PACIFIC JOURNAL OF MATHEMATICS
Vol. 34, No. 3, 1970

SPECIAL SEMIGROUPS ON THE TWO-CELL

ESMOND DEVUN

A commutative semigroup S has property (α) if (1) S is topologically a two-cell, (2) S has no zero divisors, and (3) the boundary of S is the union of two unit intervals with the usual multiplication. A characterization of semigroups having property (α) will be given. Let $(I, \cdot)$ denote the closed unit interval with the usual multiplication. Let M be a closed ideal of $(I, \cdot) \times (I, \cdot)$ such that M contains $(I \times \{0\}) \cup (\{0\} \times I)$, and $M \cap (I \times \{1\}) = \{(0, 1)\}$ or $M \cap (\{1\} \times I) = \{(1, 0)\}$. For each $a, b \in (0, 1)$ define a relation $R(a, b; M)$ on $(I, \cdot) \times (I, \cdot)$ by $(x, y) \in R(a, b; M)$ if (1) $x = y$ or (2) $x, y \in (I \times \{0\}) \cup (\{0\} \times I)$, or (3) there exists an $s \in (0, \infty)$ such that x and y are in the same component of $M \cap \{(a^{st}, b^{s-st}): 0 \leqq t \leqq 1\}$.

LEMMA. **The relation $R(a, b; M)$ is a closed congruence.**

THEOREM. **A semigroup S has property (α) if and only if there exists a, b, M such that $(I, \cdot) \times (I, \cdot)/R(a, b; M)$ is iseomorphic to S.**

A central problem in the theory of topological semigroups is to characterize those semigroups whose underlying space is fixed. In general this problem is much too difficult; however, in some special cases considerable progress has been made. For example semigroups on the unit interval with identities are completely classified in [3], [4], and [7]. Some special cases on the two-cell have also been investigated [1], [2], [5], [6] and [7].

In this paper we are concerned with commutative semigroups having property (α). A semigroup S has property (α) if (1) S is topologically a two-cell, (2) S has no zero divisors, and (3) the boundary of S is the union of two unit intervals with the usual multiplication. A description of commutative semigroups satisfying property (α) will be given.

We begin by giving a method of constructing commutative semigroups having property (α). We will show later that this method yields all commutative semigroups having property (α).

Let $(I, \cdot)$ denote the closed unit interval with the usual multiplication. Let M be a closed ideal of $(I, \cdot) \times (I, \cdot)$ such that M contains $(I \times \{0\}) \cup (\{0\} \times I)$ and $M \cap (I \times \{1\}) = \{(0, 1)\}$ or $M \cap (\{1\} \times I) = \{(1, 0)\}$. For a, b contained in the open interval $(0, 1)$ define the relation $R(a, b; M)$ on $(I, \cdot) \times (I, \cdot)$ by $(x, y) \in R(a, b; M)$ if (1) $x = y$ or (2) $x, y \in (I \times \{0\}) \cup (\{0\} \times I)$ or (3) there exists an s contained in the positive reals such that x and y are in the same component of

$M \cap \{(a^{st}, b^{s-st}): 0 \leqq t \leqq 1\}$.

LEMMA 1. *The relation $R(a, b; M)$ is a closed congruence, and hence $(I, \cdot) \times (I, \cdot)/R(a, b; M)$ is a semigroup.*

