ON THE OSCILLATION OF SOLUTIONS
OF A CLASS OF LINEAR
FOURTH ORDER DIFFERENTIAL EQUATIONS

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The purpose of this paper is to study the oscillation properties of the solutions of the differential equation

(1) $$y^{(4)} = p(t)y$$

on $(-\infty, \infty)$, where $p(t)$ is positive and continuous for all t. Three different types of oscillation are considered, and necessary and sufficient conditions are given for the existence of solutions of each type.

Oscillatory behavior of solutions of (1) has been studied by a number of people (see [5]). Leighton and Nehari [3] have studied a slightly more general class of self-adjoint linear differential equations of fourth order and have given a number of results concerning oscillation of solutions of (1). Hastings and Lazer [2] have shown that under the additional assumptions $P \varepsilon C'$ $[a, \infty)$, $p(t) > 0$ and $p'(t) \geq 0$, (1) has two linearly independent oscillatory solutions which are bounded on $[a, \infty)$. If further, $\lim_{t \to \infty} p(t) = \infty$, then all oscillatory solutions of (1) tend to zero.

We shall distinguish between oscillation in the positive sense and that in the negative sense. A nontrivial solution of (1) is called *positively* (*negatively*) *oscillatory* if its set of zeros is not bounded above (below). It is called *fully oscillatory* if its set of zeros is neither bounded above nor below. It is called *positively* (*negatively*) (*fully*) *nonoscillatory* if it is not positively (negatively) (fully) oscillatory.

In order to gain some motivation, we consider the simple example

(2) $$y^{(4)} = y ,$$

whose solutions are of the form $y = c_1 e^x + c_2 e^{-x} + c_3 \sin x + c_4 \cos x$. We note that $y = e^{-x} + \sin x$ is a positively oscillatory solution of (2) which is not negatively oscillatory. Similarly, $y = e^x + \sin x$ is a negatively oscillatory solution which is not positively oscillatory. Further, $y = \sin x$ and $y = \cos x$ are fully oscillatory.

We shall show that corresponding to the solutions $y = e^x$ and $y = e^{-x}$ of (2) there are solutions w and z of (1), respectively, such that $w(t)$, $w'(t)$, $w''(t)$, $w'''(t) > 0$ for all $t \varepsilon (-\infty, \infty)$ and $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$, and $z'''(t) < 0$ for $t \varepsilon (-\infty, \infty)$. Under an additional assumption it is shown that any other solutions w_1 and z_1

with these properties must be constant multiples of w and z, respectively. The main theorem concerning positive oscillation states that (1) has a positively oscillatory solution if and only if every positively nonoscillatory solution y of (1) has the property that either there exists a number t_0 such that sgn $y = $ sgn $y^{(j)}$, $j = 1, 2, 3$, for all $t \geq t_o$, or sgn $y^{(j)} \neq$ sgn $y^{(j+1)}$, $j = 0, 1, 2$, for all $t \, \varepsilon \, (-\infty, \infty)$. It then follows that if (1) has one positively oscillatory solution, it has three linearly independent positively oscillatory solutions. Two of these have the property that every solution which is a nontrivial linear combination of these two is also positively oscillatory. However, as suggested by the solutions of (2), given any three linearly independent solutions of (1), some linear combination of them is positively nonoscillatory. It is also shown that duals of the theorems concerning positively oscillatory solutions of (1) hold for the negatively oscillatory solutions. Finally, a necessary and sufficient condition is given for the existence of a fully oscillatory solution of (1) and it is shown that if (1) has one fully oscillatory solution, then it has two linearly independent fully oscillatory solutions such that the zeros of any two independent linear combinations of these solutions separate on $(-\infty, \infty)$. Further, any nontrivial linear combination of these two solutions is also fully oscillatory. As suggested by the solutions of (2), if y_1 and y_2 are any two independent linear combinations of these two solutions, then $y_1^{(i)}$ and $y_2^{(i)}$ cannot have common zeros, $i = 0, 1, 2, 3$.

