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# A NOTE ON COMMUTATIVE INJECTIVE RINGS

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**The purpose of this note is to determine necessary and sufficient conditions for a commutative injective ring to be a product of local rings. This reduces the study of such rings to the study of local rings, since a product of rings is injective if each factor is.**

Throught $R$ will be a ring with unit, $J$ will denote its Jacobson radical, and if $M$ is $R$ module $E(M)$ will denote its injective hull. A ring $R$ is right (sesp. left) self injective if it is injective as a right (resp. left) module over itself. One may easily verify that if $R = \pi R_i$, then $R$ is right self injective if and only if each $R_i$ is. Given an injective ring of a given type one would therefore like to realize it as a product of simpler rings. One example of a right self injective ring is the full ring of linear transformations on a vector space over a division ring. Faith [2] determined all rings which are products of full linear rings in the following:

THEOREM A. *The following are equivalent*:
(1) *$R$ is a product of left full linear rings.*
(2) *$R$ is a right self injective semiprime ring with large socle.*

If R is commutative the above theorem characterizes those rings which are products of fields. Our purpose here is to determine when a commutative injective ring may be written as a product of local rings. We prove two theorems:

THEOREM 2. *If $R$ is a commutative injective ring with large socle then $R$ is a product of local rings.*

THEOREM 6. *Let $R$ be a commutative injective ring, then $R$ is a product of local rings if and only if $R/J$ has large socle and if $x \in R$, and $x^1$ denotes the right annihilator of $x$ then $x^1 + J/J$ is not large in $R/J$.*

We will make use of the following easily available theorem [4]:

THEOREM B. *If $R$ is injective then*
(1) *$R/J$ is a regular ring*;
(2) *Idempotents left modulo $J$*;
(3) $J = \{r \in R \mid r^{\perp}$ *is large*$\}$.

The following lemma will be of central importance and may be found in [1], though we will outline a proof.

LEMMA 1. *Let $R$ be a commutative injective ring, $V_1$ and $V_2$ two distinct simple $R$ modules and $E(V_1)$, $E(V_2)$ their respective injective hulls, then* $\operatorname{Hom}_R(E(V_1), E(V_2)) = 0$.

*Proof.* Let $V_1 = R/M_1$ and $V_2 = R/M_2$ where $M_1$ and $M_2$ are maximal ideals. Using Theorem B, one can show that there is an idempotent $e \in M_1 - M_2$. Then $E(V_1)e = 0$, for if not, we can find an $x \in E(V_1)$ such that $xe = 1 + M_1$ and this would imply that $1 - e \in M_1$, a contradiction. Now let $f \in \operatorname{Hom}_R(E(V_1), E(V_2))$. If $f$ is not zero, by the definition of injective hull there is an $x \in E(V_1)$ such that $f(x) = 1 + M_2$. But $xe = 0$ so $0 = f(xe) = f(x) \cdot e = (1 + M_2)e$, which says that $e \in M_2$, a contradiction which establishes the lemma.

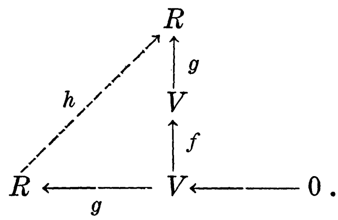
We now prove

THEOREM 2. *If $R$ is commutative injective and has large socle then $R$ is a product of local rings.*

*Proof.* The proof uses the same basic method as that of Theorem A. We let $S$ be the socle of $R$, and write $S = \Sigma \oplus V_i$ where the $V_i \approx R/M_i$ are simple. Let $V = \Sigma \oplus E(V_i)$. We first observe that $V$ is a faithful module for, if $r \in R$, by hypothesis there is an $s \in R$ such that $rs \in S$. Write $rs = \Sigma v_i$, $v_i \in V_i$, and at least one of the $v_i$, say $v_k \neq 0$. Then, $r^\perp \subset v_k^\perp$, and the map $f: rR \to V$ given by $f(r) = 1 + M_k$ is well defined, and extends to a map $\bar{f}: R \to E(V_k)$. If $f(1) = x$, then $xr \neq 0$ and $V$ is faithful.

