ON UNIQUENESS OF CENTRAL
DECOMPOSITIONS OF GROUPS

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In this paper a condition for two central decompositions of a group into centrally indecomposable factors to be isomorphic modulo the center is derived. Using this result a larger class of groups with central decompositions into centrally indecomposable factors isomorphic modulo the center than that of an earlier work of the author is determined. An example of a group having two central decompositions into centrally indecomposable factors which are not isomorphic modulo the center is also obtained.

The purpose of this paper is to extend the results of an earlier work of the author [5]. The central products defined in [5] are special cases of the generalized direct products in [4]. The term 'central product' was introduced by P. Hall in [2]. In [1] Gorenstein discussed the irreducible representations of central products in terms of those of the individual factors and also made use of them in the study of extra-special p-groups and critical subgroups of p-groups. It is of interest to note that in these cases the central products involved all have central factors whose factor groups with respect to their centers are abelian of rank ≤ 3. The results of this paper show that all such central factors are centrally indecomposable and moreover any two such decompositions are isomorphic modulo the center. It must, however, be noted that in [1] central products are used in a more general sense, in that the amalgamated subgroup is allowed to be a proper subgroup of the center, while in our case the amalgamated subgroup must be the center itself. But this does not affect the above observation.

All notations and terminology will be the same as in [5], and unless otherwise specified H shall always mean $Z(G)$. Moreover, the ascending and descending chain conditions for normal subgroups are always assumed and the rank of an abelian group is used in the sense of Prüfer (Kaplansky [3]). Thus $G = (A \times B)_{II}$ means that $G = AB$, with $[A, B] = 1$ and $A \cap B = H = Z(G)$. Two central decompositions of G are said to be isomorphic modulo H if the induced direct decompositions of G/H are isomorphic. Our problem is to find conditions for a group to have unique, up to isomorphism modulo H, central decomposition into centrally indecomposable factors.

The following results from [5] will be needed:

LEMMA 1. (*Lemma* 3.1, [5]). *If* $G = (A \times B)_H = (C \times D)_H$ *and* $x \in C$ *then there exist* $a \in A$ *and* $b \in B$ *with* a *and* b *respectively of the form* $a = cd$ *and* $b = c^K d^{-1}$ *where* $c, c^* \in C$ *and* $d \in D$ *such that* $x = ab$.

A sharper form of this result will be:

LEMMA 1a. *If* $G = (A \times B)_H = (C \times D)_H$ *and* $x \in C$ *then there exist* $c \in C$ *and* $d \in D$ *such that* $cd \in A$ *and* $c^{-1}xd^{-1} \in B$.

Proof. Let $x = ab$ with $a \in A$ and $b \in B$. Then $a = cd$ and $b = c'd'$ for some $c, c' \in C$ and $d, d' \in D$. Thus $x = cdc'd' = cc'dd'$. Hence $dd' \in C \cap D = H$. Therefore $d' = d^{-1}h$ for some $h \in H$, whence $[d, d'] = 1$. It follows that $c^{-1}xd^{-1} = c'd'$.

Lemma 1 follows from Lemma 1a by putting $c^* = c^{-1}x$.

LEMMA 2. (*Lemma* 3.4, [5]). *If* $G = (A \times B)_H = (C \times D)_H$ *and* M *is a subgroup of* A *containing* H *such that* $M \subset P_A(C)$ *and* $C = P_C(M)$, *then* $A = (M \times N)_H$ *where* $N = A \cap D$.

$P_A(C)$ and $P_C(M)$ are respectively the H-projections of C in A and M in C. In general if $G = (\prod_{i=1}^n G_i)_H$ and θ is the homomorphism mapping G onto $\bar{G} = G/H$ then the H-projection of $x \in G$ in G_i is defined to be the set of all preimages of $(x\theta)\theta_i$ under θ, where θ_i is the $\bar{G}_i$-decomposition operator of $\bar{G} = \prod_{i=1}^n \bar{G}_i$, $\bar{G}_i = G_i\theta$. This set is denoted by $P_{G_i}(x)$. Also, for $x \in G$, $\bar{x}$ shall always mean $x\theta$.

THEOREM 1. (*Theorem* 3.10, [5]). *Let* G *be a nilpotent group of class 2 with a cyclic commutator subgroup.* G *is centrally indecomposable if and only if* G/H *is of rank 2 and* $C(G)$ *is either a p-group or an infinite group.*

2. In the theory of direct decompositions it is known that if a group G has a trivial center then G has a unique Remak decomposition. An analogue to this property in the case of central decompositions will be that if the factor group $G/Z(G)$ has a trivial center then the central decomposition of G into centrally indecomposable factors is unique.

The following lemma can be proved by an easy computation:

LEMMA 2.1. *If* $G = (A \times B)_H = (C \times D)_H$ *then* $[P_A(C), P_A(D)] \subseteq H$.

LEMMA 2.2. *Let* $G = (A \times B)_H = (C \times D)_H$. *If* $A \subset C$ *then* $D \subset B$.

Proof. Assume $A \subset C$. Let $x \in D$. Then $x = ab$ with $a \in A$ and

$b \in B$. Now $[A, x] \subset [C, D] = 1$ and $[A, b] \subset [A, B] = 1$. Therefore $[A, a] = 1$. Thus $a \in Z(A) = H \subset B$, whence $x = ab \in B$.