Proof. We will first show $R(a, b; M)$ is closed. Let $(\hat{r}_n, \hat{s}_n) \in R(a, b; M)$ for $n = 1, 2, 3, \cdots$, with $(\hat{r}_n, \hat{s}_n) \rightarrow (\hat{r}, \hat{s})$. If an infinite number of the elements of the sequence satisfy (1) or (2), then $(\hat{r}, \hat{s}) \in R(a, b; M)$. Hence we can assume all of the elements of the sequence satisfy (3). This implies there exist sequences w_n, c_n, d_n such that $\hat{r}_n = (a^{w_n c_n}, b^{w_n - w_n c_n})$ and $\hat{s}_n = (a^{w_n d_n}, b^{w_n - w_n d_n})$ where w_n is a positive real number and $c_n, d_n \in [0, 1]$. Since $\hat{r}_n \rightarrow \hat{r}$ and $\hat{s}_n \rightarrow \hat{s}$, we have either (a) $w_n \rightarrow \infty$ or (b) $w_n \rightarrow w \in (0, \infty)$, $c_n \rightarrow c$ and $d_n \rightarrow d$. If (a) holds we have $a^{w_n c_n} \rightarrow 0$ or $b^{w_n - w_n c_n} \rightarrow 0$, and $a^{w_n d_n} \rightarrow 0$ or $b^{w_n - w_n d_n} \rightarrow 0$, hence $\hat{r}, \hat{s} \in (\{0\} \times I) \cup (I \times \{0\})$ and $(\hat{r}, \hat{s}) \in R(a, b; M)$. If (b) holds we use the fact that $(a^{w_n e_n}, b^{w_n - w_n e_n}) \in M$ for any e_n satisfying $\min(c_n, d_n) \leqq e_n \leqq \max(c_n, d_n)$. Let it be the case that $\min(c, d) \leqq e \leqq \max(c, d)$. Then there exists a sequence such that $\min(c_n, d_n) \leqq e_n \leqq \max(c_n, d_n)$ and $e_n \rightarrow e$. Since $(a^{w_n e_n}, b^{w_n - w_n e_n}) \in M$ and M is closed we obtain $(a^{w_n e_n}, b^{w_n - w_n e_n}) \rightarrow (a^{we}, b^{w-we}) \in M$. Hence $\hat{r}$ and $\hat{s}$ are in the same component of $(M \cap \{(a^{wt}, b^{w-wt}): 0 \leqq t \leqq 1\})$, which implies $(\hat{r}, \hat{s}) \in R(a, b; M)$.

To show that $R(a, b; M)$ is a congruence, after a moments reflection, it becomes clear that we need only show $((x, 1)\hat{r}, (x, 1)\hat{s})$ satisfies property (3) whenever $(\hat{r}, \hat{s})$ satisfies property (3) and $0 < x < 1$. Let $\hat{r} = (a^{wc}, b^{w-wc})$ and $\hat{s} = (a^{wd}, b^{w-wd})$ with $c \leqq d$. Also $\{(a^{we}, b^{w-we}): c \leqq e \leqq d\} \subset M$. Since $0 < x < 1$, there exist a $q \in (0, \infty)$ such that $(a^q, 1) = (x, 1)$. Using the fact that M is an ideal of $(I, \cdot) \times (I, \cdot)$ we see that

$$(x, 1)(a^{we}, b^{w-we}) = (a^q, 1)(a^{we}, b^{w-we}) = (a^{q+we}, b^{w-we}) = (a^{mf}, b^{m-mf}) \in M$$

for $m = q + w$ and $f = ew + q/w + q$ and $c \leqq e \leqq d$. This completes the proof.

One can observe that the map $\varphi: (I, \cdot) \times (I, \cdot) \rightarrow (I, \cdot) \times (I, \cdot)/R(a, b; M)$ which sends elements to their equivalence classes is a monotone map, and no equivalence class of $R(a, b; M)$ separates $(I, \cdot) \times (I, \cdot)$. A theorem of Whyburn [8] reveals that $(I, \cdot) \times (I, \cdot)/R(a, b; M)$ is a two-cell. Also since $(I, \cdot) \times (I, \cdot)/R(a, b; M)$ is the homomorphic image of $(I, \cdot) \times (I, \cdot)$ which is commutative, it is commutative. Furthermore, the boundary of $(I, \cdot) \times (I, \cdot)/R(a, b; M)$ equals $\varphi((I, \cdot) \times \{1\}) \cup \varphi(\{1\} \times (I, \cdot))$, and hence is the union two unit intervals with usual multiplication. Finally since $(I \times \{0\}) \cup (\{0\} \times I)$ is a completely prime ideal of $(I, \cdot) \times (I, \cdot)$, $(I, \cdot) \times (I, \cdot)/R(a, b; M)$ has no zero divisors. Thus we have proved the following:

THEOREM A. *$(I, \cdot) \times (I, \cdot)/R(a, b; M)$ is a commutative semigroup satisfying property (α).*

Now we will take a commutative semigroup S satisfying property (α) and find $a, b \in (0, 1)$ and an ideal M such that $(I, \cdot) \times (I, \cdot)/R(a, b; M)$ is iseomorphic to S.