Next, observe that $S$ is a large submodule of $V$, for if $x = (x_1, \cdots, x_n, 0, \cdots)$ is an element of $V_1$ then we can fine an $r \in R$ such that $x_1 r_1$ is contained in $V_1$ and $x_1 r_1 \neq 0$. If $x_i r_1 \neq 0$ we can find an $r_2$ such that $x_i r_1 r_2 \in V_i$ and $x_i r_1 r_2 \neq 0$. Continuing in this manner we can find an $s$ so that $xs \in S$ and $xs \neq 0$ by the choice of the last $r_k$.

Now there is a monomorphism $i: S \to R$, which may be extended to a map $g: V \to R$ by the injectivity of $R$. The above paragraph shows that $g$ is a monomorphism. Let $T = \operatorname{End}_R V$. Since $\operatorname{Hom}_R (E(V_i), E(V_j)) = 0$ if $i \neq j$ it is easy to see that $T \approx \pi T_i$ where $T_i = \operatorname{End}_R E(V_i)$. It is well known that the endomorphism ring of any indecomposable injective module is local [2, 4], so that each $T_i$ is local. Now, there is a natural ring map $G: R \to T$ given by $G(r) = r_d$ and $r_d(x) = xr$. The faithfulness of $V$ guarantees that this map is a monomorphism. We will prove the theorem if we can show that $G$ is an epimorphism. To do this we let $f \in T$ and consider the following diagram which may be completed by the map $h$:



Then if $v \in V$ we have $hg(v) = gf(v)$. But

$$hg(v) = h(1) \cdot g(v) = g(v) \cdot h(1) = g(v \cdot h(1)) .$$

Thus we have $g(v \cdot h(1) - f(v)) = 0$, but since $g$ is a monomorphism $f(v) = v \cdot h(1)$ and the theorem is proved.

COROLLARY 3. *If $R$ is commutative and injective then $R = P \times Q$ where $P$ is a product of local rings, each with simple socle, and $Q$ has zero socle.*

*Proof.* Let $S$ be the socle of $R$ and $P = E(S)$, then apply Theorem 2.

A ring $R$ is quasi-frobenius (Q.F.) if $R$ is artinian and injective. If $R$ is commutative and Q.F. then $R$ is a product of a finite number of local rings, each with simple socle. In relation to this we have:

COROLLARY 4. *If $R$ is commutative, then $R$ is a product of local* Q.F. *rings if and only if the socle of $R$ is large, and the injective hull of every simple ideal has finite length.*

*Proof.* Write $R = P \times Q$ as in Corollary 3. Then $Q = 0$, and $P = \pi P_i$ where the $P_i$ are local and have simple socle. Since the $P_i$ are local, each is equal to the injective hull of its socle, which is simple, and hence must have finite length.

Before we proceed to the next theorem we introduce the notion of the bi-endomorphism ring of a module. If $M$ is a right $R$ module, then $M$ is $S$-$R$ bi-module where $S = \text{End}_R M_R$. The ring $T = \text{End}_s M$ is called the bi-endomorphism ring of $M$. There is a natural map $G : R \to T$ where $Gr = r_d$ and $(m)r_d = mr$. If the map $G$ is an isomorphism, $M$ is said to be balanced. A well konwn theorem asserts that every module of the form $R \oplus A$ is balanced [3].

Next, a lemma which will enable us to compute an important endomorphism ring.

LEMMA 5. *Let $M = \pi M_i$, and suppose that for each $i$ there is an $r_i$ such that $M_i r_i = 0$ for $i \neq j$ and for $m_i \in M_i$ $m_i r_i = m_i$. Then* $S = \operatorname{End} M \approx \pi \operatorname{End} M_i \approx \pi T_i$. *The map is given by* $(t_1, \cdots, t_i, \cdots)$ $(m_1, \cdots, m_i \cdots) = (t_i m_i, \cdots, t_i m_i, \cdots)$.

