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ON THE HYPERPLANE SECTION THROUGH A RATIONAL POINT OF AN ALGEBRAIC VARIETY

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Let V/k be an irreducible affine algebraic variety of dimension $\geqq 3$ defined over an infinite field k with $\mathfrak{p}$ as its prime ideal in $k[X_1, \cdots, X_n]$. Let P be a rational normal point on V/k. It is proved that (1) for a generic hyperplane H_u through P, $(\mathfrak{p}, H_u)$ is a prime ideal and $(\mathfrak{p}, H_u)$ is quasi-absolutely (absolutely irreducible) if $\mathfrak{p}$ is quasi-absolutely (absolutely irreducible). (2) It is not true in general that $V \cap H_u$ is normal at P; however, $V \cap H_u$ is normal at P if the local ring of V/k at P is also Cohen-Macaulay (Theorem 8).

It is well known [11] that if V/k is a normal variety of dimension $\geqq 2$, then for almost all hyperplanes H the section $V \cap H$ is again a normal variety. This research is motivated by this result to study the following problem: If V/k is normal at a rational point P on V, will hyperplane sections of V through P be normal at P? Section 1 localizes some of the results of [11]. Section 2 describes the ideal decomposition of the generic hyperplane section through a given rational point of an irreducible variety, and Section 3 gives a negative answer to the problem of normality. As a consequence the converse of [3; Lemma 4, p. 360] is invalid in general.

1. Generalities.

In the following and the subsequent sections, a variety V/k shall mean an irreducible algebraic variety in the affine space A^n defined over a field k of arbitrary characteristic.

Recall the following definitions.

DEFINITION 1. Let V/k be a variety with $(\xi) = (\xi_1, \cdots, \xi_n)$ as a generic point over k, and let P be a point on V. Let

$$k[\xi]_p = \left\{ \frac{f(\xi)}{g(\xi)} \,|\, f, g \in k[\xi] \quad \text{and} \quad g(P) \neq 0 \right\}$$

be the local ring of V at P in the function field $k(\xi)$ of V over k. We say that P is k-normal on V if $k[\xi]_p$ is integrally closed in $k(\xi)$, that P is k-simple on V if $k[\xi]_p$ is a regular local ring, and that P is singular on V if P is not k-simple on V.

DEFINITION 2. Let V/k be a variety of dimension r, and let P be a point on V. We say that V/k is locally free of s-dimensional singularities at P if every s-dimensional subvariety of V containing P is k-simple on V.

DEFINITION 3. Let R be a finite integral domain $k[\xi_1, \cdots, \xi_n]$ over a field k or a localization thereof relative to a prime ideal of $k[\xi_1, \cdots, \xi_n]$. Let $\mathfrak{p}$ be a prime ideal of R we define

$$\begin{aligned} ht\, \mathfrak{p} &= \text{max. (length of chains of prime ideals contained in } \mathfrak{p}), \\ \operatorname{depth} \mathfrak{p} &= \text{max. (length of chains of prime ideals containing } \mathfrak{p}), \\ \dim \mathfrak{p} &= \text{transcendence degree of the quotient field of } R/\mathfrak{p} \text{ over } k, \\ \dim R &= \text{transcendence degree of the quotient field of } R \text{ over } k\,. \end{aligned}$$

It is well known that $ht\, \mathfrak{p} + \operatorname{depth} \mathfrak{p} = \dim R$ and $\dim \mathfrak{p} = \operatorname{depth} \mathfrak{p}$.

The following criterion for local normality is parallel to [11; Th. 3, p. 363] and is well known [8; (12.9), p. 41].