COROLLARY 2.3. *If* $G = (A \times B)_H = (A \times D)_H$ *then* $B = D$.

It may be noted that Corollary 2.3 can also be obtained by applying Theorem 2.5 of [5].

THEOREM 2.4. *If* G/H *has a trivial center, then the central decomposition of* G *into centrally indecomposable factors is unique.*

Proof. Let $G = (\prod_{i=1}^{n} G_i)_H = (\prod_{i=1}^{m} F_i)_H$ be any two central decompositions of G into centrally indecomposable factors. Let $P = P_{G_1}(F_1)$ and $Q = P_{G_1}(F_1')$. (Throughout this paper we shall use G_i' to denote $(\prod_{k \neq i}^{n} G_k)_H$). Clearly $G_1 = PQ$. Indeed we can assume, without loss, that $H \subsetneqq P$. Let $x \in G_1$. By Lemma 1, there exist $a \in P$, $b \in Q$ and $u \in G_1'$ such that $x = ab$ with $au \in F_1$ and $bu^{-1} \in F_1'$. Thus $u \in P_{G_1'}(F_1)$ and $u \in P_{G_1'}(F_1')$). But by Lemma 2.1, $[P_{G_1'}(F_1), P_{G_1'}(F_1')] \subset H$. Since $G_1' = P_{G_1'}(F_1) \cdot P_{G_1'}(F_1')$, it follows that $[u, G_1'] \subset H$, whence $\bar{u} \in Z(G/H) = 1$, $\bar{u}$ being the image of u in G/H. Thus $u \in H$. It follows that $au \in P$ and $bu^{-1} \in Q$. Let $A = P \cap F_1$ and $B = Q \cap F_1'$. It is clear that $G_1 = (A \times B)_H$. But G_1 is centrally indecomposable. Hence either $A = G_1$ and $B = H$ or $A = H$ and $B = G_1$. Now $B = G_1$ implies that $Q = G_1$. Since by Lemma 2.1, $[P, Q] \subset H$, we have $P/H \subset Z(G/H) = 1$. Thus $P \subsetneqq H$, contradicting the assumption $H \subsetneqq P_{G_1}(F_1)$. Therefore we must have $A = G_1$ and $B = H$. This implies that $P = G_1$ and $Q = H$, whence $b \in H$. Thus by Lemma 2 and the indecomposability of F_1 we have $G_1 = F_1$. Now by Corollary 2.3, $G_1' = F_1'$. Therefore by induction we have $F_i = G_i$ after suitable reindexing.

3. From Theorem 1 it follows immediately that if G is a p-group of class 2 such that $G/Z(G)$ is of rank 2 then G is centrally indecomposable. In fact in the case of p-groups this property can also easily be proved by considering the ranks of abelian groups. Indeed it is not difficult to prove the central indecomposability of a p-group G of class 2 with $G/Z(G)$ abelian of rank 3. M. Schick pointed out that any finite p-group which can be generated by three of less H-generators (in the sense defined below) is centrally indecomposable.

DEFINITION 3.1. Let G be any group. If the elements $a_1, \cdots, a_n$ together with H generate G, then G is said to be generated by the H-generators $a_1, \cdots, a_n$.

THEOREM 3.2. *If* G *is a finite* p-group *with* $H = Z(G)$ *such that*

G can be generated by three or fewer H-generators, then G is centrally indecomposable.

Proof. Suppose $G = (A \times B)_H$. We shall show that either A or B is H. Let ψ be the canonical homomorphism of G onto G/N where $N = \{\Phi(G), H\}$. Now $G\psi = A\psi \times B\psi$ is elementary abelian and $G\psi$ has rank at most 3. It follows that one of $A\psi$ and $B\psi$ has rank at most 1. If, say, $A\psi = (A/H)/\Phi(A/H)$ is cyclic then A/H is cyclic. This implies that A is abelian whence $A = Z(A) = H$.

4. In this section we shall derive a condition for central decompositions of G into centrally indecomposable factors to be isomorphic modulo the center and apply it to central decompositions of nilpotent groups of class 2 in which the factor group of each factor with respect to the center is abelian of rank ≤ 3.

The following lemma can be easily proved.

LEMMA 4.1. *Let $G = A \times B = C \times D$. A and C are exchangeable if and only if there exist subgroups $U \subset B$ and $V \subset D$ such that $A \times U = C \times V$ in which A and C are exchangeable.*

LEMMA 4.2. *Let $G = (A \times B)_H = (C \times D)_H$. If there exist subgroups $H \subset U \subset B$ and $H \subset V \subset D$ such that $(A \times U)_H = (C \times V)_H$ and $\bar{A}$ and $\bar{V}$ are exchangeable in $\bar{A} \times \bar{U} = \bar{C} \times \bar{V}$, then $G = (A \times U \times B \cap D)_H = (C \times V \times B \cap D)_H$.*

Proof. Since $\bar{A}$ and $\bar{V}$ are exchangeable in $\bar{A} \times \bar{U} = \bar{C} \times \bar{V}$ and $\bar{V} \subset \bar{D}$, therefore $\bar{V}\theta_A = \bar{A}$ and $\bar{V} \subseteq \bar{A}\theta_D$ where θ_A is the $\bar{A}$-decomposition operator in $\bar{G} = \bar{A} \times \bar{B}$ and θ_D is the $\bar{D}$-decomposition operator in $\bar{G} = \bar{C} \times \bar{D}$. Thus $P_A(V) = A$ and $V \subseteq P_D(A)$. Hence by Lemma 2 we have $G = (C \times V \times B \cap D)_H$. In the same way we can show that $G = (A \times U \times B \cap D)_H$.

