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AN APPROXIMATION THEOREM FOR SUBALGEBRAS OF $H\infty$

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Let E be a closed subset of the unitcircle $T = \{z : |z| = 1\}$ and denote by B_E the algebra of all functions which are bounded and continuous on the set $X = \{z : |z| \leqq 1 \;\&\; z \notin E\}$ and analytic in $D = \{z : |z| < 1\}$.

The main result of this paper (Theorem 1) is that there exist an open set V_E containing X such that every $f \in B_E$ can be approximated uniformly on X by functions being analytic in V_E.

The algebra B_E was introduced in [4] by E. A. Heard and J. H. Wells.

In [4] they characterize the interpolationsets for B_E. At the end of their paper they remark that the question of whether D is dense in the maximal ideal space $M(B_E)$ of B_E is open in case E is a proper nonempty subset of T. As a corollary of Theorem 1 we prove that D is dense in $M(B_E)$. (In proving the corollary we of course use the Carleson corona-theorem [1]).

This corollary has also been proved recently by Jaqueline Detraz in [3] where it follows from the very interesting fact that the restriction map from $M(H\infty)$ (the maximal ideal space of $H\infty(D)$) to $M(B_E)$ is onto. This is the main theorem of [3, Th. 2]. [3] contains also other results about B_E that we do not prove her. However, Theorem 2 of [3] can also be proved by using the main result of this paper together with the Carleson corona-theorem since Theorem 2 of [3] is equivalent with the fact that D is dense in $M(B_E)$. But the proof of Theorem 2 in [3] is more direct and do not involve the Carleson corona theorem.

Through the whole paper $r_0 > 1$ will be a fixed real number.

Define an open set V_E by $V_E = X \cup \{z : 1 \leqq |z| < r_0 \;\&\; \frac{z}{|z|} \notin V\}$

THEOREM 1. *For every $f \in B_E$ and every $\varepsilon > 0$ there exist a function g analytic in V_E such that $\|f - g\|_X < \varepsilon$.*

LEMMA 1. *Suppose $f \in B_E$ and e is a continuously differensiable function on T with compact support contained in $\mathbf{C} \backslash E$.*

If $f = u + iv$ and we define $u_1(\theta) = u(\theta)e(\theta)$ $(\theta \in (-\pi, \pi])$, then the function

$$f_1(z) = \frac{1}{2\pi}\int_{\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} u_1(\theta) d\theta = u_1(z) + v_1(z)$$

is in the disc-algebra $A(D)$ consisting of all continuous functions on $\bar{D}$ being analytic in D.

Proof. Let $\theta \in (-\pi, \pi]$. Then the

integral

$$I(\theta) = \frac{-1}{2\pi}\int_{\pi}^{\pi}\frac{u_1(\theta+t)-u_1(\theta-t)}{2\tan\dfrac{t}{2}}dt$$

exists because it equals the sum

$$I_1(\theta) + I_2(\theta) + I_3(\theta)$$

where

$$I_1(\theta) = \frac{-1}{2\pi}\int_{\pi}^{\pi}\frac{(u(\theta+t)-u(\theta-t))e(\theta)}{2\tan\dfrac{t}{2}}\,dt$$

$$I_2(\theta) = \frac{-1}{2\pi}\int_{\pi}^{\pi}\frac{u(\theta+t)\,(e(\theta+t)-e(\theta))}{2\tan\dfrac{t}{2}}\,dt$$

and

$$I_3(\theta) = \frac{-1}{2\pi}\int_{\pi}^{\pi}\frac{u(\theta-t)\,(e(\theta)-e(\theta-t))}{2\tan\dfrac{t}{2}}\,dt\,.$$

It is well-known that the existence if $I(\theta)$ is equivalent with the existence of $v_1{}^*(\theta)^{\text{def}}\ \lim_{r\to 1} v_1\,(\mathrm{re}^{i\theta})$ and that $I(\theta) = v_1{}^*(\theta)$ if $I(\theta)$ or $v_1{}^*(\theta)$ exists. (See [5] at pages 78 and 79).

Using the results mentioned in [5] we get that $I_1(\theta) = v(\theta)\cdot e(\theta)$. A change of variable in $I_3(\theta)$ shows that $I_3(\theta) = I_2(\theta)$ and $I_2(\theta)$ exists since $e \in C_0^1(T)$ and u is bounded.

Since $v_1{}^*(\theta)$ exists for all θ, Lemma 1 is proved if we can show that $v_1{}^*$ is continuous on T. For then f_1 has the continuous boundary values $f_1(\theta) = u_1(\theta) + iv_1{}^*(\theta)$ and

$$\int_T e^{in\theta} f_1(\theta)d\theta = 0 \qquad\qquad \text{for } n = 1, 2, \cdots.$$

By the remarks above it is sufficient to show that $I_2(\theta)$ is continuous on T.

The proof of this depends on the fact that $e \in C_0^1(T)$ and that for a fixed $f \in L^1(T)$ the map $x \to f_x$(where $f_x(y) = f(yx^{-1})$ when $x, y \in T$) from T *to* $L^1(T)$ is uniformly continuous. We omit the details.

Proof of Theorem 1. Let $T \quad E = \bigcup_{k=1}^{\infty} I_k$ where each I_k is an open interval (arc) and $I_k \cap I_j = \emptyset$ if $k \neq j$.

Consider a fixed I_k. Construct open intervals $\{K_{kn}\}_{n=1}^{\infty}$ such that $I_k = \bigcup_{n=1}^{\infty} K_{kn}$. We also require that each $z \in I_k$ is not contained in more than two such intervals and that $K \cap \bar{K}_{kn} \neq \emptyset$ only for finitely many n if K is a compact subset of I_k. We choose non-negative functions $e_{kn} \in C_0^1(T)$ with the support of e_{kn} contained in K_{kn} and such that $\sum_{n=1}^{\infty} e_{kn}(z) = 1$ if $z \in I_k$.