PROPOSITION 1. *Let V/k be a variety of dimension r defined over a field k, and let P be a point of dimension s on V. P is k-normal on V if and only if* (1) *V/k is locally free of $(r-1)$-dimensional singularities at P,* (2) *every nonzero principal ideal $(a) \cdot k[\xi]_p$ is unmixed of dimension $r - s - 1$.*

PROPOSITION 2. *Let V/k, (ξ), and P be the same as those in Proposition 1, let $k[\xi]_p^*$ be the integral closure of $k[\xi]_p$, and let $\mathfrak{C}_p$ be the conductor of $k[\xi]_p$. If V is locally free of $(r-1)$-dimensional singularities at P and if $\mathfrak{C}_p \neq (1)$, then every nonzero element of $\mathfrak{C}_p$ generates a mixed principal ideal.*

Proof. Let $\alpha \in k[\xi]_p^*$ not in $k[\xi]_p$, and let $c \in \mathfrak{C}_p$, whence $c\alpha \in k[\xi]_p$, say $c\alpha = b$, $b \in k[\xi]_p$. Then $(c) \cdot k[\xi]_p$ must be mixed. Indeed, if $(c)\, k[\xi]_p$ were unmixed, and let $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$ be the associated prime ideals of $(c)\, k[\xi]_p$, then $\dim \mathfrak{p}_i = r - s - 1$, for $i = 1, 2, \cdots, t$. α is integral over $k[\xi]_p$, hence integral over $(k[\xi]_p)_{\mathfrak{p}_i}$ for $i = 1, 2, \cdots, t$. By hypothesis $(k[\xi]_p)_{\mathfrak{p}_i}$ is a regular local ring of dimension 1, for $i = 1, 2, \cdots, t$, therefore $(k[\xi]_p)_{\mathfrak{p}_i}$ is integrally closed for $i = 1, 2, \cdots, t$. Hence $\alpha \in \bigcap_{i=1}^t (k[\xi]_p)_{\mathfrak{p}_i}$ and $b \in (\bigcap_{i=1}^t (c)(k[\xi]_p)_{\mathfrak{p}_i}) \cap k[\xi]_p = \bigcap_{i=1}^t \mathfrak{q}_i$, where $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$ is a primary decomposition of $(c)\, k[\xi]_p$. Thus $b \in (c)\, b[\xi]_p$, i.e., $\alpha \in k[\xi]_p$, a contradiction.

Let V/k be a variety of dimension r defined over a field k with (ξ) as a generic point, and let P be a point on V. Let u be an indeterminate over $k(\xi)$, it is well known that V is a variety over $k(u)$ with (ξ) as a generic point of V over the pure transcendental extension field $k(u)$. Let $k(u)[\xi]_p = \{f(u; \xi)/g(u; \xi) \mid f, g \in k(u)[\xi]$ and $g(u; p) \neq 0\}$

be the local ring of V at P over $k(u)$. We have, by [10, (d), p. 64], the following lemma.

LEMMA 1. *$k[\xi]_p$ is integrally closed if and only if $k(u)[\xi]_p$ is integrally closed.*

Recall the definition of the ground form of an unmixed r-dimensional ideal $\mathfrak{A}'$, [11; p. 373], as following: Let $\mathfrak{A}$ be an unmixed r-dimensional ideal in the polynomial ring $k[X_1, \cdots, X_n]$, we form $r+1$ linear forms in the X_i's with indeterminates coefficients u_{ij}: $z_i = u_{i1}x_1 + \cdots + u_{in}X_n$, $i = 1, 2, \cdots, r+1$, and consider the ideal $\mathfrak{A} \cdot k(u)[X] \cap k(u)[z_1, \cdots, z_{r-1}]$, where $k(u)[X] = k(u_{11}, \cdots u_{r+1n})[X_1, \cdots, X_n]$, which is a principal ideal $(E(z_1, \cdots, z_{r+1}; u))$ in $k(u)[X]$. If E is normalized so as to be a polynomial in the u_{ij} and primitive in them, so that E is defined to within a factor in k, then E is the elementary divisor form or the ground form of $\mathfrak{A}$. The polynomial E is integral in any z_i over the other z_i's and is a polynomial in $z_1, \cdots, z_{r+1}$ of least degree in z_{r+1}, which is in $\mathfrak{A} \cdot k(u)[X]$. If $\mathfrak{A}$ is prime, then its ground form is irreducible, the converse is not true in general; but $\mathfrak{A}$ is primary if and only if its ground form is a power of an irreducible polynomial [9; Th. 9, p. 252]. $\mathfrak{A}$ is prime and absolutely irreducible if and only if (E) is prime and absolutely irreducible [9; Th. 15, p. 259]. If $\mathfrak{A}$ is prime and quasi-absolutely irreducible, then (E) is prime and quasi-irreducible [11, p. 373].