2. **Positive oscillation.** First we state two lemmas. See [2] and [3] for proofs.

LEMMA 1. *If $y(t)$ is a nontrivial solution of (1) and a is a number such that $y(a), y'(a), y''(a), y'''(a) \geq 0$, then $y(t), y'(t), y''(t), y'''(t) > 0$ for all $t > a$.*

LEMMA 2. *If $y(t)$ is a nontrival solution of (1) and a is number such that $y(a) \geq 0, y'(a) \leq 0, y''(a) \geq 0$ and $y'''(a) \leq 0$, then $y(t) > 0$, $y'(t) < 0, y''(t) > 0$ and $y'''(t) < 0$ for all $t < a$.*

We note if y is a solution of (1), then so is $-y$. Hence it follows from Lemma 1 that $y(a), y'(a), y''(a), y'''(a) \leq 0$ (but not all zero) implies $y(t), y'(t), y''(t), y'''(t) < 0$ for all $t > a$. Similarly, it follows from Lemma 2 that if y is a nontrivial solution such that $y(a) \leq 0$, $y'(a) \geq 0, y''(a) \leq 0$ and $y'''(a) \geq 0$, then $y(t) < 0, y'(t) > 0, y''(t) < 0$ and $y'''(t) > 0$ for all $t < a$.

Throughout the remainder of this paper we let $z_0, z_1, z_2,$ and z_3 denote solutions of (1) defined on $(-\infty, \infty)$ by the initial conditions

$$z_i^{(j)}(0) = \delta_{ij} = \begin{cases} 0, \ i \neq j \\ 1, \ i = j \end{cases}$$

for $i, j = 0, 1, 2, 3$.

THEOREM 1. *There exists a solution w of (1) such that $w(t)$, $w'(t)$, $w''(t)$, $w'''(t) > 0$ for all t.*

Proof. For each natural number n, let c_{on}, c_{1n}, c_{2n} and c_{3n} be numbers satisfying

$$c_{0n}z_0 \ (-n) + c_{1n}z_1 \ (-n) + c_{2n}z_2 \ (-n) + c_{3n}z_3 \ (-n) = 0 \ ,$$
$$c_{0n}z_0' \ (-n) + c_{1n}z_1' \ (-n) + c_{2n}z_2' \ (-n) + c_{3n}z_3' \ (-n) = 0 \ ,$$
$$c_{0n}z_0'' \ (-n) + c_{1n}z_1'' \ (-n) + c_{2n}z_2'' \ (-n) + c_{3n}z_3'' \ (-n) = 0 \ ,$$
$$c_{0n}z_0'''(-n) + c_{1n}z_1'''(-n) + c_{2n}z_2'''(-n) + c_{3n}z_3'''(-n) > 0 \ ,$$
$$c_{0n}^2 + c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = 1 \ .$$

Let $y_n = c_{0n}z_0 + c_{1n}z_1 + c_{2n}z_2 + c_{3n}z_3$. The existence of numbers c_{0n}, c_{1n}, c_{2n}, and c_{3n}, satisfying the above conditions, is easy to verify. For one can choose such numbers satisfying the first three equations and the last equation. Since z_0, z_1, z_2 and z_3 are linearly independent, y_n is a nontrivial solution of (1). It follows from the Uniqueness Theorem that $y_n'''(-n) \neq 0$. If $y_n'''(-n) < 0$, then $-c_{0n}$, $-c_{1n}$, $-c_{2n}$, and $-c_{3n}$ satisfy the required conditions.

