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OSCILLATION THEOREMS FOR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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It is the purpose of this paper to show that oscillation of the linear second order equation

$$(r(t)x')' + p(t)x = 0 \tag{1}$$

implies oscillation of the equation

$$(r_1(t)x')' + a(t)p_1(t)x = 0 \tag{2}$$

for a large class of positive functions $a(t)$, where the following condition holds for all large t:

$$r(t) \geqq r_1(t) > 0,\ p(t) \leqq p_1(t)\ . \tag{H}$$

We shall also assume that the functions $r(t), r_1(t), p(t), p_1(t)$, and $a(t)$ are continuous on some half line $[T, +\infty)$.

Recently, Fink and St. Mary [2] have shown that to each $p(t) \in C[T, +\infty)$ one can associate a number λ_0, $0 \leqq \lambda_0 \leqq +\infty$, such that the equation $(rx')' + \lambda px = 0$ is oscillatory provided the number λ satisfies $\lambda > \lambda_0$ and nonoscillatory if $0 \leqq \lambda < \lambda_0$. We shall show that λ may be replaced by a class of positive functions $a(t)$. Thus, one may associate with each $p(t)$ a wide class of functions which are oscillation preserving. For an extensive bibliography concerning oscillation and nonoscillation criteria for (1) we refer the reader to [4]. We wish to remark that results obtained here are immediate consequences of the Sturm Comparison Theorem if $p(t)$ and $p_1(t)$ are nonnegative. (See [3]).

We begin with a comparison theorem on a finite interval $[c, d]$.

THEOREM 1.1 *Let $a(t) \in C^{(1)}[c, d]$ satisfy $a(t) \geqq 1$ and $a'(t) \geqq 0$ on $[c, d]$ and assume condition* (H) *holds on $[c, d]$. Let $y(t)$ and $z(t)$ be solutions of* (1) *and* (2), *respectively, with*

$$\frac{r(c)y'(c)}{y(c)} \geqq \frac{r_1(c)z'(c)}{z(c)} > 0 \tag{3}$$

and assume $y'(d) = 0$ with $y'(t) > 0$ on $[c, d)$. Then $z'(t_0) = 0$ for some $c < t_0 \leqq d$.

Proof. We remark first that the expression on the right [left] of (3) is to be replaced by $+\infty$ in case $z(c) = 0$ [$y(c) = 0$]. Assume first that $r \equiv r_1$ and $p \equiv p_1$ on $[c, d]$ and that equality holds in (3). If the lemma is not true, we may assume $z'(t) > 0$ on $(c, d]$. We have

$$\begin{aligned} 0 &\leqq \int_c^d r(t)(a(t) - 1)(y'(t))^2 dt \\ &= -r(c)y(c)y'(c)(a(c) - 1) + \int_c^d p(t)(y(t))^2(a(t) - 1)dt \\ &\quad - \int_c^d r(t)y(t)y'(t)a'(t)dt \leqq \int_c^d p(t)(y(t))^2(a(t) - 1)dt \end{aligned}$$

since $r(t)y(t)y'(t)a'(t) \geqq 0$ on $[c, d]$.

Now letting $w(t) = r(t)(z(t)y'(t) - y(t)z'(t))$ we have

$$\begin{aligned} \int_c^d p(t)(y(t))^2(a(t) - 1)dt &= \int_c^d w'(t)y(t)/z(t)dt \\ &= -r(d)(y(d))^2 z'(d)/z(d) - \int_c^d (w(t))^2/r(t)(z(t))^2 dt < 0 , \end{aligned}$$

and this is a contradiction. Now to extend the result to the case $r(t) \geqq r_1(t) > 0$ and $p(t) \leqq p_1(t)$, consider the equations

$$(r(t)z')' + a(t)p(t)z = 0 \tag{4}$$

$$(r_1(t)u')' + a(t)p_1(t)u = 0 \tag{5}$$

and let $z(t)$, $u(t)$ be the solutions of (4) and (5), respectively, satisfying $z'(t) > 0$ on $[c, d)$, $z'(d) = 0$ and