PROPOSITION 3. *Let V/k be an r-dimensional variety defined over a field k with $\mathfrak{p}$ as its prime ideal in $k[X]$ $(=k[X_1, \cdots, X_n])$. Let p be a point on V and let E be the ground form of $\mathfrak{p}$. Then V is k-normal at p if and only if $(\mathfrak{p}, \partial E/\partial z_{r+1}) \cdot k(u)[X]_p$ is unmixed.*

Proof. By Lemma 1, V is k-normal at P if and only if V is $k(u)$-normal at P. By [13; Lemma 2, p. 132] $V/k(u)$ is free of $(r-1)$-dimensional singularities at P. Let (ξ) be a generic point of $V/k(u)$, and pass to $k(u)[\xi]$, we assert that $k(u)[\xi]_p$ is integrally closed if and only if $(\partial \bar{E}/\partial \bar{z}_{r+1}) \cdot k(u)[\xi]_p$ is unmixed, where the bar denotes residue. By the proof of [11; Th. 5, p. 365], we have $\partial \bar{E}/\partial \bar{z}_{r+1} \in \mathfrak{C}$, the conductor of $k(u)[\xi]$ in its integral closure $k(u)[\xi]^*$. Let $\mathfrak{C}_p$ be the conductor of $k(u)[\xi]_p$ in its integral closure $k(u)[\xi]^*_p$. By [15; Lemma, p. 269], $\mathfrak{C} \cdot k(u)[\xi]_p = \mathfrak{C}_p$. Therefore $\partial \bar{E}/\partial \bar{z}_{r+1} \in \mathfrak{C}_p$. By Proposition 2, we have that $k(u)[\xi]_p$ is integrally closed if and only if $(\partial \bar{E}/\partial \bar{z}_{r+1}) \cdot k(u)[\xi]_p$ is unmixed.

2. Irreducibility of generic hyperplane section through a normal point. Let V/k be a variety of dimension $r \geqq 2$. Let $P \in V$ be a rational point. We are studying the generic hyperplane section of V through P. Without loss of generality, we may assume once for all in the sequel that V passes through (0) the origin of the affine space and that $P=(0)$. We shall denote the prime ideal of V/k by $\mathfrak{p}$ in the sequel. Let $u_1, \cdots, u_n$ be n indeterminates over k, and let H_u be the generic hyperplane through (0) defined by $u_1X_1 + \cdots + u_nX_n = 0$. We shall use H_u in two senses whenever it is proper: (1) H_u means the linear polynomial $u_1X_1 + \cdots + u_nX_n$ in $k(u)[X]$ $(=k(u_1, \cdots, u_n)[X_1, \cdots, X_n])$, (2) H_u stands for the hyperplane defined by $u_1X_1 + \cdots + u_nX_n = 0$. Let $k(u) = k(u_1, \cdots, u_n)$, V is a variety over $k(u)$ and $V \cap H_u$ is defined over $k(u)$. Let $(\mathfrak{p}, H_u) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$ be an irredundant primary decomposition with $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$ as the associated prime ideals. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$, $s \leqq t$, be the isolated prime ideals. Since $(0) \in V$, $(\mathfrak{p}, H_u) \subset (X_1, \cdots, X_n) \cdot k(u)[X]$. Hence $(X_1, \cdots, X_n) \cdot k(u)[X]$ must contain at least one of the $\mathfrak{p}_i$, $i \leqq s$, say $\mathfrak{p}_1$. Let us denote $\mathfrak{p}_1$ by $\mathfrak{p}_u$ and let W_u be the variety over $k(u)$ of $\mathfrak{p}_u \cdot W_u$ is of dimension $r-1$ as it is well known that any component of $V \cap H$, where H is a hypersurface, is of dimension $r-1$. Let (ξ) be a generic point of W_u over $k(u)$. Since $\operatorname{tr.deg}_{k(\xi)} k(u;\xi) + \operatorname{tr.deg}_k k(\xi) = \operatorname{tr.deg}_k k(u;\xi) = \operatorname{tr.deg}_k k(u) + \operatorname{tr.deg}_{k(u)} k(u;\xi) = n + r - 1$ and $\operatorname{tr.deg}_{k(\xi)} k(u;\xi) \leqq n-1$, we have $\operatorname{tr.deg}_{k(\xi)} k(u;\xi) \geqq r$. But $(\xi) \in V$, therefore $\operatorname{tr.deg}_k k(\xi) = r$. We thus have