THEOREM 4.3. *Let $G = (\prod_{i=1}^{n} G_i)_H = (\prod_{i=1}^{m} F_i)_H$ be any two central decompositions of G into centrally indecomposable factors. If for each pair of factors G_i and F_j such that $\bar{G}_i$ and $\bar{F}_j$ are not exchangeable in the induced direct decompositions of $\bar{G}$ there exist centrally indecomposable subgroups $H \subset U_i \subset G_i'$ and $H \subset V_j \subset F_j'$ such that the decompositions $\bar{G}_i \times \bar{U}_i = \bar{F}_j \times \bar{V}_j$ are exchangeable, then any two central decompositions are isomorphic modulo H.*

Proof. If each pair of factors G_i and F_j are exchangeable in the induced direct decompositions of $\bar{G}$ then clearly the two central decompositions are isomorphic modulo H. Hence we need only consider

the case when there exist G_i and F_j such that $\bar{G}_i$ and $\bar{F}_j$ are not exchangeable in the induced direct decompositions of $\bar{G}$. Thus by hypothesis there exist centrally indecomposable subgroups U_i and V_j such that the decompositions $\bar{G}_i \times \bar{U}_i = \bar{F}_j \times \bar{V}_j$ are exchangeable. Now $\bar{G}_i$ and $\bar{F}_j$ are not exchangeable in $\bar{G}_i \times \bar{G}'_i = \bar{F}_j \times \bar{F}'_j$. Therefore, by Lemma 4.1, $\bar{G}_i$ and $\bar{F}_j$ are not exchangeable in $\bar{G}_i \times \bar{U}_i = \bar{F}_j \times \bar{V}_j$, whence $\bar{G}_i$ and $\bar{V}_j$ must be exchangeable. Thus, by Lemma 4.2, we have

$$G = (G_i \times U_i \times G'_i \cap F'_j)_H$$
$$= (F_j \times V_j \times G'_i \cap F'_j)_H .$$

Let $G'_i \cap F'_j = (\prod_{k=1}^{r} N_k)_H$, where the N_k's are centrally indecomposable. Since U_i is centrally indecomposable, therefore, considering $G'_i = (\prod_{k=i}^{n} G_k)_H = (U_i \times \prod_{k=1}^{r} N_k)_H$ and applying induction on n, there exists G_λ such that $\bar{G}_\lambda \approx \bar{U}_i$ and $(\prod_{k=1}^{r} N_k)_H$ and $(\prod_{k \neq i, \lambda}^{n} G_k)_H$ are isomorphic mudulo H. In the same way there exists F_μ such that $\bar{F}_\mu \approx \bar{V}_j$ and $(\prod_{k=1}^{r} N_k)_H$ and $(\prod_{k \neq j, \mu}^{m} F_k)_H$ are isomorphic modulo H. It follows immediately that $m = n$ and $G = (\prod_{i=1}^{n} G_i)_H = (\prod_{i=1}^{n} F_i)_H$ are isomorphic modulo H.

LEMMA 4.4. *Let $G = (A \times B)_H = (C \times D)_H$. Also let $C_1 = P_C(A)$ and $D_1 = P_D(A)$. If $P_A(C_1) = A$ then $(A \times P_B(C_1))_H = (C_1 \times D_1)_H$. Moreover $\bar{A}$ and $\bar{C}_1$ are exchangeable in $\bar{A} \times \overline{P_B(C_1)} = \bar{C}_1 \times \bar{D}_1$.*

Proof. Clearly $A \subset (C_1 \times D_1)_H$. Also $P_B(C_1) \subset C_1 A \subset (C_1 \times D_1)_H$. Therefore $(A \times P_B(C_1))_H \subset (C_1 \times D_1)_H$. On the other hand,

$$C_1 \subset P_A(C_1) \cdot P_B(C_1) \subset (A \times P_B(C_1))_H .$$

Moreover, because of Lemma 1, we can easily see that $P_B(C_1) = P_B(D_1)$, whence $(A \times P_B(C_1))_H = (C_1 \times D_1)_H$.

Now $1 = [D_1 \cap A, P_A(C_1)] = [D_1 \cap A, A]$. Therefore $D_1 \cap A = H$. Moreover, since $P_B(C_1) = P_B(D_1)$, it follows that $C_1 \cap B \subset P_B(D_1)$. Let $b \in C_1 \cap B$. Then there exists $d \in D$ such that $d = ab$ for some $a \in A$. An easy computation will give

$$1 = [C_1, d] = [P_A(C_1), a][P_B(C_1), b]$$
$$= [A, a][P_B(D_1), b] = [A, a] .$$

Therefore $a \in H$, whence $b \in C_1 \cap D_1 = H$. Hence $C_1 \cap P_B(C_1) = H$. Applying Lemma 2.1, we have $[P_A(C_1), P_A(D_1)] \subset H$ and

$$[P_B(C_1), P_B(D_1)] \subset H .$$

Since $A = P_A(C_1)$ and $P_B(C_1) = P_B(D_1)$ it follows immediately that

$$[\bar{A}, \bar{D}_1] = [\bar{C}_1, \overline{P_B(C_1)}] = 1 ,$$

whence $\bar{A}$ and $\bar{C}_1$ are exchangeable in $\bar{A} \times \overline{P_B(C_1)} = \bar{C}_1 \times \bar{D}_1$.