Since for each natural number n, $c_{0n}^2 + c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = 1$, there exists a sequence n_k of natural numbers such that the sequences $\{c_{0n_k}\}, \{c_{1n_k}\}, \{c_{2n_k}\}$ and $\{c_{3n_k}\}$ converge to numbers c_0, c_1, c_2 and c_3, respectively, satisfying

$$c_0^2 + c_1^2 + c_2^2 + c_3^2 = 1 \ .$$

Let $w = c_0z_0 + c_1z_1 + c_2z_2 + c_3z_3$. Now, suppose that $w^{(j)}(t_0) < 0$ for some $j = 0, 1, 2, 3$, and for some number t_0. Then, since $\{y_{n_k}^{(j)}(t_0)\}$ converges to $w^j(t_0)$, there exists a natural number N such that $y_{n_k}^{(j)}(t) < 0$ for all $n_k > N$. But this leads to a contradiction since for $-n_k < t_0$, $y_{n_k}^{(j)}(t_0) > 0$ by Lemma 1, as $y_{n_k}^{(j)}(-n_k) \geqq 0$ for all $j = 0, 1, 2, 3$. This shows that $w^{(j)}(t) \geqq 0$ for all $j = 0, 1, 2, 3, 4$, and all t. Further, since w is a nontrivial solution, there is no number τ such that $w^{(j)}(\tau) = 0$ for all $j = 0, 1, 2, 3$. Therefore, it follows that w satisfies the requirements of the theorem.

THEOREM 2. *There exists a solution $z(t)$ of (1) such that $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$ and $z'''(t) < 0$ for all t.*

Proof. The proof is similar to that of Theorem 1. We modify

each y_n to satisfy the conditions $y_n(n) = y_n'(n) = y_n''(n) = 0$ and $y_n'''(n) < 0$, and use Lemma 2.

Throughout this paper, w and z represent the solutions of (1) given in Theorems (1) and (2), respectively.

LEMMA 3. *If y is a positively nonoscillatory solution of (1) which has a zero in common with some positively oscillatory solution of (1), then there exists a number t_0 such that y, y', y'' and y''' have the same constant sign on $[t_0, \infty)$.*

Proof. Since y is a positively nonoscillatory solution of (1), there exists a number t_0 such that y, y', y'' and y''' are each of constant sign on $[t_0, \infty)$. Let u be a positively oscillatory solution and τ a number such that $u(\tau) = y(\tau) = 0$. An application of Rolle's Theorem to the function u/y shows that there exists a number $s > t_0$ such that

$$u'(s)y(s) - u(s)y'(s) = 0 \ .$$

Hence there exist nonzero constants c_1 and c_2 satisfying

$$c_1 u (s) + c_2 y (s) = 0 \ ,$$
$$c_1 u'(s) + c_2 y'(s) = 0 \ .$$

Let $Y(t)$ be the solution of (1) given by

$$Y(t) = c_1 u(t) + c_2 y(t) \ .$$

Since u is positively oscillatory and y is not, $Y \not\equiv 0$. Thus we can not have $Y''(s) = Y'''(s) = 0$. Further, sgn $Y''(s) =$ sgn $Y'''(s)$. For, otherwise, Y would be of constant sign to the left of s, by Lemma 2, contradicting $Y(\tau) = 0$. Hence, sgn $Y(t) =$ sgn $Y^{(j)}(t)$, $j = 1, 2, 3$, for all $t > s$, by Lemma 1. In order to complete the proof of this lemma, it is sufficient to show that sgn $y''(t) =$ sgn $y'''(t)$ on $[t_0, \infty)$. Suppose sgn $y''(t) \neq$ sgn $y'''(t)$ on $[t_0, \infty)$. Then y'' is bounded on $[t_0, \infty)$. But this is impossible since u is positively oscillatory and sgn $Y'''(t) =$ sgn $Y^{(4)}(t)$ for $t > s$ implies that $\lim_{t \to \infty} | Y''(t) | = \infty$. This completes the proof.