We begin this section by letting the boundary of S equal $U \cup V$ where U and V are unit intervals with the usual multiplication. Without much difficulty it can be shown that $S = U \cdot V$ and $U \cap V = \{z, i\}$ where z is the zero for S and i is the identity for S. Letting $f: (I, \cdot) \to U$ and $g(I, \cdot) \to V$ be iseomorphisms and defining $h: (I, \cdot) \times (I, \cdot) \to S$ by $h(x, y) = f(x) \cdot g(y)$, we see that h is a continuous homomorphism from $(I, \cdot) \times (I, \cdot)$ onto S.

LEMMA 2. *If $h(x_1, y_1) = h(x_2, y_2) \neq z$, then one and only one of the following holds:*

(1) $x_1 = x_2$ *and* $y_1 = y_2$

(2) $(x_1 - x_2)(y_1 - y_2) < 0$.

Proof. Let $h(x_j, 1) = u_j$ and $h(1, y_j) = v_j$, $j = 1, 2$. If (1) is not true, then suppose $x_1 > x_2$. This is the case if and only if there exist $u \in U$, $u \neq i$ such that $uu_1 = u_2$. Now $y_1 \geqq y_2$ if and only if there exist v such that $vv_1 = v_2$. Since $h(x_1, y_1) = h(x_2, y_2)$ we have $u_1v_1 = u_2v_2$ or $u_1v_1 = (u_1v_1)(uv)$ which implies $u_1v_1 = (u_1v_1) \cdot u^n \cdot v^n$ for $n = 1, 2, 3, \cdots$. Hence, $u_1v_2 = (u_1v_1) \cdot \lim u^n \cdot \lim v^n = z$. This is a contradiction. Note for $x \neq 0$, $\{h(x, y): 0 \leqq y \leqq 1\}$ is an arc in S.

LEMMA 3. *If $s \in S \backslash \{z\}$, then there exist $(x_1, y_1), (x_2, y_2) \in h^{-1}(s)$ such that for all $(x, y) \in h^{-1}(s)$ we have $x_1 \geqq x \geqq x_2$ and $y_2 \geqq y \geqq y_1$.*

Proof. Set $x_1 = \sup \{x: h(x, y) = s\}$. Construct a sequence $(q_n, r_n) \in h^{-1}(s)$ with $q_{n+1} \geqq q_n$ such that $\lim q_n = x_1$. Noting that $r_{n+1} \leqq r_n$, set $y_1 = \lim r_n$. Since $s = h(q_n, r_n)$ and $(q_n, r_n) \to (x_1, y_1)$ we have $h(x_1, y_1) = \lim h(q_n, r_n) = s$. This implies x_1 is the maximum x and y_1 is the minimum y such that $h(x, y) = s$. A similar argument yields an $(x_2, y_2) \in h^{-1}(s)$.

LEMMA 4. *If $s \in S \backslash \{z\}$, then $\pi_1(h^{-1}(s))$ is connected.*

Proof. Let $x_1 < x < x_2$ with $(x_1, y_1), (x_2, y_2) \in h^{-1}(s)$. We will show there exist a $\bar{y}$ such that $h(x, \bar{y}) = s$. The arc $\{h(x, y): 0 \leqq y \leqq 1\}$ must intersect one of the two arcs $\{h(x_1, y): y_1 \leqq y \leqq 1\}$ and $\{h(x_2, y): y_2 \leqq y \leqq 1\}$. Suppose it intersects the latter, then there exist y, y' such that $h(x, y') = h(x_2, y)$. Hence, if one chooses $\bar{y} = y'y''$ where $yy'' = y_2$, then $h(x, \bar{y}) = h(x, y'y'') = h(x, y')h(1, y'') = h(x_2, y)h(1, y'') = h(x_2, yy'') = h(x_2, y_2) = s$. This completes the proof.

REMARK 1. By using Lemma 2 we note that the $\bar{y}$ obtained in the proof above is unique.

