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COMPACT, DISTRIBUTIVE LATTICES OF FINITE BREADTH

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Necessary and sufficient conditions are given for a complete distributive lattice with topology to be embeddable in the product of n complete chains, where the embedding is required to be simultaneously algebraic and topological. Corollaries are (i) a characterization of those compact topological lattices which can be embedded in an n-cell and (ii) the fact that breadth provides a bound on the topological dimension for compact, distributive topological lattices of finite breadth.

These corollaries prove conjectures of Anderson [2] and Dyer and Shields [7], respectively.

In considering topologies on a lattice, there are two possible approaches. The "extrinsic" point of view presupposes a *given* Hausdorff topology for which the lattice operations are continuous, much as in the study of topological groups (cf. [2], [7]). Such a structure is commonly called a *topological lattice*, and a typical problem is to relate its topological dimension to the algebraic lattice structure.

The "intrinsic" point of view, on the other hand, is to examine topologies on a lattice which arise naturally from the lattice structure. Prominent examples are the Frink and Birkhoff interval topologies and the topology generated by order convergence [4, Ch. 10]. Such topologies usually fail to give a topological lattice in the above sense, even for complete distributive lattices. A typical problem is to study conditions under which an intrinsic topology does give a topological lattice.

There is one important class of lattices for which both points of view merge, namely, those compact topological lattices which are known to be embeddable, topologically and algebraically, in a product of complete chains. Specifically, if L is such a lattice, all the commonly used intrinsic topologies on L coincide and constitute the only compact Hausdorff topology under which L is a topological lattice.

In this connection, the following complementary facts are known. Let L be a distributive topological lattice with compact topology. By an "algebraic" embedding of L we mean an embedding which preserves the binary lattice operations.

1. If L has infinite breadth, it may not be possible to embed L topologically and algebraically in any product of chains [12].

2. If L has finite breadth n, then L can be embedded topolo-

gically and algebraically in a product of infinitely many complete chains [cf. 2, Corollary 3].

3. If L has finite breadth n, then L can be algebraically embedded in a product of n chains ([6, Th. 1.2], augmented by [9, Th. 7, p. 261]).

These facts lead to an obvious question: If L has finite breadth n, can L be embedded simultaneously topologically and algebraically in a product of n complete chains?

Our main theorem answers this question in the affirmative. Indeed, we show that the embedding exists even under a mild intrinsic condition in place of the compact Hausdorff condition. One corollary to this embedding theorem is the proof of a conjecture of Anderson [2], to the effect that a compact, distributive, metrizable topological lattice of finite breadth n can be embedded topologically and algebraically in the n-cell I^n. Another corollary is the fact that a compact, distributive topological lattice L of finite breadth n has topological dimension at most n (equal to n, if L is connected). For the case of metrizable lattices, this fact was conjectured by Dyer and Shields [7, p. 117]. (Lawson [13] has independently proved a generalization of the same conjecture, without the requirement of distributivity.) Putting these corollaries together, we obtain the result that a connected, compact, metrizable distributive topological lattice of finite breadth can be embedded topologically in an n-cell of the same dimension—an interesting fact, in view of the difficulty of finding such embeddings for spaces without lattice structure.

In the construction of our embedding, the lattice is coordinatized by using chains of its own prime elements.

2. Coordinate chains.

We recall several definitions: A finite subset B of a lattice L is *meet-irredundant* if no proper subset of B has the same meet as B. The *breadth* of L, Br (L), is the supremum of the cardinalities of its finite meet-irredundant subsets. An element p of L is *meet-irreducible* in L if $x \wedge y = p$ implies $x = p$ or $y = p$. The element p is *prime* in L if $x \wedge y \leqq p$ implies $x \leqq p$ or $y \leqq p$. If L is distributive, these two conditions on p are equivalent. If L has a greatest (least) element, let us call it 1(0).