If C is centrally indecomposable then by Lemma 2 we must have $C = C_1$. Thus we obtain:

COROLLARY 4.5. *Let* $G = (A \times B)_H = (C \times D)_H$, *where* C *is centrally indecomposable, and let* $C_1 = P_C(A)$ *and* $D_1 = P_D(A)$. *If* $P_A(C_1) = A$ *then* $(A \times P_B(C))_H = (C \times D_1)_H$. *Moreover* $\bar{A}$ *and* $\bar{C}$ *are exchangeable in* $\bar{A} \times P_B(C) = \bar{C} \times \bar{D}_1$.

LEMMA 4.6. *Let* G *be a* p-*group of class* 2 *and let*

$$G = \left(\prod_{i=1}^{n} G_i \right)_H = \left(\prod_{i=1}^{m} F_i \right)_H$$

be two central decompositions of G *into centrally indecomposable factors. If for some* i, j *the abelian group* $\bar{G}_i$ *is of rank* ≤ 3 *then there exist subgroups* U_i *and* B_j *with* $H \subset U_i \subset G'_i$ *and* $H \subset B_j \subset F'_j$ *such that the decompositions* $\bar{G}_i \times \bar{U}_i = \bar{F}_j \times \bar{B}_j$ *are exchangeable.*

Proof. Let $i = j = 1$. Because of Corollary 2.3 we need only prove the case of $G_1 \neq F_1$. Since $\bar{G}_1$ is of rank ≤ 3 we can let $G_1 = \{s_1, s_2, s_3, H\}$. If $\bar{G}_1$ is of rank 2 we shall let $s_3 = 1$. Now, by Lemma 1a, there exist $g_i \in G_1$ and $u_i \in G'_1$, $i = 1, 2, 3$, such that $g_i u_i \in F_1$ and $g_i^{-1} s_i u_i^{-1} \in F'_1$. With proper choice of u_i we can let $g_i = s_1^{\alpha i 1} s_2^{\alpha i 2} s_3^{\alpha i 3}$. Since $1 = [g_i u_i, g_i^{-1} s_i u_i^{-1}] = [g_i, s_i]$, writing $[s_1, s_2] = h_1$, $[s_1, s_3] = h_2$ and $[s_2, s_3] = h_3$, we have:

(1) $$h_1^{\alpha_{12}} = h_2^{-\alpha_{13}}$$

(2) $$h_1^{\alpha_{21}} = h_3^{\alpha_{23}}$$

(3) $$h_2^{-\alpha_{31}} = h_3^{\alpha_{32}} .$$

Also $[g_i u_i, g_j^{-1} s_j u_j^{-1}] = [g_j u_j, g_i^{-1} s_i u_i^{-1}] = 1$. Therefore

$$[g_i, g_j^{-1} s_j] = [u_i, u_j^{-1}] = [g_j, g_i^{-1} s_i^{-1}]$$

whence $[g_i, s_j] = [g_j, s_i]^{-1}$. Thus we obtain the following relations:

(4) $$h_1^{\alpha_{11} - \alpha_{22}} = h_2^{\alpha_{23}} h_3^{\alpha_{13}}$$

(5) $$h_2^{\alpha_{11} - \alpha_{33}} = h_1^{\alpha_{32}} h_3^{-\alpha_{12}}$$

(6) $$h_3^{\alpha_{22} - \alpha_{33}} = h_1^{-\alpha_{31}} h_2^{-\alpha_{21}} .$$

Let $A_1 = \{g_i u_i, H; i = 1, 2, 3\}$ and $B_1 = \{g_i^{-1} s_i u_i^{-1}, H; i = 1, 2, 3\}$. We shall show that at least one of $P_{G_1}(A_1)$ and $P_{G_1}(B_1)$ coincides with G_1.

If $\bar{G}_1$ is of rank 2 then $h_2 = h_3 = 1$. Thus from relations (1), (2) and (4), we have $h_1^{\alpha_{12}} = h_1^{\alpha_{21}} = h_1^{\alpha_{11} - \alpha_{22}} = 1$. This implies that $p \mid \alpha_{12}, \alpha_{21}$

and $\alpha_{11} \equiv \alpha_{22} \bmod p^k$ where ord $h_1 = p^k$. Now $g_i = s_1^{\alpha_{i1}} s_2^{\alpha_{i2}} \in P_{G_1}(A_1)$, $i = 1, 2$, and $s_1^{1-\alpha_{11}} s_2^{-\alpha_{12}}$, $s_1^{-\alpha_{21}} s_2^{1-\alpha_{22}} \in P_{G_1}(B_1)$. Since p does not divide both α and $\alpha - 1$ it follows immediately that at least one of $P_{G_1}(A_1)$ and $P_{G_1}(B_1)$ coincides with G_1.