LEMMA 5. *If $s \in S \backslash \{z\}$, then for all $(x_1, y_1), (x_2, y_2) \in h^{-1}(s)$ we have $(\sqrt{x_1x_2}, \sqrt{y_1y_2}) \in h^{-1}(s)$.*

Proof. Suppose $x_2 > x_1$, then $x_1 < \sqrt{x_1x_2} < x_2$, and there exist a unique y such that $h(\sqrt{x_1x_2}, y) = s$. Now $s^2 \neq z$, and $h(x_1x_2, y_1y_2) = s^2 = h(x_1x_2, y^2)$. Hence $y = \sqrt{y_1y_2}$.

REMARK 2. Note that $h^{-1}(z) = I \times \{0\} \cup \{0\} \times I$.

LEMMA 6. *If $s \in S \backslash \{z\}$, then there exist $(x_1, y_1), (x_2, y_2) \in h^{-1}(s)$ such that $h^{-1}(s) = \{(x_1^t x_2^{1-t}, y_1^t y_2^{1-t}): 0 \leqq t \leqq 1\}$.*

Proof. Let $(x_1, y_1), (x_2, y_2)$ be the ordered pairs obtained in Lemma 3. By inducting on the previous lemma we see $\{(x_1^d x_2^{1-d}, y_1^d y_2^{1-d}): 0 \leqq d \leqq 1, d \text{ a dyadic rational}\} \subset h^{-1}(s)$. Taking the closure of this set we get $\{(x_1^t x_2^{1-t}, y_1^t y_2^{1-t}): 0 \leqq t \leqq 1\} \subseteqq h^{-1}(s)$. Since $h^{-1}(s)$ cannot property include this set, they are equal.

Let $J = \{s: s \in S \text{ and } h^{-1}(s) \text{ is not a point}\}$. Note that J is an ideal of S, and hence $h^{-1}(J)$ and $h^{-1}(J)^*$ are ideals of $(I, \cdot) \times (I, \cdot)$.

LEMMA 7. *If $s \in J \backslash \{z\}$, then there exist $a, b \in (0,1)$ such that $h^{-1}(s) \subset \{(a^t, b^{1-t}): 0 \leqq t \leqq 1\}$.*

Proof. Let $(x_1, y_1), (x_2, y_2)$ be the ordered pairs obtained in Lemma 3. We know $x_1 > x_2 > 0$ and $y_2 > y_1 > 0$. Both x_1 and y_2 cannot be equal to 1 for if both were we would have $h(1, y_1) = h(x_2, 1)$ contradicting the fact that $U \cap V = \{z, i\}$. We shall assume $y_2 \neq 1$, hence there exist β such that $0 < \beta < 1$ and $y_1^{1-\beta} = y_2$, also $0 < x_1 \leqq 1$ and hence there exist γ such that $0 \leqq \gamma < 1$ and $x_2^\gamma = x_1$. Setting $a = (x_2 x_1^{\beta-1})^{1/\beta}$ and $b = (y_2^{-\gamma} y_1)^{1/(1-\gamma)}$, it can be shown by simple algebraic manipulation that $a, b \in (0, 1)$ and $h^{-1}(s) \subset \{(a^t, b^{1-t}): 0 \leqq t \leqq 1\}$.

Note that there exist t_1 and t_2 such that $h^{-1}(s) = \{(a^t, b^{1-t}): 0 \leqq t_1 \leqq t \leqq t_2 \leqq 1\}$.

We will now show that the $a, b \in (0, 1)$ obtained in the previous theorem is somewhat unique.

LEMMA 8. *If* $s, s' \in J\backslash\{z\}$, *and suppose* $h^{-1}(s) = \{(a^t, b^{1-t}): t_1 \leqq t \leqq t_2\}$, *then there exists* $w \in (0, \infty)$ *such that* $h^{-1}(s') = \{(a^{wt}, b^{w-wt}): t_1' \leqq t \leqq t_2'\}$.