We shall say that "the operations of L are continuous with respect to order convergence" if $x_\alpha \to a$ and $y_\beta \to b$ imply $x_\alpha \vee y_\beta \to a \vee b$ and dually, where $\to$ denotes order convergence. Birkhoff [4, p. 248] would say that L is then a "topological lattice with respect to order convergence." This condition, however, does *not* in general imply that the operations of L are continuous with respect to the (open-set) topology generated by order-convergence, i.e., the topology whose closed sets are the order-closed subsets of L. As an example, consider the complete Boolean lattice L of all regular open subsets of the unit interval. By [4, Corollary 1, p. 249] the operations of L are continuous with respect to order convergence. On the other hand, if the operations of L were continuous with respect to the open-set topology generated by order convergence, then the diagonal

$$\{(x, y) \in L \times L: x \vee y' = 1, x \wedge y' = 0\}$$

would be closed in $L \times L$. Then L would be Hausdorff, contrary to [8, Th. 1]. Continuity with respect to order convergence is thus a fairly mild condition.

DEFINITION 2.1. Let L be a complete lattice. A chain C of L will be called a *coordinate chain* if

(1) C consists of prime elements of L and includes 1,

(2) C is closed under arbitrary meets in L.

REMARK. By (2), C is a complete chain with respect to its own internal order structure [4, p. 112, Th. 3]. However, infinite joins in C may not agree with infinite joins in L. By (2), infinite meets in C and L do agree, and this fact will be used freely below.

DEFINITION 2.2. If C is a coordinate chain in L, we define $\sigma_C: L \to C$ by $\sigma_C(x) = \inf \{p \in C: p \geqq x\}$.

LEMMA 2.3. *Let L be a complete lattice whose operations are continuous with respect to order convergence. With respect to the internal order structure of C, σ_C is an (algebraic) lattice homomorphism and preserves arbitrary joins. Moreover, σ_C preserves arbitrary meets (and so is a complete homomorphism) if and only if*

(*) *for all $p, q \in C$ with $q < p$, q is not the limit of a decreasing net of elements of $\sigma_C^{-1}(p)$.*

Proof. Without difficulty one sees that σ_C is an order-preserving map of L onto C. Therefore the inequality $\sigma_C(x \wedge y) \leqq \sigma_C(x) \wedge \sigma_C(y)$ holds. To get the reverse inequality, observe that $x \wedge y \leqq \sigma_C(x \wedge y)$, which is prime. Then $x \leqq \sigma_C(x \wedge y)$ or $y \leqq \sigma_C(x \wedge y)$, say the former. By the definition of $\sigma_C(x)$, we have $\sigma_C(x) \leqq \sigma_C(x \wedge y)$; hence

$$\sigma_C(x) \wedge \sigma_C(y) \leqq \sigma_C(x) \leqq \sigma_C(x \wedge y) ,$$

as desired. Now suppose that $x_0 = \bigvee_\alpha x_\alpha$ is any join in L, and let $a = \bigvee_\alpha \sigma_C(x_\alpha)$ (join in C). Since σ_C is order-preserving, $\sigma_C(x_0) \geqq a$. On the other hand, $a \geqq x_\alpha$ for all α, so that $a \geqq x_0$, $a = \sigma_C(a) \geqq \sigma_C(x_0)$. Therefore $\sigma_C(x_0) = a$ as desired, and σ_C preserves arbitrary joins.

Now suppose that σ_C preserves arbitrary meets. If $x_\alpha \downarrow q \in C$ with $\sigma_C(x_\alpha) = p$, then $q = \sigma_C(q) = \sigma_C(\bigwedge_\alpha x_\alpha) = \bigwedge_\alpha \sigma_C(x_\alpha) = p$. Thus (*) holds.

Conversely, suppose $(*)$ holds and $x_0 = \bigwedge_\alpha x_\alpha$, where $\{x_\alpha : \alpha \in \Gamma\}$ is any given family in L. Since σ_C is known to preserve finite meets, it is no restriction to assume that the set of x_α is closed under finite meets and that the x_α form a downward-directed net. Let $q = \sigma_C(x_0)$ and let $p = \bigwedge_\alpha \sigma_C(x_\alpha)$. (By (2) of Definition 2.1, this meet is the same in C as in L.) Since σ_C is order-preserving, we must have $q \leqq p$. Define y_α to be $(x_\alpha \vee q) \wedge p$. Note that $\sigma_C(y_\alpha) = (\sigma_C(x_\alpha) \vee q) \wedge p = p$. Since the operations of L are continuous with respect to order convergence, we have $\bigwedge_\alpha y_\alpha = ((\bigwedge_\alpha x_\alpha) \vee q) \wedge p = (x_0 \vee q) \wedge p = q$. From $(*)$ it follows that $p = q$; i.e., σ_C preserves arbitrary meets.