If $\bar{G}_1$ is of rank 3 we shall show that $p \mid \alpha_{ij}$ for all $i \neq j$ and either $p \mid \alpha_{ii}$ for all i or $p \nmid \alpha_{ii}$ for all i. Thus, as in the case of $\bar{G}_1$ of rank 2, we have at least one of $P_{G_1}(A_1)$ and $P_{G_1}(B_1)$ coincides with G_1.

Suppose $p \nmid \alpha_{13}$. Then by (1) $h_2 \in \{h_1\}$. Now from (4)

$$h_3^{\alpha_{13}} = h_1^{\alpha_{11} - \alpha_{22}} h_2^{-\alpha_{23}} \in \{h_1\} \ ,$$

whence $C(G_1) = \{h_1\}$. Since G_1 is centrally indecomposable, therefore by Theorem 1, this is not possible. Hence $p \mid \alpha_{13}$. In the same way we can show that $p \mid \alpha_{ij}$ for all $i \neq j$.

Suppose now $p \mid \alpha_{11}$ and $p \nmid \alpha_{22}$. Then from (4) we have $h_1 \in \{h_2, h_3\}$. Let $h_1 = h_2^\mu h_3^\nu$. If $p \nmid \alpha_{33}$ then from (5) we have $h_2^{\alpha_{11} - \alpha_{33}} = h_2^{\mu \alpha_{32}} h_3^{\nu \alpha_{32}} h_3^{-\alpha_{12}}$. Since $p \mid \alpha_{32}$, it follows that $h_2 \in \{h_3\}$, whence $C(G_1)$ is cyclic. On the other hand if $p \mid \alpha_{33}$ then using (6) we have $h_3 \in \{h_2\}$. Hence either $p \mid \alpha_{ii}$ for all i or $p \nmid \alpha_{ii}$ for all i.

Now $P_{F_1}(G_1) = A_1$ and $P_{F_1'}(G_1) = B_1$. If $P_{G_1}(A_1) = G_1$ then by Corollary 4.5 the lemma follows immediately. If $P_{G_1}(B_1) = G_1$ then applying Lemmas 4.4 and 4.1 we obtain our required result.

THEOREM 4.7.   *Let G be a p-group of class 2.   If*

$$G = \left(\prod_{i=1}^{n} G_i \right)_H = \left(\prod_{i=1}^{m} F_i \right)_H$$

are two central decompositions of G into centrally indecomposable factors in which the factor group of each factor with respect to the center is abelian of rank ≤ 3, then the two central decompositions are isomorphic modulo H.

   Proof. If for all i, j, $\bar{G}_i$ and $\bar{F}_j$ are exchangeable in $\prod_{i=1}^{n} \bar{G}_i = \prod_{i=1}^{m} \bar{F}_i$ then the theorem is immediate. Therefore let $\bar{G}_i$ and $\bar{F}_j$ be not exchangeable. Since $\bar{G}_i$ is abelian of rank ≤ 3, therefore by Lemma 4.6, there exist subgroups $H \subset U_i \subset G_i'$ and $H \subset B_j \subset F_j'$ such that the decompositions $\bar{G}_i \times \bar{U}_i = \bar{F}_j \times \bar{B}_j$ are exchangeable. Now $\bar{G}_i$ and $\bar{F}_j$ are of rank at most 3. Thus, by Theorem 3.2, U_i and B_j are centrally indecomposable. Moreover, by Theorem 4.3, the two central decompositions can be shown isomorphic modulo H.

Since a finite nilpotent group is the direct product of its sylow subgroups the following theorems are obvious.

THEOREM 4.8. *If G is a finite nilpotent group such that any two central decompositions of each of its sylow subgroups into centrally indecomposable factors are isomorphic modulo H_p, where H_p is the center of the sylow p-subgroup, then any two central decompositions of G into centrally indecomposable factors are isomorphic modulo H.*

THEOREM 4.9. *If G is a finite nilpotent group of class 2 then any two central decompositions of G into centrally indecomposable factors, in which the factor group of each factor with respect to the center is abelian of rank $\leqq 3$, are isomorphic modulo the center.*

Applying Theorem 4.9 and Lemmas 4.6 and 4.2 it is not difficult to obtain:

THEOREM 4.10. *If G is a finite nilpotent group such that the factor group of each of the sylow subgroups with respect to its center is abelian of rank $\leqq 7$, then any two central decompositions of G into centrally indecomposable factors are isomorphic modulo the center.*

Thus we note that any central decompositions of a p-group of class 2 generated by seven H-generators into centrally indecomposable factors are isomorphic modulo the center. In § 6 we have an example of a p-group of class 2 generated by nine H-generators which admits central decompositions into centrally indecomposable factors that are not isomorphic modulo the center. The question whether there exist a nilpotent group of class 2 generated by eight H-generators admitting central decompositions into centrally indecomposable factors that are not isomorphic modulo the center is still undecided.

5. In this section we shall show that if a p-group G admits a central decomposition with each of its factors generated by two H-generators then any two central decompositions of G into centrally indecomposable factors are isomorphic modulo the center.