THEOREM 3. *The following two conditions are equivalent*:
(A) *There exists a positively oscillatory solution of (1).*
(B) *If y is any positively nonoscillatory solution of (1), then either there exists a number t_0 such that*

$$\text{sgn } y(t) = \text{sgn } y^{(j)}(t), \quad j = 1, 2, 3 \ ,$$

for all $t \geq t_0$, or

$$\text{sgn } y^{(j)}(t) \neq \text{sgn } y^{(j+1)}(t), \quad j = 0, 1, 2,$$

for all $t \; \varepsilon \; (-\infty, \infty)$.

Proof. Assume (A) holds, and let y be a positively nonoscillatory solution. Then there exists a number t_0 such that y, y', y'' and y''' are each of constant sign on $[t_0, \infty)$. We note that if $\text{sgn } y^{(j)}(t) \neq \text{sgn } y^{(j+1)}(t), j = 0, 1, 2$, for all $t \; \varepsilon \; (t_0, \infty]$, then $\text{sgn } y^{(j)}(t) \neq \text{sgn } y^{(j+1)}(t)$, $j = 0, 1, 2$, for all $t \; \varepsilon \; (-\infty, \infty)$, by Lemma 2. Let u be a positively oscillatory solution and $s > t_0$ with $u(s) = 0$. Let c be a nonzero constant such that

$$y(s) + cz(s) = 0 .$$

Let $Y = y + cz$. If $Y \equiv 0$, then y satisfies condition (B). Suppose $Y \not\equiv 0$. Assume, without loss of generality that $y > 0$ on $[t_0, \infty)$. It is easy to verify that if y does not satisfy condition (B), then we must have y, y', $y'' > 0$ and $y''' < 0$ on $[t_0, \infty)$. Thus, $\lim_{t \to \infty} y(t) = \infty$, and consequently, Y is positively nonoscillatory since z is bounded on $[t_0, \infty)$. Hence by Lemma 3, there exists a number t_1 such that $\text{sgn } Y = \text{sgn } Y^{(j)}$, $j = 1, 2, 3$, for $t > t_1$. It follows that $\lim_{t \to \infty} | Y''(t) | = \infty$ since $\text{sgn } Y''' = \text{sgn } Y^{(4)}$ for $t > t_1$. But this is a contradiction since y'' and z'' are both bounded on $[t_0, \infty)$, as $y'' \, y''' < 0$ and $z'' \, z''' < 0$. This completes the proof of the first half of Theorem 3.

In order to prove that (B) implies (A), we construct two linearly independent positively oscillatory solutions u^+ and v^+ similar to what was done in [2]. Some of the details have to be modified since we do not require $p(t)$ to satisfy the same hypothesis as in [2]. For the sake of completeness we go through the construction of u^+ and v^+.

For each natural number n, let b_{0n}, b_{3n}, c_{2n} and c_{3n} be numbers satisfying

$$(3) \qquad\qquad b_{0n}^2 + b_{3n}^2 = c_{2n}^2 + c_{3n}^2 = 1 ,$$

$$(4) \qquad\qquad b_{0n}z_0(n) + b_{3n}z_3(n) = 0 ,$$

$$(5) \qquad\qquad c_{2n}z_2(n) + c_{3n}z_3(n) = 0 .$$

Define $u_n(t)$ and $v_n(t)$ to be the solutions of (1) given by

$$u_n(t) = b_{0n}z_0(t) + b_{3n}z_3(t) ,$$

$$v_n(t) = c_{2n}z_2(t) + c_{3n}z_3(t) .$$

By (3) there exists a sequence $\{n_k\}$ of natural numbers and numbers b_0, b_3, c_2, c_3 such that the sequences $\{b_{0n_k}\}$, $\{b_{3n_k}\}$, $\{c_{2n_k}\}$, and $\{c_{3n_k}\}$ converge to b_0, b_3, c_2 and c_3, respectively, where $b_0^2 + b_3^2 = c_2^2 + c_3^2 = 1$. Let u^+ and v^+ be the solutions of (1) given by

$$u^+(t) = b_0 z_0(t) + b_3 z_3(t) \ ,$$

$$v^+(t) = c_2 z_2(t) + c_3 z_3(t) \ .$$

Suppose u^+ is positively nonoscillatory. Since (B) holds and $(u^+)'(0) = 0$, there exists a number t_0 such that for all $t > t_0$,

$$\operatorname{sgn} u^+(t) = \operatorname{sgn} (u^+)^{(j)}(t), \ j = 1, 2, 3 \ .$$