$$\frac{r(c)z'(c)}{z(c)} \geqq \frac{r_1(c)u'(c)}{u(c)} > 0 .$$

Suppose that $u'(t) > 0$ on $(c, d]$. Then multiplying (4) by $u(t)$ and (5) by $z(t)$, integrating, and subtracting yields

$$\begin{aligned} &r_1(d)u'(d)z(d) + r(c)z'(c)u(c) - r_1(c)u'(c)z(c) \\ &+ \int_c^d (r(t) - r_1(t))u'(t)z'(t)dt + \int_c^d a(t)z(t)u(t)(p_1(t) - p(t))dt = 0 , \end{aligned} \tag{6}$$

which is a contradiction since the left hand side of (6) is positive. Therefore, $u'(t_0) = 0$ for some $c < t_0 \leqq d$. This proves the theorem.

We now introduce the following definition:

CONDITION (A). The function $\theta(t) \in C[T, +\infty)$ is said to satisfy condition (A) provided

$$\liminf_{t\to\infty} \int_T^t \theta(s)ds \geqq 0 \text{ for all large } T .$$

LEMMA 1.2. *Assume $p(t)$ satisfies condition* (A) *and let* $\int^{\infty} ds/r(s) = +\infty$. *Let $y(t)$ be a solution of* (1) *with $y(t) > 0$ for all $t \geqq t_0$. Then there is a $t_1 \geqq t_0$ such that $y'(t) > 0$ on $[t_1, +\infty)$ or $y' \equiv 0$ for all large t.*

Proof. Assume $y' \not\equiv 0$ for all large t. If the lemma is not true, assume first that $y'(t) < 0$ for all $t \geqq T$, $T \geqq t_0$. We may assume by condition (A) that T is sufficiently large so that

$$\int_T^t p(s)ds \geqq 0 \text{ for all } t \geqq T .$$

For if no such T exists, let $T \geqq t_0$ be fixed but arbitrary and define

$$T_1 = \sup\left\{t > T: \int_T^t p(s)ds < 0\right\} .$$

Now if $T_1 = +\infty$, then choose $t_n \to +\infty$ such that

$$\int_T^{t_n} p(s)ds < 0 \text{ for all } n .$$

This contradicts the assumption that $p(t)$ satisfies condition (A) by the arbitrariness of T. Hence, $T_1 < +\infty$ which implies

$$\int_{T_1}^t p(s)ds \geqq 0 \text{ for all } t \geqq T_1 ,$$

contradicting the assumption that no such T_1 exists. Then

$$\int_T^t p(s)y(s)ds = y(t)\int_T^t p(s) - \int_T^t y'(s)\int_T^s p(\sigma)d\sigma ds \geqq 0,\ t \geqq T , \tag{7}$$

so that integrating (1) we have by (7)

$$y'(t) \leqq r(T)y'(T)/r(t),\ t \geqq T . \tag{8}$$

Now an integration of (8) for $t \geqq T$ shows that $y(t) \to -\infty$, a contradiction.

Assume next that $y'(T_n) = 0$ for $T_n \to +\infty$. Let $v(t) = -r(t)y'(t)/y(t)$, $t \geqq t_0$ so that

$$v'(t) = p(t) + (v(t))^2/r(t),\ t \geqq t_0 . \tag{9}$$

Integrating (9) between T_n and T_{n+1}, summing on n, and using the fact that $v(t) \not\equiv 0$ for all large t contradicts the assumption that $p(t)$ satisfies condition (A). This proves the lemma.