LEMMA 5.1. *Let $G = (\prod_{i=1}^{n} G_i)_H = (\prod_{i=1}^{m} F_i)_H$. If there exists F_i such that $\bar{F}_i$ is exchangeable with $\bar{G}_j$ for all j in the induced direct decompositions of $\bar{G}$ then G is nilpotent of class 2.*

Proof. Let F_1 be such that $\bar{F}_1$ is exchangeable with every $\bar{G}_j$ in the induced direct decompositions of $\bar{G}$. This implies that $\bar{F}_1\theta_j = \bar{G}_j$ for all j, where θ_j is the $\bar{G}_j$-decomposition operator. Moreover $1 = [\bar{F}_1, \bar{G}_j] = [\bar{F}_1\theta_j, \bar{G}_j]$ for all j, whence G_j is nilpotent of class 2 for all j. Therefore G is nilpotent of class 2.

LEMMA 5.2. *Let G be a p-group and let $G = (\prod_{i=1}^{n} G_i)_H = (\prod_{i=1}^{m} F_i)_H$ where each G_i is generated by two H-generators. If there exists an F_i such that $\bar{F}_i$ is exchangeable with $\bar{G}_j$ for all j in $\prod_{i=1}^{n} \bar{G}_i = \prod_{i=1}^{m} \bar{F}_i$, then $C(G)$ is cyclic.*

Proof. Let F_1 be such a factor. Then by Lemma 5.1 G is nilpotent of class 2. Thus for $G_i = \{a_i, b_i, H\}$ we have $[a_i, b_i] = h_i \in Z(G) = H$. Now $\bar{F}_1$ exchangeable with $\bar{G}_i$ implies that $\bar{F}_1$ is of rank 2. Moreover if $F_1 = \{a, b, H\}$ then $\{\bar{a}\theta_i, \bar{b}\theta_i\} = \bar{G}_i$, where θ_i is the $\bar{G}_i$-decomposition operator. Thus without loss of generality we can let $a = a_1 a_2, \cdots, a_n$ and $b = b_1 b_2, \cdots, b_n$. Now $a_i = x_i y_i$ for some $x_i \in F_1$ and $y_i \in F_1'$. Let $x_i = a^{\alpha_i} b^{\beta_i} k_i$, $k_i \in H$. Then $y_i = a^{-\alpha_i} b^{-\beta_i} a_i k_i'$, $k_i' \in H$. But $[b, y_i] = 1$. This implies that

$$h_1^{\alpha_i} \cdots h_{i-1}^{\alpha_i} h_i^{\alpha_i - i} h_{i+1}^{\alpha_i} \cdots h_n^{\alpha_i} = 1 \qquad \text{for all } i .$$

Let p^r be the highest order of the h_i's. Then for each pair of α_i and α_j, $i \neq j$, at least one is a multiple of the other modulo p^r. Let $\alpha_j \equiv k\alpha_i \bmod p^r$ (say). This implies that,

$$1 = h_1^{k\alpha_i} \cdots h_n^{k\alpha_i} h_i^{-k} = h_1^{\alpha_j} \cdots h_n^{\alpha_j} h_i^{-k} .$$

Therefore $h_i^k = h_1^{\alpha_j} \cdots h_n^{\alpha_j}$. But $h_1^{\alpha_j} \cdots h_n^{\alpha_j} = h_j$. Hence $h_j \in H_i = \{h_i\}$. Thus for each pair of h_i and h_j either $H_i \subseteqq H_j$ or $H_j \subseteqq H_i$, whence $C(G)$ is cyclic.

THEOREM 5.3. *If G is a p-group admitting a central decomposition into centrally indecomposable factors with each factor generated by two H-generators, then any two central decompositions of G into centrally indecomposable factors are isomorphic modulo the center.*

Proof. Let $G = (\prod_{i=1}^{n} G_i)_H = (\prod_{i=1}^{m} F_i)_H$, where G_i is generated by two H-generators for all i. If every $\bar{G}_i$ is exchangeable with every $\bar{F}_j$ in $\prod_{i=1}^{n} \bar{G}_i = \prod_{i=1}^{m} \bar{F}_i$ then our theorem is true. Therefore we shall assume that there exists a pair of G_i and F_j such that $\bar{G}_i$ and $\bar{F}_j$ are not exchangeable. Let $i = j = 1$. Let $G_1 = \{s_1, s_2, H\}$. Then, by Lemma 1a, there exist elements $a_i = g_i u_i \in F_1$ and $b_i = g_i^{-1} s_i u_i^{-1} \in F_1'$, $g_i \in G_1$ and $u_i \in G_1'$, $i = 1, 2$. Let $U = \{u_1, u_2, H\}$, $A = \{a_1, a_2, H\}$ and $B = \{b_1, b_2, H\}$. It is easy to see that $C = (G_1 \times U)_H = (A \times B)_H$. We shall show that $\bar{G}_1$ and $\bar{B}$ are exchangeable in $\bar{G}_1 \times \bar{U} = \bar{A} \times \bar{B}$. If C is nilpotent of class 2, then, by Lemma 4.6, $\bar{G}_1$ is exchangeable with either $\bar{A}$ or $\bar{B}$. But, by Corollary 4.5, $\bar{G}_1$ and $\bar{A}$ exchangeable will imply $\bar{G}_1$ and $\bar{F}_1$ exchangeable in $\bar{G}_1 \times \bar{U} = \bar{F}_1 \times \bar{B}$, whence by Lemma 4.1 contradicting the inexchangeability of $\bar{G}_1$ and $\bar{F}_1$ in