Let τ be any number greater than t_0. Since $\{u_{n_k}(\tau)\}$, $\{u'_{n_k}(\tau)\}$, $u''_{n_k}(\tau)$, and $u'''_{n_k}(\tau)$ converge to $u^+(\tau)$, $u^{(+)}{}'(\tau)$, $(u^+)''(\tau)$, and $(u^+)'''(\tau)$, respectively, there exists a natural number N such that

$$\operatorname{sgn} u_{n_k}(\tau) = \operatorname{sgn} u_{n_k}^{(j)}(\tau), \ j = 1, 2, 3 \ ,$$

for all $n_k > N$. Hence, by Lemma 1,

$$\operatorname{sgn} u_{n_k}(t) = \operatorname{sgn} u_{n_k}{}^{(j)}(t), \ j = 1, 2, 3 \ ,$$

for all $t > \tau$ and $n_k > N$. But this is a contradiction since $u_{n_k}(n_k) = 0$ for all natural numbers n_k. Therefore, u^+ is positively oscillatory. Simillarly, v^+ is also positively oscillatory (note: $v^+(0) = 0$). This completes the proof of Theorem 3.

COROLLARY. *If* (1) *has a positively oscillatory solution and y is any positively nonoscillatory solution, then either* $\lim_{t \to \infty} |y(t)| = \infty$ *or* $\lim_{t \to -\infty} |y(t)| = \infty$.

REMARK. An argument, similar to the one given to show that u^+ is positively oscillatory, can be given to show that any nontrivial linear combination of u^+ and v^+ is positively oscillatory.

We note that u^+ and v^+ are linearly independent since, otherwise, we would have $u^+ = k z_3$, $k \neq 0$; contradicting the fact that u^+ is positively oscillatory.

THEOREM 4. *If* (1) *has one positively oscillatory solution, then it has three linearly independent positively oscillatory solutions.*

Proof. The existence of one positively oscillatory solution implies condition (B) of Theorem 3 and hence the existence of two linearly independent positively oscillatory solutions u^+ and v^+, as constructed in the proof of Theorem 3. In order to find a third positively oscillatory solution, let c be a nonzero constant satisfying

$$u^+(0) + c z(0) = 0 \ .$$

Let $y^+(t) = u^+(t) + c z(t)$. Suppose y^+ is positively nonoscillatory. Since

condition (B) holds and $y^+(0) = 0$, there exists a number t_0 such that for all $t > t_0$,

$$\text{sgn } y^+(t) = \text{sgn } (y^+)^{(j)}(t), \quad j = 1, 2, 3 \ .$$

Therefore, $\lim_{t \to \infty} |y^+(t)| = \infty$, which is a contradiction since u^+ is positively oscillatory and z is bounded on $[t_0, \infty)$. That y^+, u^+, and v^+ are linearly independent follows. For, otherwise, z would be a linear combination of u^+ and v^+ and hence positively oscillatory.

REMARK. We note that in view of Theorems (3) and (4), condition (B) is equivalent to the existence of three linearly independent positively oscillatory solutions.

The following theorem gives a necessary condition for the essential uniqueness of the solution $z(t)$.

THEOREM 5. *Suppose that* (1) *has no positively oscillatory solution. If y is any solution satisfying*

$$\text{sgn } y^{(j)}(t) \neq \text{sgn } y^{(j+1)}(t), \quad j = 0, 1, 2 \ .$$

for all t, then $y = c\ z$ for some constant c.