*Proof.* Let $f \in S$ and $f(x_1, \cdots, x_i, \cdots) = (y_1, \cdots, y_i, \cdots)$. Then, by multiplying by $r_i$ we have $f(0, \cdots, x_i, \cdots) = (0, \cdots, y_i, \cdots)$. If we let $w_i$ denote the injection $w_i \colon M_i \to M$ and $\pi_i$ the projection $M \to M_i$, and let $t_i \in \operatorname{End} M_i$ be given by $\pi_i f w_i$, we have $f(x_1, \cdots, x_n, \cdots) = (t\ x_1, \cdots, t_i x_i, \cdots)$ and the lemma is proved.

THEOREM 6. *Let $R$ be a commutative injective ring, then $R$ is a product of local rings if and only if socle $R/J$ is large and for all $x \in R$ $x^{\perp} + J/J$ is not large in $R/J$.*

*Proof.* Sufficiency. If $R = \pi R_i$ then it is easy to see that $J = \pi J_i$ where $J_i$ is the Jecobson radical of $R_i$. So, $R/J$ is a product of fields and has large socle. Now suppose $(x_1, \cdots, x_i, \cdots)^{\perp} + J$ is large in $R/J$. Let $a_i$ be that element of $R$ which is 1 on the $i^{th}$ coordinate and 0 everywhere else. Since these $a_i$ are simple modulo $J$, $(x_1, \cdots, x_i, \cdots)^{\perp} + J \ni a_i + J$. Thus, there is an element $(r_1, \cdots, r_i, \cdots) \in x^{\perp}$ such that $(r_1, \cdots, r_i, \cdots) - a_i \in J$, so, $(r_1, \cdots, r_i, \cdots) = (j_1, J_2, \cdots, j_i\text{-}1, j_{1+1}, \cdots)$, $j_i \in J$, and we have $(r_1 j_1, \cdots, r_i(j_i\text{-}1), \cdots) = 0$. This implies that $r_i = 0$, for all $i$ and the contradiction establishes the sufficiency.

Necessity. We first deal with the hypothesis that $R/J$ has large socle. Now $R/J$ is a regular ring by Theorem B, so each simple ideal of $R/J$ is generated by an idempotent. If $r \in R$ denote the image of $r$ under the canonical epimorphism in $R/J$ by $\bar{r}$. Then, since idempotents lift modulo $J$ by Theorem B we may assume that every simple submodule of $R/J$ is generated by $\bar{e}$ where $e$ is an idempotent in $R$. If $S$ is the socle of $R/J$ we have $S = \Sigma e \bar{e}_i\ R/J$ where the $e_i$ are idempotents. Now $e_i R / e_i R \cap J = v_i$ is a simple module. Let $v = \Sigma \bigoplus E(v_i)$, we claim that $V$ is a faithful module. To see this, we note first that if $x \in R$, then there is an $e_i$ such that $xe_i \neq 0$. If not, we would have $x^{\perp} + J \supset S$ contrary to the hypothesis. It is also true that $e_i R \cap J$ is the unique maximal submodule of $e_i R$. To see this, let $e_i r \in e_i R$ and suppose $e_i r \notin J$. Then the simplicity of $e_i R / e_i R \cap J$ implies that there is an $S$ and $j \in J$ such that $e_i = e_i rs + j$. We then have that $e_i - 1 = e_i rs + j\text{-}1$ and we get $0 = e_i rs + e_i(j\text{-}1)$. Or, $0 = e_i rs(1\text{-}j)^{-1} + e$ whence $e_i = \text{-}e_i rs(1\text{-}j)^{-1}$ or $e_i R = e_i r R$. Now let $M_i$ be that maximal ideal with $e_i M_i \subset J$. Using the commutativity of $R$, if $x \in R$ we have $x^{\perp} = e_i R \cap x^{\perp} \bigoplus (e_i\text{-}1) R \cap x^{\perp}$. Now $xe_i \neq 0$ implies that $e_i R \cap x^{\perp} \subset J$, and the definition of $M_i$ yields that

$(e_i\text{-}1) \in M_i$, thus $x^\perp \subset M_i$. Using the representation of $v_i$ as $R/M_i$, we may define a map $f : xR \to E(R/M_i)$ by $f(x) = 1 + M_i$. Extending this map to all of $R$ we let $f(1) = y$ and we have $yx = 1 + M_i \neq 0$. Thus, $V$ is faithful.