EXAMPLE 1. Let $L = \{(x, y) \in I \times I : y \leqq x\}$ and let

$$C = (\{1\} \times I) \cup \{(0, 0)\} ,$$

where I is the unit interval. Then C violates $(*)$.

EXAMPLE 2. Let $L = ([0, 1/2] \times [0, 1/2]) \cup ([1/2, 1] \times I) \subseteqq I \times I$. The set of primes of L is $C(1) \cup C(2)$ where $C(1) = \{(x, y) \in L: (0 \leqq x < 1/2$ and $y = 1/2)$ or $(1/2 \leqq x \leqq 1$ and $y = 1)\}$ and $C(2) = \{(x, y) \in L: x = 1$ and $0 \leqq y \leqq 1\}$. Note that $C(1)$ is a complete lattice with respect to its internal order structure but is not a complete subset of L. The order-closure $C(1)^*$ on the other hand is a complete subset of L. $C(1)^*$ is not a coordinate chain nor does $\sigma_{C(1)^*}$ satisfy $(*)$. $C(1)$ is a coordinate chain and $\sigma_{C(1)}$ does satisfy $(*)$.

EXAMPLE 3. More generally, let $f, g: I \to I$ be any nondecreasing functions on the unit interval, continuous or not, and let L be the closure of the region in $I \times I$ lying between the graphs of the functions $\max(f, g)$ and $\min(f, g)$. Then L is a complete lattice. It is instructive to examine coordinate chains in L, for various choices of f, g.

LEMMA 2.4. *Suppose that L is a complete lattice with finite breadth and that the operations of L are continuous with respect to order convergence. Then every element of L is the meet of finitely many meet-irreducible elements (prime elements, if L is distributive).*

Proof. Let $x_0 \in L$. If x_0 is meet irreducible then we are done. Suppose that x_0 is not meet irreducible. Then we can find a finite set $\{x'_1, \cdots, x'_m\} \subseteqq L$ so that

(1) $x'_1 \wedge \cdots \wedge x'_m = x_0$

(2) $\{x'_1, \cdots, x'_m\}$ is meet-irredundant and

(3) m is the maximum cardinality of those sets satisfying (1) and (2). Let $X = \{x \in L: x \wedge x'_2 \wedge \cdots \wedge x'_m = x_0\}$. Let C be a maximal

chain in X. L is complete so C has a supremum; let it be x_1. Since $\wedge$ is continuous under order convergence, $x_1 \in X$. It follows that $\{x_1, x'_2, \cdots, x'_m\}$ satisfies (1), (2) and (3). Suppose that $x_1 = a \wedge b$ for some $a, b \in L$. Since $\{a, b, x'_2, \cdots, x'_m\}$ satisfies (1) and has cardinality greater than m, some element of the set is meet-redundant. Since $a \wedge b = x_1$ no x'_i can be meet-redundant. So we may assume that b is redundant. $\{a, x'_2, \cdots, x'_m\}$ satisfy (1), (2) and (3) and $a \geqq x_1$. By the maximality of x_1 we then have $a = x_1$. Hence x_1 is meet-irreducible. By the same procedure we can also obtain meet-irreducibles $x_2, \cdots, x_m$ such that $x_1 \wedge \cdots \wedge x_m = x_0$. The proof is thus complete.

If L is a lattice and $x \in L$, let us say that x is a *local maximum* of L if there is no decreasing net $\{x_\alpha : \alpha \in \Gamma\}$ in L such that $x_\alpha > x$ for all α and $\bigwedge_\alpha x_\alpha = x$.

LEMMA 2.5. *Let L be a complete, distributive lattice of finite breadth n whose operations are continuous with respect to order convergence. Then there exist n coordinate chains $C(1), \cdots, C(n)$, in L such that*

(a) $\bigcup_{i=1}^n C(i)$ *contains every prime in L;*

(b) *if $x \in L$ then $x = \sigma_{C(1)}(x) \wedge \cdots \wedge \sigma_{C(n)}(x)$;*

(c) *if $p \in C(i)$ is a local maximum of $C(i)$, then p is in fact a local maximum of L.*

Proof. Let P be the set of primes of L and let $\mathscr{S}$ be the collection of all n-tuples of coordinate chains whose union is P. Thus

$$\mathscr{S} = \{(C(1), \cdots, C(n)) : \textstyle\bigcup_i C(i) = P\} .$$

$\mathscr{S}$ is partially ordered by coordinatewise set inclusion. Since any intersection of coordinate chains is a coordinate chain it follows readily that any totally ordered subset of $\mathscr{S}$ is bounded below in $\mathscr{S}$. Thus, in order to conclude by Zorn's Lemma that $\mathscr{S}$ has a minimal element, we need only show that $\mathscr{S} \neq \varnothing$.