Proof. Suppose that y and z are linearly independent. Let c be a constant such that $z(0) + cy(0) = 0$. Let $Y = z + cy$. Since Y is positively nonoscillatory, there exists a number t_0 such that none of Y, Y', Y'' and Y''' change sign on $[t_0, \infty)$. Assume, without loss of generality that Y and, hence, $Y^{(4)}$ are positive on $[t_0, \infty)$. We note that Y is bounded on this interval since y and z are both bounded. Therefore, we must have $Y''' < 0$, $Y'' > 0$ and $Y' < 0$ for all $t \geq t_0$. But this contradicts the fact that $Y(0) = 0$, thus proving the theorem.

THEOREM 6. *Let y_1, y_2 and y_3 be any three linearly independent solutions of* (1). *If* (1) *has a positively oscillatory solution, then some linear combination of y_1, y_2 and y_3 is positively nonoscillatory.*

Proof. It is easy to verify that w, z, u^+ and v^+ form a basis for the solutions of (1), where u^+ and v^+ are the positively oscillatory solutions of Theorem 3. For, z is bounded on $[0, \infty)$ and $\lim_{t \to \infty} w(t) = \infty$; while, w is bounded on $(-\infty, 0]$ and $\lim_{t \to -\infty} z(t) = \infty$. Hence y_1, y_2 and y_3 can be written as

$$y_1 = a_1 u^+ + a_2 v^+ + a_3 w + a_4 z \ ,$$
$$y_2 = b_1 u^+ + b_2 v^+ + b_3 w + b_4 z \ ,$$

$$y_3 = c_1 u^+ + c_2 v^+ + c_3 w + c_4 z \ .$$

Let d_1, d_2 and d_3 be numbers (not all zero) satisfying

$$d_1 a_1 + d_2 b_1 + d_3 c_1 = 0 \ ,$$
$$d_1 a_2 + d_2 b_2 + d_3 c_2 = 0 \ .$$

Then $y = d_1 y_1 + d_2 y_2 + d_3 y_3$ is a linear combination of w and z and hence positively nonoscillatory.

3. **Negative oscillation.** We note that $y(t)$ is a negatively oscillatory solution of (1) if and only if $Y(t)$ is a positively oscillatory solution of

(1') $$Y^{(4)} = PY \ ,$$

where $Y(t) = y(-t)$ and $P(t) = p(-t)$. Further,

$$\operatorname{sgn} y(t) = \operatorname{sgn} y^{(j)}(t), \ j = 1, 2, 3 \ ,$$

for all $t \ \varepsilon \ (-\infty, t_0]$ (and hence for t, by Lemma 1) if and only if

$$\operatorname{sgn} Y^{(j)}(t) \neq \operatorname{sgn} Y^{(j+1)}(t), \ j = 0, 1, 2 \ ,$$

for all $t \ \varepsilon \ [-t_0, \infty)$ (and hence for all t, by Lemma 2). Similarly,

$$\operatorname{sgn} y^{(j)}(t) \neq \operatorname{sgn} y^{(j+1)}(t), \ j = 0, 1, 2 \ ,$$

for all $t \ \varepsilon \ (-\infty, t_0]$ if and only if

$$\operatorname{sgn} Y(t) = \operatorname{sgn} Y^{(j)}(t), \ j = 1, 2, 3 \ ,$$

for all $t \ \varepsilon \ [-t_0, \infty)$.

Thus, one can reduce the study of negatively oscillatory solutions of (1) to that of positively oscillatory solutions. We now state the duals of the preceding theorems concerning positively oscillatory solutions for negatively oscillatory solutions. The proofs follow from the above observations and the results established in the previous section.

LEMMA 3'. *If y is a negatively nonoscillatory solution of* (1) *which has a zero in common with some negatively oscillatory solution of* (1), *then there exists a number t_0 such that*

$$\operatorname{sgn} y^{(j)}(t) \neq \operatorname{sgn} y^{(j+1)}(t), \ j = 0, 1, 2 \ ,$$

for all $t \ \varepsilon \ (-\infty, t_0]$.