Having carried out this construction for $k = 1, 2, 3 \cdots$ we renumerate the double-sequence $\{e_{kn}\}$ to a sequence $\{\alpha_j\}$ by defining $\alpha_1 = e_{11}, \alpha_2 = e_{12}, \alpha_3 = e_{21}, \alpha_4 = e_{31}, \alpha_5 = e_{22}$ and so on.

The sequence $\{K_{kn}\}$ is renumerated in the same manner to a sequence $\{K_j\}$ so that support $\alpha_j \subset K_j$ for $j = 1, 2, \cdots$

For each N we let S_N denote the union of the supports of the functions α_j for $j \geqq N$.

By W_j we mean the compact set of all points in $\{z : |z| \leqq r_0\}$ except those z such that $|z| > 1$ and such that the linesegment from the origin to z intersects K_j.

The construction of the sets K_j guarantees that for each compact subset K of V_E there exists a number N such that $K \subset W_j$ for $j \geqq N$.

Let now $f \in B_E$. We can without loss of generality assume $f = u + iv$ where $v(0) = 0$. Define $u_j(\theta) = u(\theta) \cdot \alpha_j(\theta) \; j = 1, 2, \cdots$ $\theta \in (-\pi, \pi]$.

Now let

$$f_j(z) = \frac{1}{2\pi} \int_{\pi}^{\pi} H(\theta, z) u_j(\theta) d\theta$$

where

$$H(\theta, z) = \frac{e^{i\theta} + z}{e^{i\theta} - z} .$$

The function f_j is analytic outside the support of u_j, but by Lemma 1 we can view f_j as a function continuous on $\bar{D}$ and analytic in D.

But then it is easy to see that we in fact have that f_j is continuous on W_j and analytic in the interior of W_j.

Let $\varepsilon > 0$ be given. Choose polynomials p_j such that

$$\| f_j - p_j \|_{W_j} < \frac{\varepsilon}{2j} \qquad \text{for } j = 1, 2, \cdots$$

Define now $f_E(z) = 1/2\pi \int_E H(\theta, z) u(\theta) d\theta$ for $z \in \mathbf{C} \backslash E$. f_E is analytic but not necessarily bounded in $\mathbf{C} \backslash E$.

Let K be a compact subset of V_E. Then there exists a number N_1 such that $K \subset W_j$ if $j \geqq N_1$. We can choose a number $N_2 \geqq N_1$ such that the distance from S_{N_2} to K is positive.

Then we have that

$$(^*):\quad \|\sum_M^M f_j\|_K \to 0$$

as $M \geqq N \geqq N_2$ and $N \to \infty$ because $\sup|H(\theta, z)| < \infty$ $z \in K$, $\theta \in S_{N_2}$ and the Lebesgue measure of S_N tends to zero as $N \to \infty$.

Define $P_N(z) = \sum_1^N f_j(z)$ for all z and let $F_N(z) = \sum_1^N f_j(z)$ if $z \in X$.

From $(^*)$ and the fact that $\|p_j - f_j\|_{W_j} < \varepsilon/2j$ it follows that P_N is a uniform Cauchy-sequence on compact subsets of V_E.

Thus

$$P(z) = \lim_{N\to\infty} P_N(z) \qquad (z \in V_E)$$

is analytic in V_E. In the same way that the formula $(^*)$ was proved we get that on compact subsets K of X we have that

$$\|f_E + F_N - f\|_K \longrightarrow 0$$

as $N \to \infty$.

Let now $z \in X$. Then we have that

$$\begin{aligned}&|f_E(z) + P(z) - f(z)| \\ &\quad \leqq |P - P_N(z)| + |P_N(z) - F_N(z)| + |f_E(z) + F_N(z) - f(z)|\,.\end{aligned}$$

Let now $N \to \infty$. Then we get that

$$|f_E(z) + P(z) - f(z)| \leqq \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon\,.$$

Since $f_E + P$ is analytic in V_E and $z \in X$ was arbitrary the theorem is proved.

COROLLARY. *D is dense in the maximal ideal space $M(B_E)$ of B_E.*

Proof. If the corollary is not true then there exists $m \in M(B_E)$ and functions $f_1, \cdots f_n \in B_E$ such that $m(f_i) = 0$ $i = 1, 2, \cdots n$ and such that $\sum_1^n |f_i| \geqq \delta$ in D for some $\delta > 0$. It is not difficult by Theorem 1 to see that we can assume $f_1, \cdots f_n$ to be analytic in V_E.

Then we can construct an open simply connected set $V \subset V_E$ containing X such that $f_1 \cdots f_n$ are bounded in V and $\sum_1^n |f_i| \geqq \delta/2$ in V. By the Carleson corona theorem we can find bounded analytic functions $g_1, \cdots g_n$ in V such that $f_1g_1 + \cdots + f_ng_n \equiv 1$ in V. Since

$g_1, \cdots g_n$ restricted to X are in B_E we have the contradiction

$$1 = m(1) = \sum_{i=1}^{n} m(f_i) \cdot m(g_i) = 0 \ .$$

References

1. L. Carleseon, Ann. of Math. **76** (1962), 547-559.
2. Jaqueline Detraz, Comptes Rendus Acad. Sci. Paris **269**, 688-691.
3. ———, Comptes Rendus Acad. Sci. Paris **269**, 833-835.
4. E. A. Heard and J. H. Wells, Pacific. J. Math. **28** (1969), 543-553.
5. K. Hoffman, *Banach Spaces of Analytic functions*, Prentice Hall, Englewood Cliffs, N. J., 1962.

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