LEMMA 2. *If* $\dim V \geqq 2$, *a generic point of* W_u *over* $k(u)$ *is also a generic point of* V *over* k.

LEMMA 3. *If* $\xi_j \neq 0$, *then* $u_1, \cdots u_{j-1}, u_{j+1}, \cdots, u_n$ *are algebraically independent over* $k(\xi)$.

Proof. Say

$$i = 1, \operatorname{tr.deg}_{k(u_2,\cdots,u_n)} k(u_1, \cdots, u_n; \xi) + \operatorname{tr.deg}_k k(u_2, \cdots u_n) = n + r -- 1\,.$$

Therefore $\operatorname{tr.deg}_{k(u_2,\cdots,u_n)} k(u_1,\ldots,u_n;\xi) = r$.
Since

$$\frac{u_2\xi_2 + \cdots + u_n\xi_n}{\xi_1} \in k(u_2, \cdots, u_n; \xi_1, \cdots, \xi_n)\,,$$

we have $k(u_1, \cdots, u_n; \xi) = k(u_2, \cdots, u_n; \xi)$. Now

$$\operatorname{tr.deg}_{k(\zeta)} k(u_2, \cdots, u_n; \zeta) + r = r + n - 1\,.$$

Therefore $\operatorname{tr.deg}_{k(\xi)} k(u_2, \cdots, u_n; \xi) = n-1$, i.e., $u_2, \cdots, u_n$ are algebraically independent over $k(\xi)$.

PROPOSITION 4. *Let* $(\xi), \mathfrak{p}_u$ *and* W_u *be as above. Then* $(\mathfrak{p}, H_u)$:

$(X_1, \cdots, X_u)^\rho = \mathfrak{p}_u$ *for sufficiently large integers* ρ, *where* $(X_1, \cdots, X_n) = (X, \cdots, X_n) \cdot k(u)[X]$.

Proof. Let $F(u_1, \cdots, u_n; X) \in \mathfrak{p}_u$ be a polynomial, we may assume $F(u_1, \cdots, u_n; X) \in k[u_1, \cdots, u_n][X]$. If $\xi_1 \neq 0$, $F(u_1, \cdots, u_n; \xi) = 0$ implies that $F(-(u_2\xi_2 + \cdots + u_n\zeta_n/\xi_1), u_2, \cdots, u_n; \xi) = 0$. Hence there exists a nonnegative integer σ such that X_1^σ.

$$F\Big(-\frac{u_2X_2 + \cdots + u_nX_n}{X_1}, u_2, \cdots, u_n; X\Big) \in k(u_2, \cdots, u_n)[X]$$

vanishes at (ξ). By Lemma 3, the prime ideal determined by (ξ) in $k(u_2, \cdots, u_n)[X]$ is $\mathfrak{p}k(u_2, \cdots, u_n)[X]$. Thus

$$X_1^\sigma F\Big(-\frac{u_2X_2 + \cdots + u_nX_u}{X_1}, u_2, \cdots, u_n; X\Big) \in \mathfrak{p} \cdot k(u_1, \cdots, u_n)[X]$$