THEOREM 3'. *The following two conditions are equivalent* :

(A') *There exists a negatively oscillatory solution of* (1).

(B') *If* y *is a negatively nonoscillatory solution of* (1), *then either there exists a number* t_0 *such that*

$$\operatorname{sgn} y^{(j)}(t) \neq \operatorname{sgn} y^{(j+1)}(t), \ j = 0, 1, 2 \ ,$$

for all $t \ \varepsilon \ (-\infty, t_0]$, *or*

$$\operatorname{sgn} y = \operatorname{sgn} y^{(j)}, \ j = 1, 2, 3 \ ,$$

for all $t \ \varepsilon \ (-\infty, \infty)$.

THEOREM 4'. *If* (1) *has one negatively oscillatory solution, then it has three linearly independent negatively oscillatory solutions.*

THEOREM 5'. *Suppose that* (1) *has no negatively oscillatory solution. If* y *is any solution such that*

$$\operatorname{sgn} y(t) = \operatorname{sgn} y^{(j)}(t), \ j = 0, 1, 2 \ ,$$

for all t, *then* $y = cw$ *for some constant* c.

THEOREM 6'. *Let* y_1, y_2 *and* y_3 *be any three linearly independent solutions of* (1). *If* (1) *has a negatively oscillatory solution, then some linear combination of* y_1, y_2 *and* y_3 *is negatively nonoscillatory.*

4. Full oscillation.

THEOREM 7. *The following two conditions are equivalent* :
(A'') *There exists a fully oscillatory solution of* (1).
(B'') *Conditions* (B) *and* (B') *both hold.*

Prooof. The fact that (A'') implies (B'') follows from Theorems (3) and (3'). Suppose that (B'') holds. For each natural number n, let c_{0n}, c_{1n} and c_{2n} be numbers satisfying

$$c_{0n}z_0(n) + c_{1n}z_1(n) + c_{2n}z_2(n) = 0 \ ,$$
$$c_{0n}z_0(-n) + c_{1n}z_1(-n) + c_{2n}z_2(-n) = 0 \ ,$$
$$c_{0n}^2 + c_{1n}^2 + c_{2n}^2 = 1 \ .$$

Similarly, let b_{1n}, b_{2n} and b_{3n} be numbers satisfying

$$b_{1n}z_1(n) + b_{2n}z_2(n) + b_{3n}z_3(n) = 0 \ ,$$
$$b_{1n}z_1(-n) + b_{2n}z_2(-n) + b_{3n}z_3(-n) = 0 \ ,$$
$$b_{1n}^2 + b_{2n}^2 + b_{3n}^2 = 1 \ .$$

Define $U_n(t)$ and $V_n(t)$ to be the solutions of (1) given by

$$U_n(t) = c_{0n}z_0(t) + c_{1n}z_1(t) + c_{2n}z_2(t) \ ,$$

$$V_n(t) = b_{1n}z_1(t) + b_{2n}z_2(t) + b_{3n}z_3(t) \ .$$

There exist subsequences $\{c_{0n_k}\}$, $\{c_{1n_k}\}$ and $\{c_{2n_k}\}$ of $\{c_{0n}\}$, $\{c_{1n}\}$ and $\{c_{2n}\}$, respectively, and numbers c_0, c_1 and c_2 such that

$$\lim_{n_k \to \infty} \{c_{0n_k}\} = c_0, \ \lim_{n_k \to \infty} \{c_{1n_k}\} = c_1 \ , \qquad \lim_{n_k \to \infty} \{c_{2n_k}\} = c_2 \ ,$$

$$c_0^2 + c_1^2 + c_2^2 = 1 \ .$$

Let $U(t) = c_0z_0(t) + c_1z_1(t) + c_2z_2(t)$. One can verify, using an argument similar to the ones used in proofs of Theorems (3) and (3'), that U is fully oscillatory.