THEOREM 1.3. *Let equation* (1) *be oscillatory and assume condi-*

tion (H) *holds for all large t. Let $a(t) \in C^{(1)}[T, +\infty)$ satisfy $a(t) \geqq 1$ and $a'(t) \geqq 0$ for $t \geqq T$ and let $a(t)p_1(t)$ satisfy condition* (A). *Assume also that $\int^{\infty} ds/r_1(s) = +\infty$. Then equation* (2) *is oscillatory.*

Proof. If equation (2) is nonoscillatory, we may assume that $z(t)$ is a solution of (2) with $z(t) > 0$ and $z'(t) > 0$ for $t \geqq T$. Otherwise, if $z' \equiv 0$ for all large t, then $p_1 \equiv 0$ for all large t so (1) cannot be oscillatory by the Sturm Comparison Theorem. But if $y(t)$ is a solution of (1) with $y(t_1) = y'(t_2) = 0$, $t_2 > t_1 \geqq T$ and $y'(t) > 0$ on $[t_1, t_2)$, then $z'(t)$ must vanish on $(t_1, t_2]$ by Theorem 1.1, a contradiction.

COROLLARY 1.4. *Let equation* (1) *be nonoscillatory, let $p(t)$ satisfy condition* (A), *and let $a(t) \in C^{(1)}[T, +\infty)$, $r_1(t)$, $p_1(t)$, $r(t)$, $p(t)$ satisfy the following conditions for all $t \geqq T$:*

$$0 < a(t) \leqq 1,\ a'(t) \leqq 0,\ r(t) \leqq r_1(t),\ p_1(t) \leqq p(t)$$

and assume $\int^{\infty} ds/r(s) = +\infty$. Then equation (2) *is nonoscillatory.*

Proof. If $y(t)$ is a nonoscillatory solution of (1) with $y(t) > 0$ for $t \geqq T$ and $y' \equiv 0$ for all large t, then $p(t) \equiv 0$ for all large t so the result follows from the Sturm Comparison Theorem. If $y' \not\equiv 0$ for all large t then a proof similar to Theorem 1.3 is valid (using the analogue of Theorem 1.1).

EXAMPLE 1.5. Willett [5] has shown that

$$x'' + \lambda((2 + t \sin t)/2t^2)x = 0 \tag{10}$$

is oscillatory if $\lambda > \sqrt{18} - 4$ and nonoscillatory if $\lambda < \sqrt{18} - 4$. Since $p(t) = (2 + t \sin t)/2t^2$ satisfies condition (A) we see that with $a(t) = 1/(t + \sin t)$,

$$x'' + \gamma a(t)p(t)x = 0 \tag{11}$$

is nonoscillatory for all $\gamma \geqq 0$.

REMARK. The differentiability assumptions on $a(t)$ can be replaced by nonincreasing or nondecreasing in the previous results. We wish also to remark that techniques similar to the above may be used to obtain oscillation and boundedness results for the second order nonlinear equation

$$x'' + p(t)x^{2n+1} = 0 \tag{12}$$

and its generalizations under the assumption that $p(t)$ satisfies condition (A) (see [1]).

2. In this section we shall weaken the assumptions on $p(t)$ and strengthen the assumptions on $a(t)$ and $r_1(t)$. We begin with a comparison theorem on a finite interval.

THEOREM 2.1. *Let $y(t)$ be a nonnull solution of* (1) *satisfying $y(c) = y(d) = 0$, let $a(t) \in C^{(1)}[c, d]$ satisfy $a(t) \geqq 1$ on $[c, d]$ and assume $r_1(t)a'(t)$ is nonincreasing on $[c, d]$. Assume also that condition* (H) *holds on $[c, d]$. Then every solution of* (2) *has a zero on (c, d).*

Proof. We may assume $p \equiv p_1$ and $r \equiv r_1$ on $[c, d]$ since the solution to $(r_1(t)x')' + p_1(t)x = 0$ satisfying $x(c) = 0 \neq x'(c)$ must vanish again on $(c, d]$ by the Sturm Comparison Theorem. Let $z(t)$ be the solution of (2) satisfying $z(c) = 0 \neq z'(c)$ and assume $z(t) > 0$ on $(c, d]$. Then we have

$$
\begin{aligned}
0 \leqq \int_c^d r(t)(a(t) - 1)(y'(t))^2 dt &= \int_c^d p(t)(y(t))^2(a(t) - 1)dt \\
&\quad - \int_c^d r(t)a'(t)y(t)y'(t)dt \\
&= \int_c^d p(t)(y(t))^2(a(t) - 1)dt + \frac{1}{2}\int_c^d (y(t))^2 d(r(t)a'(t)) \\
&\leqq \int_c^d p(t)(y(t))^2(a(t) - 1)dt \, .
\end{aligned}
$$

This leads to a contradiction by an argument similar to Theorem 1.1 and proves the result.