$$\prod_{i=1}^{n} \bar{G}_i = \prod_{i=1}^{m} \bar{F}_i .$$

Therefore $\bar{G}_1$ and $\bar{B}$ are exchangeable in $\bar{G}_1 \times \bar{U} = \bar{A} \times \bar{B}$. If C is not nilpotent of class 2, then, since U is nilpotent of class 2, we must have G_1 not nilpotent of class 2. This means that $\bar{G}_1$ is directly indecomposable. Therefore by Remak-Krull-Schmidt theorem $\bar{G}_1$ must be exchangeable with either $\bar{A}$ or $\bar{B}$. Again as it is in the case of C being nilpotent of class 2 the exchangeability of $\bar{G}_1$ and $\bar{A}$ is impermissible. Hence $\bar{G}_1$ and $\bar{B}$ are exchangeable. Thus by Lemma 4.1, $\bar{G}_1$ and $\bar{B}$ are exchangeable in $\bar{G}_1 \times P_{G_1'}(F_1) = \bar{F}_1 \times \bar{B}$.

We shall now show that $P_{G_1'}(F_1) = V$ is either centrally indecomposable or G is nilpotent of class 2 with cyclic $C(G)$. Let

$$V = \left(\prod_{i=1}^{r} V_i \right)_H$$

where each V_i is centrally indecomposable.

Case 1. $V \neq G_1'$. Then by induction on n we must have $r < n - 1$. Thus $(G_1 \times V_1 \times \cdots \times V_r)_H$ has $r + 1 < n$ factors. Hence again by induction this decomposition is isomorphic modulo H to $(F_1 \times B)_H$, whence $r = 1$. Therefore V is centrally indecomposable.

Case 2. $V = G_1'$. Since $\bar{F}_1$ and $\bar{V}$ are exchangeable, therefore $\bar{F}_1$ and $\bar{V}_i$ are not exchangeable for all i, unless $r = 1$, in which case V is centrally indecomposable. Thus as in the first part of the proof, for each i there exist $W_i \subseteqq V_i'$ and $B_i^* \subseteqq F_1'$ such that $(V_i \times W_i)_H = (F_1 \times B_i^*)_H$ with $\bar{V}_i$ and $\bar{B}_i^*$ exchangeable in $\bar{V}_i \times \bar{W}_i = \bar{F}_1 \times \bar{B}_i^*$. Now suppose $W_i \neq V_i'$ for some i. Since by induction the decompositions $G_1' = (\prod_{i=2}^{n} G_i)_H = (\prod_{i=1}^{r} V_i)_H$ are isomorphic modulo H, therefore, applying the argument of Case 1, we have $r = 1$, whence V is centrally indecomposable. Thus we need only consider the case of $W_i = V_i'$ for all i. This means,

$$G = (V_i \times W_i)_H = (F_1 \times B_i^*)_H = (F_1 \times B)_H$$

for all i.

Thus by Corollary 2.3, $B_i^* = B$ for all i. Thus $\bar{V}_i$ and $\bar{B}$ are exchangeable in $\bar{V}_i \times \bar{V}_i' = \bar{F}_1 \times \bar{B}$ for all i. Since $\bar{G}_1$ and $\bar{B}$ are exchangeable in $\bar{G}_1 \times \bar{G}_1' = \bar{F}_1 \times \bar{B}$, therefore, by Lemmas 5.1 and 5.2, G must be nilpotent of class 2 with cyclic $C(G)$.

Now, if $V = P_{G'}(F_1)$ is centrally indecomposable then by induction on n and Theorem 4.3 the decompositions $(\prod_{i=1}^{n} G_i)_H = (\prod_{i=1}^{m} F_i)_H$ are isomorphic modulo H. If G is nilpotent of class 2 with cyclic $C(G)$ then by Theorem 1 we reach the same conclusion.

COROLLARY 5.4. *If G is a finite nilpotent group with each of*

its sylow subgroups generated by at most four H_p-generators, H_p being the center of the sylow p-subgroup, then any two central decompositions into centrally indecomposable factors are isomorphic modulo the center.

6.   We shall now give an example of a p-group of class 2 admitting two central decompositions into centrally indecomposable factors which are not isomorphic modulo the center.

Let G be generated by a_i, b_i, c_i, $i = 1, 2, 3$, satisfying the following relations:

$$a_i^2 = b_i^2 = c_i^2 = 1$$

$$[a_i, b_j] = [a_i, c_j] = [b_i, c_j] = 1$$

$$[a_i, a_j] = [b_i, b_j] = [c_i, c_j] \in Z(G)$$

$$[a_i, a_j]^2 = 1 \ .$$

Thus G is a nilpotent 2-group with center $Z(G) = H = \{h_1, h_2, h_3\}$ where $h_1 = [a_1, a_2]$, $h_2 = [a_1, a_3]$ and $h_3 = [a_2, a_3]$. Now let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$ and $C = \{c_1, c_2, c_3\}$. Then $G = (A \times B \times C)_H$. On the other hand if we let

$$F_1 = \{a_1b_1c_1, \ a_2b_2c_2, \ a_3b_3c_3\}$$

and

$$F_2 = \{a_1c_1, \ a_2c_2, \ a_3c_3, \ b_1c_1, \ b_2c_2, \ b_3c_3\}$$

then $G = (F_1 \times F_2)_H$. By Theorem 3.2, A, B, C, and F_1 are all centrally indecomposable. We need to show that F_2 is centrally indecomposable. To prove this we shall first show that the subgroup $T = (A \times B)_H$ has a unique central decomposition into centrally indecomposable factors.