It is easy to see that an $R$ module is faihful if and only if there is an index set $I$ and a monomorphism $0 \to R \to M^I$ where $M^I$ denotes the direct product. Clearly if there is such a monomorphism $M$ is faithful, and conversely, if $M$ is faithful there is a collection of elements $m_i \in M$ with $\bigcap m_i^\perp = 0$. Then simply send $1 \in R$ to $(m_1, \cdots, m_i, \cdots)$ in $M^I$. Thus, there is a set $I$ and a monomorphism $0 \to R \to M^I$, or, using the injectivity of $R$, we have $V^I \approx R \oplus A$.

By grouping the $E(V_i)$ together we may write $V = \pi U_i$ where $U_i = E(V_i)^J$ for some index set $J$. Let $A = \text{End}_R V$, then we claim that $A = \pi A_i$ where $A_i = \text{End } U_i$. We wish to apply Lemma 5, and our choice for the $r_i$ are the $e_i$. Suppose $i \neq j$, then we must have $U_j \cdot e_i = 0$. If this is not true, we have an $x \in E(V_j)$ such that $xe_i = 1 + M_j$. But since the $\bar{e}_i$ give a direct sum modulo $J$ we have $e_i \cdot e_j \in J$, which implies that $e_i \in M_j$, but this contradicts the above equation since $(xe_i)e_j = 1 + M_j = e_i + M_j = 0$. We also have that if $y \in E(R/M_i)$, then $ye_i = y$. For if not, let $(y - ye_i)\, r = 1 + M_i$, then get $e_i \in M_i$ by multiplying both sides by $e_i$, but this is a contradiction, since $1 - e_i \in M_i$. This establishes our contention.

Consider the module ${}_A V$ and let $T = \text{End}_A V$. The above has established that each $U_i$ is a left $A$ module, and if we let $T_i$ denote the biendomorphism ring of $U_i$, then Lemma 5 applies if we take $r_i = \pi_i$, the $i^{th}$ projection, so $T = \pi T_i$. But $V$ is of the form $R \oplus A$, so $R = T = \pi T_i$. To complete the proof we need only establish that each $T_i$ is local. It is clear that every biendomorphism of $U_i$ is given by right multiplication so that $T_i = R/U_i^\perp$, and we will be finished if we show that $T_i$ is local. To do this it suffices to determine $E(R/M_i)^\perp$.

Claim: $E(R/M_i)^\perp = \{r \in R \mid r^\perp \nsubseteq M_i\}$.

If $r^\perp \nsubseteq M_i$ and $yr = 1 + M_i$ we have a contradiction. If $E(R/M_i)$ $r = 0$ then $r^\perp \nsubseteq M_i$ or else we have $f : rR \to R/M_i$ given by $f(r) = 1 + M_i$ and extending to all of $R$ gives a $y$ such that $yr = 1 + M_i \neq 0$.

Now let $I = E(R/M_i)^\perp$. We claim that $R/I$ is local. First, $I \subset M_i$. Now if $I \subset M_i$ we show $M_i = M_i$. Let $m \in M_i$, then $m = e_i m + (1\text{-}e_i)$ $m_{i-1}$. First, $e_i m \in J$, and second $(1\text{-}e_i)m \in I$. The former assertion is true by the definition of $M_i$ and the second is true by the remarks above, since $(1\text{-}e_i)m^\perp \ni e_i$ and $e_i \notin M_i$. Thus, $M_i = J + I$. But if $M_i \supset I$ then $M_i \supset J$ so $M_i \supset I + J = M_i$ and the theorem is proved.

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