COROLLARY 2.2. *Assume equation* (1) *is oscillatory and for all $t \geqq T$ assume that $a(t) \in C^{(1)}[T, +\infty)$ satisfies $a(t) \geqq 1$ with $r_1(t)a'(t)$ nonincreasing. Let condition* (H) *hold for all $t \geqq T$. Then equation* (2) *is oscillatory.*

COROLLARY 2.3. *Assume equation* (1) *is oscillatory and let* $\{a_n(t)\}_{n=1}^{\infty}$ and $\{r_n(t)\}_{n=1}^{\infty}$ *be a sequence of continuous functions with $a_n(t) \in C^{(1)}[T, +\infty)$, $a_n(t) \geqq 1$, $r_n(t)a_n'(t)$ nonincreasing, and $r(t) \geqq r_1(t) \geqq \cdots \geqq r_n(t) \geqq \cdots > 0$ for all $t \geqq T$.*

Let

$$\lim_{n\to\infty} r_n(t) \equiv r_0(t) > 0$$

and

$$\lim_{n\to\infty} \prod_{i=1}^{n} a_i(t) \equiv a(t)$$

where we assume the convergence is uniform on compact subsets of $[T, +\infty)$. *Then the equation*

$$(r_0(t)x')' + a(t)p(t)x = 0 \tag{13}$$

is oscillatory.

Analogous results are true for the case when (1) is nonoscillatory:

COROLLARY 2.4. *Let* (1) *be disconjugate on the interval* $[c, d]$. (*That is, the only solution of* (1) *with more than one zero on* $[c, d]$ *is the zero solution.*) *Let* $a(t) \in C^{(1)}[c, d]$ *satisfy* $0 < a(t) \leqq 1$ *and assume* $r_1(t)a'(t)/(a(t))^2$ *is nondecreasing. Let* $r(t) \leqq r_1(t)$ *and* $p(t) \geqq p_1(t)$ *on* $[c, d]$. *Then equation* (2) *is disconjugate on* $[c, d]$.

EXAMPLE 2.5. Let $r(t) \equiv r_1(t) \equiv 1$ and assume $x'' + p(t)x = 0$ is oscillatory. Then

$$x'' + t^{\alpha}(\log t)^{\beta}p(t)x = 0 \tag{14}$$

is oscillatory for all $\alpha, \beta \geqq 0$.

In a certain sense, the conditions imposed on $a(t)$ in Theorem 2.1 and the following corollaries cannot be weakened. For example, if $p \equiv p_1$ is positive on $[c, d]$ and $r \equiv r_1 \equiv 1$, then using the ordinary Sturm Comparison Theorem with $a(t) \equiv 1$ on some subinterval $[c, t_0]$, $c < t_0 < d$, $a(d) < 1$, and $a''(t) \leqq 0$ on $[c, d]$, we see that equation (2) oscillates slower than equation (1).

To show the necessity of the requirement that $r_1(t)a'(t)$ be nonincreasing in the above results is less trivial. To do this, we appeal to some recent results announced by Wong [6], where it is shown that

$$x'' + \left(\frac{(\log t - 1)\sin\sqrt{2t}}{t\log t}\right)x = 0 \tag{15}$$

is oscillatory. However, it is also shown in [6] that

$$x'' + \left(\frac{\sin\sqrt{2t}}{t}\right)x = 0 \tag{16}$$

is nonoscillatory. Here, with $a(t) = (\log t)(\log t - 1)^{-1}$, all the assumptions, except for $a'(t)$ nonincreasing, of Corollary 2.2 are satisfied. Yet the conclusion of Corollary 2.2 does not hold for this example.

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