** (1962), 3-42.

8. Kurt Kreith, *A remark on a comparison theorem of Swanson*, Proc. Amer. Math. Soc. **20** (1969), 549-550.
9. L.M. Kuks, *Sturm's theorem and oscillation of solutions of strongly elliptic systems*, Soviet Math. Dokl. **3** (1962), 24-27.
10. V.B. Lidskiĭ, *Oscillation theorems for canonical systems of differential equations* (Russian), Dokl. Akad. Nauk SSSR **102** (1955), 877-880.
11. M. Morse, *A generalization of the Sturm separation and comparison theorems in n-space*, Math. Ann. **103** (1930), 52-69.
12. Mauro Picone, *Un teorema sulle soluzioni delle equazioni lineari ellittiche autoaggiunte alle derivate parziali del secondo-ordine*, Rend. Acc. Lincei (1) **20** (1911), 213-219.
13. C.A. Swanson, *Comparison theorems for quasilinear elliptic differential inequalities*, J. of Differential Equations (to appear)
14. ———, *Comparison and oscillation theory of linear differential equations*, Mathematics in Science and Engineering, Volume 48, Academic Press, New York, 1968.

Received July 14, 1969. Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant Nr. AFOSR-69-1791.

THE UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, B.C., CANADA