Proof. Let $h(x_1, y_1) = s$ and $h(x_2, y_2) = s'$. From the previous lemma we know there exist $c, d \in (0, 1)$ such that $h^{-1}(s') = \{(c^t, d^{1-t}): t_1' \leqq t \leqq t_2'\}$. For $(x, y) \in h^{-1}(s)\cdot(x_2, y_2)$ we have $h(x, y) = ss'$, also for $(x', y') \in (x_1, y_1)h^{-1}(s')$ we have $h(x', y') = ss'$. But $h^{-1}(s)\cdot(x_2, y_2) = \{(a^{u\delta}, b^{u-u\delta}): \delta_1 \leqq \delta \leqq \delta_2\}$ and $(x_1, y_1)h^{-1}(s') = \{(c^{v\eta}, d^{v-v\eta}): \eta_1 \leqq \eta \leqq \eta_2\}$. However, there exist $p, q \in (0, 1)$ such that $h^{-1}(ss') = \{(p^\lambda, q^{1-\lambda}): \lambda_1 \leqq \lambda \leqq \lambda_2\}$. This implies $a^u = p = c^v$, $b^u = q = c^v$ or $c = a^{u/v}$, $d = b^{u/v}$.

NOTATION. Let $\operatorname{Comp}(a^w, b^{1-w})$ be the component of $h^{-1}(J) \cap \{(a^t, b^{1-t}): 0 \leqq t \leqq 1\}$ containing (a^w, b^{1-w}).

LEMMA 9. *If* $s \in J\backslash\{z\}$, *and if* $\{(a^t, b^{1-t}): t_1 \neq t_2$ *and* $t_1 \leqq t \leqq t_2\} \subset h^{-1}(s)$, *then* $h^{-1}(s) = \operatorname{Comp}(a^{t_1}, b^{1-t_1})$.

Proof. Let $(a^w, b^{1-w}) \in \operatorname{Comp}(a^{t_1}, b^{1-t_1})$ and suppose $w < t_1$ and $h(a^w, b^{1-w}) = s' \neq s$. Now $\{h(a^t, b^{1-t}): w \leqq t \leqq t_1\}$ is a curve in J containing s and s'. Also for each $q \in [w, t_1]$ there exist β_q, γ_q such that $\beta_q < \gamma_q$ and $h^{-1}(a^q, b^{1-q}) = \{(a^t, b^{1-t}): \beta_q \leqq t \leqq \gamma_q\}$. Moreover, for $s_1, s_2 \in$ and $s_1 \neq s_2$ we have $h^{-1}(s_1) \cap h^{-1}(s_2) = \varnothing$. Hence $\{h^{-1}(s): s \in \{h(a^t, b^{1-t}): w \leqq t \leqq t_1\}\}$ is an uncountable collection of disjoint closed intervals contained in the interval $\{(\mathrm{a}^t, b^{1-t}): 0 \leqq t \leqq 1\}$. This is impossible.

LEMMA 10. *If* $s \in J$, *then* $sS = sU = sV$.

Proof. If $s = z$, then $zS = zU = zV = \{z\}$. Let $s \neq z$ and $h(x, y) = s = h(x', y')$ with $x > x'$ and $y' > y$. Choose x'', y'' such that $xx'' = x'$ and $y'y'' = y$. Let $(\bar{x}, 1) \in \{(t, 1): x'' \leqq t \leqq 1\}$. We will show there exists $(1, \bar{y}) \in \{(1, s): y'' \leqq s \leqq 1\}$ such that $s\cdot h(\bar{x}, 1) = s\cdot h(1, \bar{y})$. Now $s\cdot h(\bar{x}, 1) = h(x, y)\cdot h(\bar{x}, 1) = h(x\bar{x}, y)$ and $x \geqq x\bar{x} \geqq x'$. Hence there exists a unique $\tilde{y}$ such that $y \leqq \tilde{y} \leqq y'$ and $h(x\bar{x}, \tilde{y}) = s$. Choose $\bar{y}$ such that $\bar{y}\tilde{y} = y$. We see $y'' \leqq \bar{y} \leqq 1$, and

$$\begin{aligned} s\cdot h(1, \bar{y}) &= h(x\bar{x}, \tilde{y})\cdot h(1, \bar{y}) = h(x\bar{x}, \bar{y}\tilde{y}) = h(x\bar{x}, y) \\ &= h(x, y)\cdot h(\bar{x}, 1) = s\cdot h(\bar{x}, 1)\ . \end{aligned}$$