Suppose T has another central decomposition into centrally indecomposable factor. Then by Theorem 4.10, $T = (D \times E)_H$ is isomorphic modulo H to $T = (A \times B)_H$. Now, by Lemma 1a, there exist elements $d_i = g_i u_i \in D$ and $e_i = g_i^{-1} a_i u_i^{-1} \in E$, where $g_i = a_1^{\alpha_{i1}} a_2^{\alpha_{i1}} a_3^{\alpha_{i3}}$ and $u_i \in B$. Computing the same way as it is in Lemma 4.6, we obtain the relations (1)–(6) of § 4. But in G, $h_i^\alpha = h_j^\beta$ for $i \neq j$ if and only if $\alpha \equiv \beta \equiv 0 \bmod 2$. Therefore $\alpha_{ij} \equiv 0 \bmod 2$ for $i \neq j$ and $\alpha_{11} \equiv \alpha_{22} \equiv \alpha_{33} \bmod 2$. Thus if $\alpha_{11} \equiv 1 \bmod 2$ then $A = D$ and if $\alpha_{11} \equiv 0 \bmod 2$ then $A = E$, whence, by Corollary 2.3, T has a unique decomposition into centrally indecomposable factors.

Now suppose $F_2 = (U \times V)_H$.

Case 1.   There exists one of $\bar{U}$ and $\bar{V}$ which is not exchangeable

with one of $\bar{A}$, $\bar{B}$ and $\bar{C}$. Let $\bar{A}$ and $\bar{U}$ (say) be not exchangeable. Then, by Lemma 4.6, there exist subgroups $H \subset Q \subset A'$ and $H \subset R \subset U'$ such that $(A \times Q)_H = (U \times R)_H$ with $\bar{A}$ and $\bar{R}$ exchangeable in

$$\bar{A} \times \bar{Q} = \bar{U} \times \bar{R} \, .$$

Thus,

$$\begin{aligned} G &= (A \times Q \times A' \cap U')_H \\ &= (U \times R \times A' \cap U')_H \, . \end{aligned}$$

But $(Q \times A' \cap U')_H = (B \times C)_H$ which has a unique central decomposition into centrally indecomposable factors. Therefore $Q = B$ or C. Let $Q = B$ (say). This implies that $U = B$ and $A = R$, whence $B \subset F_2$. But it is easy to see that $B \cap F_2 = H$. Hence this case does not exist.

Case 2. Every factor of $\bar{G} = \bar{A} \times \bar{B} \times \bar{C}$ is exchangeable with every factor of $\bar{G} = \bar{F}_1 \times \bar{U} \times \bar{V}$. Now the exchangeability of $\bar{U}$ and $\bar{C}$ implies that there exists $u = w_1(a_i)w_2(b_i)c_1 \in U$ where $w_1(a_i)$ and $w_2(b_i)$ are words on a_i's and b_i's respectively. In fact for convenience we shall regard w_1 and w_2 as functions. Since $[a_1b_1c_1, u] = 1$ we have either

(1) $[a_1, w_1(a_i)] = [b_1, w_2(b_i)]^{-1} = [b_1, w_2(b_i)] \neq 1$

or

(2) $[a_1, w_1(a_i)] = [b_1, w_2(b_i)] = 1 \, .$

Suppose (1) is true. Then since A and B are isomorphic under the map $\psi \colon a_i \to b_i$ with ψ fixing H, therefore, $[a_1, w_1(a_i)] = [b_1, w_2(b_i)]$ implies that $[b_1, w_1(b_i)] = [b_1, w_2(b_i)]$, whence $w_1(b_i) = w_2(b_i)h$ for some $h \in H$. Thus $u = w_1(a_i)w_1(b_i)c_1h^{-1}$. Now $\bar{G}$ is the free abelian product of the cyclic subgroups $\{\bar{a}_i\}$, $\{\bar{b}_i\}$, $\{\bar{c}_i\}$, $i = 1, 2, 3$. Clearly the length of $\bar{u}$ on $\bar{a}_i$, $\bar{b}_i$, $\bar{c}_i$, $i = 1, 2, 3$ is odd. But it is not difficult to see that every element of $\bar{F}_2$ is of even length. Hence (1) cannot be true.

Now, if (2) is true, then $w_1(a_i) \in Z_A(a_1) = \{a_1, H\}$ and

$$w_2(b_i) \in Z_B(b_1) = \{b_1, H\} \, .$$

Thus u is of the form $a_1b_1c_1h$, a_1c_1h, b_1c_1h or c_1h for some $h \in H$. Since $a_1b_1c_1h$, $c_1h \notin F_2$, therefore u can be assumed to be either a_1c_1 or b_1c_1. In the same way there exist $v \in V$ such that v can be assumed to be either a_1c_1 or b_1c_1. Since $U \cap V = H$, therefore, if $a_1c_1 \in U$ then $b_1c_1 \in V$ or vice versa. Now F_2 is generated by a_1c_1, a_2c_2, a_3c_3, b_1c_1, b_2c_2, b_3c_3, whence U or V is generated by a proper choice of three of these elements. But any such choice will not satisfy $[U, V] = 1$. Hence F_2 is centrally indecomposable.

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