First we observe that any set of pairwise noncomparable elements of P has cardinality at most n. Indeed, the meet of any $(n+1)$ elements of P is redundant, and in any redundant meet of primes, two primes must be comparable [cf. 4, Lemma 1, p. 58]. Then by Dilworth's decomposition theorem [6, Th. 1.1], P is the union of n chains. Extend these to maximal chains $D(1), \cdots, D(n)$. It is readily seen that the $D(i)$ are coordinate chains.

Suppose that $(C(1), \cdots, C(n))$ is a minimal element in $\mathscr{S}$. By construction (a) holds, and (b) then follows immediately from Lemma 2.4. Suppose that $p \in C(i)$ violates (c); i.e., p is a local maximum of $C(i)$ but is not a local maximum of L. Let $C(i)' = C(i) \backslash \{p\}$. $C(i)'$ is a coordinate chain. We claim that $C(i)' \cup (\bigcup_{j \neq i} C(j)) = P$. Since p is not a local maximum of L we can find a downward directed net $\{x_\alpha\}$ of elements of $L \backslash \{p\}$ which converges to p. For each j, let $q_j = \bigwedge_\alpha \sigma_{C(j)}(x_\alpha)$. By (b) we have $p = q_1 \wedge \cdots \wedge q_n$, so $p = q_j$ for some j. Moreover, $j \neq i$; otherwise p would be a limit of elements of $C(i)$ greater than p; hence p would not be a local maximum of $C(i)$. Then $(C(1), \cdots, C(i)', \cdots, C(n))$ is in $\mathscr{S}$. This contradicts the minimality of $(C(1), \cdots, C(n))$. Thus (c) holds.

LEMMA 2.6. *Let L and $C(1), \cdots, C(n)$ be as in the previous lemma. Then for each i, $\sigma_{C(i)}$ is a complete lattice homomorphism of L onto $C(i)$ (with respect to the internal order structure of $C(i)$).*

Proof. By induction on n. If $n = 1$ then $L = C(1)$ and $\sigma_{C(1)}$ is the identity map. Suppose next that the lemma holds for all lattices of breadth less than n. We shall use Lemma 2.3 to show that $\sigma = \sigma_{C(1)}$ is a complete homomorphism; the cases of $\sigma_{C(2)}, \cdots, \sigma_{C(n)}$ are similar. Suppose that $p, q \in C(1)$ with $p > q$, where p and q violate $(*)$. Let $\mathrm{Br}\,(\sigma^{-1}(p)) = m$. By Lemma 2.5 (b), the sublattice $\sigma^{-1}(p)$ can be coordinatized algebraically by the $n - 1$ chains $C(2), \cdots, C(n)$, with the value of $\sigma = \sigma_{C(1)}$ being fixed at p. Therefore $m \leqq n - 1$. Let L' be the order closure of $\sigma^{-1}(p)$ in L. L' is a complete, distributive lattice of breadth $m \leqq n - 1$ whose operations are continuous with respect to order convergence. Since p, q do not satisfy $(*)$, $q \in L'$. Let $C(1)', \cdots, C(m)'$ be the chains constructed for L' by means of Lemma 2.5. By the induction hypothesis, each $\sigma_{C(i)'}$ is a complete lattice homomorphism on L'. Suppose that $\{x_\alpha : \alpha \in \Gamma\}$ is a decreasing net in $\sigma^{-1}(p)$ which converges to q. q is not a local maximum of L. Hence by Lemma 2.5(c) there is a decreasing net $\{p_\beta : \beta \in \Lambda\}$ which converges to q, such that $p_\beta \in C(1)$ and $q < p_\beta < p$ for every $\beta \in \Lambda$. By Lemma 2.5(a) for L', a cofinal subnet of $\{p_\beta : \beta \in \Lambda\}$ is contained in some $C(i)'$, say $C(1)'$. Thus $q \in C(1)'$. Choose a particular $p_\beta \in C(1)'$. Since $x_\alpha \to q$, by our induction hypothesis we have $\sigma_{C(1)'}(x_\alpha) \to \sigma_{C(1)'}(q) = q$. So for some $\alpha \in \Gamma$, $\sigma_{C(1)'}(x_\alpha) \leqq p_\beta$, giving $p > p_\beta \geqq \sigma_{C(1)'}(x_\alpha) \geqq x_\alpha$. Then in L, $p > p_\beta = \sigma(p_\beta) \geqq \sigma(x_\alpha) \geqq p$. This is a contradiction. Thus $\sigma = \sigma_{C(1)}$ does satisfy $(*)$, and hence by Lemma 2.3 is a complete homomorphism.