The same method yields for each $(1, \bar{y}) \in \{(1, s): y'' \leqq s \leqq 1\}$ an $(\bar{x}, 1) \in \{(t, 1): x'' \leqq t \leqq 1\}$ such that $s\cdot h(1, \bar{y}) = s\cdot h(\bar{x}, 1)$. Let $s' \in S$. Then there exist m, n positive integers and x_0, y_0 such that $x'' \leqq x_0 \leqq 1$, $y'' \leqq y_0 \leqq 1$ and such that $h(x_0^n, y_0^m) = s'$. Hence $s \cdot s' = s \cdot h(x_0^n, y_0^m) = s \cdot h(x_0, 1)^n h(1, y_0)^m$. But there exist $\hat{x}, \hat{y}$ such that $x'' \leqq \hat{x} \leqq 1$ and $y'' \leqq \hat{y} \leqq 1$ and

$$s \cdot h(x_0, 1)^n \cdot h(\hat{x}, 1)^m = s \cdot h(x_0, 1)^n \cdot h(1, y_0)^m = s \cdot h(1, \hat{y})^n \cdot h(1, y_0)^m .$$

That is $s \cdot U = s \cdot S = s \cdot V$.

LEMMA 11. $h^{-1}(J) \cap (\{1\} \times I) = \{(1, 0)\}$ *or* $h^{-1}(J) \cap (I \times \{1\}) = \{(0, 1)\}$.

Proof. Suppose this is false. Then there exist $(x, 1), (1, y) \in h^{-1}(J)$ and $0 < x < 1$ and $0 < y < 1$. From the previous theorem, letting $h(x, 1)$ represent the element s, we obtain $x' \neq 0$ such that $h(x, 1)h(1, y) = h(x, 1) \cdot h(x', 1) = h(xx', 1)$. Also letting $h(1, y)$ represent the element s, we get $y' \neq 0$ such that $h(x, 1)h(1, y) = h(1, y')h(1, y) = h(1, yy')$. So $h(xx', 1) = h(1, yy')$. But this contradicts the assumption that $U \cap V = \{z, i\}$.

LEMMA 12. *If* $(1, d) \in h^{-1}(J)^*$, *then* $(1, c) \in h^{-1}(J)$ *for* $0 \leqq c \leqq d$.

Proof. Let $(1, d) \in h^{-1}(J)^*$. One sees immediately that $\{(x, y): 0 \leqq x < 1, 0 \leqq y < d\} \subset h^{-1}(J)$. Let $a, b \in (0, 1)$ be as in Lemma 7. For $0 < c < d$ we have $(1, c) = (1, b^w)$, and hence there exists t_1 such that $\{(a^{tw}, b^{w-tw}): 0 < t < t_1\} \subset h^{-1}(J)$. From Lemma 9 there exists an $s \in S$ such that $h(a^{wt}, b^{w-wt}) = s$ for $t \in (0, t_1)$. Using the continuity of h we get $\lim_{t \to 0} h(a^{wt}, b^{w-wt}) = h(1, b^w) = s$. That is $(1, c) \in h^{-1}(J)$. For $c = 0$, $h(1, c) = h(1, 0) = z$ which is always in J.

The same method of proof also shows that if $(d, 1) \in h^{-1}(J)^*$, then $(c, 1) \in h^{-1}(J)$ for $0 \leqq c < d$.

COROLLARY 13. *If* $(x, 1), (1, y) \in h^{-1}(J)^*$, *then* $x = 0$ or $y = 0$.

Let S be a commutative semigroup satisfying property (α). If $J \neq \{z\}$, then there exist $a, b \in (0, 1)$ which satisfies the conditions of Lemma 7. If $J = \{z\}$, let $a = 1/2$, $b = 1/2$. From Theorem A we see that $(I, \cdot) \times (I, \cdot)/R(a, b, h^{-1}(J)^*)$ is a commutative semigroup satisfying property (α). Moreover, the following theorem holds.