The solutions $V_n(t)$ give rise to a solution $V(t)$ of the form

$$V(t) = b_1z_1(t) + b_2z_2(t) + b_3z_3(t) \ ,$$

which is fully oscillatory. Similarly, any linear combination of U and V is also fully oscillatory.

REMARK. The solutions U and V of Theorem 7 are linearly independent. To see this it is sufficient to show that $c_0 \neq 0$. Suppose $c_0 = 0$, so that $U = c_1z_1 + c_2z_2$. Since $U(0) = U'''(0) = 0$, either $\operatorname{sgn} U^{(j)} = \operatorname{sgn} U^{(j+1)}$, $j = 1, 2, 3$, for all $t > 0$, or $\operatorname{sgn} U^{(j)} \neq \operatorname{sgn} U^{(j+1)}$, $j = 0, 1, 2$, for all $t < 0$. Hence U is not fully oscillatory, a contradiction.

THEOREM 8. *Let U and V be the solutions of Theorem 7. If Y_1 and Y_2 are two independent linear combinations of U and V, then $Y_1^{(j)}$ and $Y_2^{(j)}$ cannot have any common zeros, $j = 0, 1, 2, 3$.*

Proof. Suppose that $Y_1^{(j)}(s) = Y_2^{(j)}(s) = 0$ for some $j = 0, 1, 2$, and some number s. Then there exist nonzero constants c_1 and c_2 satisfying

$$(6) \qquad c_1Y_1^{(j)}(s) + c_2Y_2^{(j)}(s) = 0 \ ,$$

$$(7) \qquad c_1Y_1^{(j+1)}(s) + c_2Y_2^{(j+1)}(s) = 0 \ .$$

Let $Y = c_1Y_1 + c_2Y_2$. Then by Lemmas (1) and (2), Y is either positively or negatively nonoscillatory; contradicting the fact that every linear combination of U and V is fully oscillatory. If $j = 3$, the proof is similar; we replace (7) by

$$(7') \qquad c_1Y_1^{(j-1)}(s) + c_2Y_2^{(j-1)}(s) = 0 \ .$$

THEOREM 9. *If (1) has one fully oscillatory solution, then it has two independent fully oscillatory solutions U and V such that every nontrivial linear combination of U and V is fully oscillatory, and*

the zeros of any two independent linear combinations of U and V separate on $(-\infty, \infty)$. Further, U, V, z and w form a basis for the solutions of (1).

Proof. We simply prove that the zeros of any two independent linear combinations Y_1 and Y_2 of U and V separate on $(-\infty, \infty)$, since the remaining assertions are now obvious. Assuming that U and V are the solutions of Theorem 7, let t_1 and $t_2 (t_1 < t_2)$ be any two consecutive zeros of Y_1. Suppose that Y_2 has no zero between t_1 and t_2. Then by Theorem 8, Y_2 does not vanish in the interval $[t_1, t_2]$. Thus, $(Y_1/Y_2)'$, and hence $Y_2 Y_1' - Y_1 Y_2'$, vanishes at some point s between t_1 and t_2. Therefore, there exist nonzero constants c_1 and c_2 satisfying

$$c_1 Y_1(s) + c_2 Y_2(s) = 0 \ ,$$
$$c_1 Y_1'(s) + c_2 Y_2'(s) = 0 \ .$$

But this leads to a contradiction, since the solution $y = c_1 Y_1 + c_2 Y_2$ cannot be fully oscillatory by Lemmas (1) and (2).

REMARK. Since U, V, w and z form a basis for the solutions of (1), it follows that any fully oscillatory solution of (1) which is bounded on $(-\infty, \infty)$ must be a linear combination of U and V. One might conjecture that every fully oscillatory solution is a linear combination of U and V. We note that this is not true for positively or negatively oscillatory solutions. In fact, $y = \sin x + e^{-x}$ is a positively oscillatory solution of (2) which is not a linear combination of $\sin x$ and $\cos x$.

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