3. The embedding theorem.

THEOREM 3.1. *Let L be a complete, distributive lattice of finite breadth n. Then the following five conditions on L are equivalent.*

(1) *The operations of L are continuous with respect to order convergence.*

(2) *Under some topology, L is a compact, Hausdorff topological lattice.*

(3) *The interval and order topologies on L coincide and make L a compact, Hausdorff topological lattice.*

(4) *L can be embedded, via a complete lattice isomorphism, in some product of complete chains.*

(5) *L can be embedded, via a complete lattice isomorphism, in a product of n complete chains.*

Moreover, if these equivalent conditions hold, then all compact Hausdorff topologies which make L a topological lattice coincide with the interval and order topologies, and the embeddings of (4) *and* (5) *are topological embeddings.*

Proof. We shall show (5)$\Rightarrow$(4)$\Rightarrow$(3)$\Rightarrow$(2)$\Rightarrow$(1)$\Rightarrow$(5). Of these, the implications (5)$\Rightarrow$(4) and (3)$\Rightarrow$(2) are trivial. The implication (4)$\Rightarrow$(3) follows from the easily proven fact that agreement of interval and order convergence holds for complete chains and is preserved under formation of products of complete chains [cf. 4, Ex. 11, p. 250] and under relativization to complete sublattices. The implication (2)$\Rightarrow$(1) follows from the fact that any compact Hausdorff topology is weaker than the order topology [16]. For (1)$\Rightarrow$(5), construct coordinate chains $C(1), \cdots, C(n)$ as in Lemma 2.5, and let $\varphi: L \rightarrow C(1) \times \cdots \times C(n)$ be given by $\varphi(x) = (\sigma_{C(1)}(x), \cdots, \sigma_{C(n)}(x))$. Then φ is one-to-one by Lemma 2.5(b) and is a complete lattice homomorphism by Lemma 2.6.

Finally, if (1) through (5) hold, all compact Hausdorff topologies are trapped between the interval and order topologies [16] and therefore coincide, by (3). Moreover, φ, being complete, is continuous with respect to order convergence, and has a compact domain and Hausdorff range. Therefore φ is a topological embedding. The proof of Theorem 3.1 is thus complete.

We next show that in the case of *connected* topological lattices, the requirement of compactness in the main embedding theorem can be replaced by the weaker hypothesis of local compactness. We include some auxiliary facts.

COROLLARY 3.2. *Let L be a locally compact and connected, distributive topological lattice of finite breadth n. Then*

(i) *L has a neighborhood base consisting of compact lattice intervals;*

(ii) *L can be embedded in the direct product P of its own compact intervals;*

(iii) *L can be embedded in the real cube $I^{\mathrm{Hom}(L,I)}$* (cf. [2, Corollary 3]);

(iv) *L can be embedded in a product of n complete chains.*

Here all embeddings are simultaneously algebraic and topological. Hom(L, I) is the set of all continuous homomorphisms of L into the real unit interval I.