THEOREM B. *The semigroups* S *and* $(I, \cdot) \times (I, \cdot)/R(a, b, h^{-1}(J)^*)$ *are iseomorphic.*

Proof. Consider the diagram

$$\begin{tikzcd} (I, \cdot) \times (I, \cdot) \arrow[r, "h"] \arrow[d, "\varphi"'] & S \\ (I, \cdot) \times (I, \cdot)/R(a, b, h^{-1}(J)^*) \arrow[ur, "h\varphi^{-1}"'] & \end{tikzcd}$$

when h and φ are the maps described earlier. We will show the relation $h\varphi^{-1}$ is an iseomerphism. To prove this we need only show that for $(x, y) \in (I, \cdot) \times (I, \cdot)$, $\varphi^{-1}\varphi(x, y) = h^{-1}h(x, y)$. Let $(x, y) \in (I, \cdot) \times (I, \cdot)$. If $x = 0$ or $y = 0$ then $\varphi^{-1}\varphi(x, y) = (\{0\} \times I) \cup (I \times \{0\}) = h^{-1}h(x, y)$. Also if $\varphi^{-1}\varphi(x, y) = \{(x, y)\}$, then $h^{-1}h(x, y) = \{(x, y)\}$. Suppose $\varphi^{-1}\varphi(x, y)$ is not a point and $\varphi^{-1}\varphi(x, y) =$ the component of $(h^{-1}(J)^* \cap \{(a^{tw}, b^{w-tw}): 0 \leqq t \leqq 1\})$ containing $(x, y) = \{(a^{wt}, b^{w-wt}): t_1 \leqq t \leqq t_2, t_1 \neq t_2\}$. Hence $\{(c, d): c < a^{wt}, d < b^{w-wt}$, for some $t \in [t_1, t_2]\} \subset h^{-1}(J)$. Let $w_n = w^{1+(1/n)}$, $n = 1, 2, 3, \cdots$, then $a^{w_n t} < a^{wt}$ and $b^{w_n - tw} < b^{w-tw}$ for $t_1 \leqq t \leqq t_2$. This implies $\{(a^{w_n t}, b^{w_n - w_n t}): t_1 \leqq t \leqq t_2\} \subset h^{-1}(J)$. Using Lemma 9 we see

$$h(a^{w_n t_1}, b^{w_n - w_n t_1}) = h(a^{w_n t}, b^{w_n - w_n t}) = h(a^{w_n t_2}, b^{w_n - w_n t_2})$$

for $t_1 \leqq t \leqq t_2$. Also $\lim h(a^{w_n t}, b^{w_n - w_n t}) = h(a^{wt}, b^{w-wt}) = h(x, y)$ for $t_1 \leqq t \leqq t_2$. And we have $h^{-1}h(x, y) = \varphi^{-1}\varphi(x, y)$. The induced homomorphism theorem implies $h\varphi^{-1}$ is an iseomorphism.

I wish to thank Professor Haskell Cohen for his understanding help. Also I would like to thank Dr. J. T. Borrego for his helpful comments.

Bibliography

1. D. R. Brown, *Topological semilattices on the two-cell*, Pacific J. Math. **15** (1965) 35-46.
2. H. Cohen and H. S. Collins, *Affine semigroups*, Trans. Amer. Math. Soc. **93** (1959), 97-113.
3. H. Cohen and L. I. Wade, *Clans with zero on an interval*, Trans. Amer. Math. Soc. **88** (1958), 523-535.
4. W. M. Faucett, *Compact semigroups irreducibly connected between two idempotents*, Proc. Amer. Math. Soc. **6** (1955), 741-747.
5. J. A. Hildebrant, *On uniquely divisible semigroups on the two-cell*, Pacific J. Math. **23** (1967), 91-95.
6. Anne Lester, *Some semigroups on the two-cell*, Proc. Amer. Math. Soc. **10** (1959), 648-655.
7. P. S. Mostert and A. L. Shields, *On the structure of semigroups on a compact manifold with boundary*, Ann. of Math. **65** (1957), 117-143.
8. G. T. Whyburn, *Analytic topology*, Providence, Rhode Island; Amer. Math. Soc. Colloquium Publications, Vol. 28, 1942.

Received February 17, 1969. This paper contains part of a doctoral dissertation written under the direction of Professor Haskell Cohen.

WICHITA STATE UNIVERSITY