Proof. (i) Theorem 1 of [1] states that a locally compact, connected topological lattice is locally convex. Furthermore, a distributive topological lattice of finite breadth is "locally a lattice," i.e., has a neighborhood base consisting of sublattices [7, proof of Th. 3]. By applying these three local properties of L in succession, we can shrink any neighborhood of a point in L to a compact neighborhood which is the convex hull of a compact sublattice, i.e., is a compact interval. (ii) It is sufficient to show that if K is closed in L and $x \in L \backslash K$, then there is a compact interval $[a, b]$ in L and a continuous homomorphism $\varphi: L \rightarrow [a, b]$ such that $\varphi(x) \notin \varphi(K)^*$ (Embedding Lemma, [11, p. 116]). Accordingly, given K and x, use (i) to choose $[a, b]$ with x in the interior of $[a, b] \subseteqq L \backslash K$, and let $\varphi(y) = (a \vee y) \wedge b$ for $y \in L$. $\varphi(x) = x$, whereas Lemma 6(ii) of [1], combined with its dual, states that $\varphi(K)$ is contained in the boundary of $[a, b]$. (iii) Each compact interval M of L is a compact, connected, distributive, topological lattice of finite breadth, and so can be embedded in the cube $I^{\mathrm{Hom}(M,I)}$, according to [2, Corollary 3]. Then by (ii), L can be embedded in *some* real cube, hence in $I^{\mathrm{Hom}(L,I)}$. (iv) The closure of the image of L in P of (ii) or the cube of (iii) satisfies the hypotheses of Theorem 3.1-(4).

We are now able to prove and generalize a conjecture first formulated by Dyer and Shields [7, p. 447] and sharpened by Anderson [2, p. 62].

COROLLARY 3.3. *Let L be a metrizable, distributive topological lattice of finite breadth n which is either* (a) *compact or* (b) *locally compact, separable, and connected. Then L can be embedded, topologically and algebraically, in the n-cell* I^n.

Proof. By Theorem 3.1-(5) or Corollary 3.2-(iv), L can be embedded in a product of n chains. It therefore suffices to show that if $\varphi: L \rightarrow C$ is a continuous homomorphism of L onto a chain C, then C can be embedded in the unit interval I.

Assume hypothesis (a). We claim that C has at most countably many gaps (closed intervals with only two elements): It is well known that the continuous image of a compact metric space is again such a space; hence C is such a space. Therefore the base of open intervals in C can be thinned to a countable base. It is easy to see that any endpoint of a gap in C must also be an endpoint of a basic set. Thus the claim is verified. Now, fill each gap of C with a copy of the real open unit interval, and let D be the chain thus obtained. D is a complete, connected chain with a countable topologically dense subset. In a connected chain, a topologically dense subset and $0, 1$ (if present) together form an order-dense subset. But a complete, connected chain with such a subset is the completion of the rationals by cuts [4, p. 200]. Thus $D = I$.

Assume hypothesis (*b*). Then $C = \varphi(L)$ is a separable connected chain. As before, C then has a countable order-dense subset and $D = C \cup \{0, 1\}$ is complete [4, Th. 14, p. 243]. Again, $D = I$.

We are now ready to relate breadth to dimension. Let dim be covering dimension as defined in [15] and let cd be codimension as defined in [5]. Recall that for a locally compact Hausdorff space X, $\mathrm{cd}\, X \leqq \dim X$.

COROLLARY 3.4 (*cf. Lawson* [13]). *Suppose that L is a distributive topological lattice of finite breadth n.* (i) *If L is compact, then* $\dim L \leqq n$. (ii) *If L is connected and either compact or locally compact and metrizable, then* $n = \dim L = \mathrm{cd}\, L$.

Proof. Assume any group of hypotheses. By Theorem 3.1 or Corollary 3.2, L can be embedded in a product of n compact chains $P = C(1) \times \cdots \times C(n)$. By [10, Th. 5.1], $\dim P \leqq \Sigma_i \dim C(i) = n$. Since L is fully normal, locally compact, and hence has the starfinite property, $L \subseteqq P$ implies $\dim L \leqq \dim P \leqq n$ [14, Th. 1]. Let L be connected. The method of proof of [3, Th. 1] shows that a connected distributive topological lattice of breadth n contains a product of n compact connected chains and hence has codimension at least n. Thus $n \leqq \mathrm{cd}\, L \leqq \dim L \leqq n$.

The following fact can be extracted from Corollaries 3.3 and 3.4.

COROLLARY 3.5. *Let L be a compact, connected metrizable distributive topological lattice of finite breadth. Then L can be embedded homeomorphically in a cell of the same topological dimension as L.*

PROBLEM. To what extent can the requirement of connectedness be relaxed in Corollaries 3.2 and 3.3?

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