for sufficiently large σ. But

$$X_1^\sigma F\Big(-\frac{u_2X_2 + \cdots + u_nX_n}{X_1}, u_2, \cdots, u_n; X\Big) - X_1^\sigma F(u_1, \cdots, u_n; X) \equiv 0$$

mod $(u_1X_1 + \cdots + u_nX_n) \cdot k(u)[X]$ for sufficiently large σ. We have $X_1^\sigma F(u_1, \cdots, u_n; X) \in (\mathfrak{p}, H_u) \cdot k(u)[X]$ for sufficiently large σ. The above discussion is symmetric with respect to those $\xi_i \neq 0$. Therefore for any $\xi_i \neq 0$, we have $X_i^{\sigma_i}F(u_1, \cdots, u_n; X) \in (\mathfrak{p}, H_u)$ for sufficiently large integer σ_i and for all $F \in \mathfrak{p}_u$. For any j such that $\xi j = 0$, $X_j \in \mathfrak{p}$. Thus $X_j^{\sigma_j}F \in (\mathfrak{p}, H_u)$ for any positive integer σ_j and for all $F \in \mathfrak{p}_u$. Thus $(\mathfrak{p}, H_u){:}\,(X_1, \cdots, X_n)^\rho \supset \mathfrak{p}_u$ for sufficiently large integer ρ. We now show the other inclusion. Let $g(u_1, \cdots, u_n; X)$ be an element in $(\mathfrak{p}, H_u){:}\,(X_1, \cdots, X_n)^\rho$. Then for any $h(u_1, \cdots, u_n; X) \in (X_1, \cdots, X_n)^\rho$, $h(u; X) \cdot g(u; X) \in (\mathfrak{p}, H_u)$. Therefore, there exists $m_i(u; X), n(u; X) \in k(u)[X]$ such that $h(u; X)g(u; X) = \sum_{i=1}^{s} m_i(u; X) \cdot F_i(X) + n(u; X)H_u$, where $(F_1, \cdots, F_s) \cdot k[X] = \mathfrak{p}$. Thus $h(u; \xi)g(u; \xi) = 0$. If $g(u; \xi) \neq 0$, then $h(u; X) = 0$ at (ξ) for all $h(u; X) \in (X_1, \cdots, X_n)^\rho$, which implies that $(\xi) = (0)$, a contradiction. Thus $g(u; X) = 0$ at (ξ) and therefore $\mathfrak{p} \supset (p, H_u){:}\,(X_1, \cdots, X_n)^\rho$.

COROLLARY. $(\mathfrak{p}, H_u)$ *has only one isolated component.*

Proof. Suppose $\mathfrak{p}_2$ is another isolated component, by Proposition 4, we have $(\mathfrak{p}, H_u){:}\,(X_1, \cdots, X_n)^{\rho'} = \mathfrak{p}_2$, for sufficiently large integer ρ'. Hence we have $\mathfrak{p}_2 = (\mathfrak{p}, H_u) = (X_1, \cdots, X_n)^\rho = \mathfrak{p}_u$.

THEOREM 1. *If* V/k *is of dimension* $r \geqq 2$, *then* $(\mathfrak{p}, H_u) \cdot k(u)[X]$

is either a prime ideal $\mathfrak{p}_u$ *or an intersection of the prime ideal* $\mathfrak{p}_u$ *with a primary ideal of which* $(X_1, \cdots, X_n) \cdot k(u)[X]$ *is its radical.*

Proof. Let $\mathfrak{B} = (\mathfrak{p}, H_u)$ and let $\mathfrak{B} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$ be the irredundant primary representation of $\mathfrak{B}$ with $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$ as the associated prime ideals. By the corollary, there exists only one isolated prime component, say $\mathfrak{q}_i$, and denote $\mathfrak{p}_1$ by $\mathfrak{p}_u$. Let $\mathfrak{m} = (X_1, \cdots, X_n) \cdot k(u)[X]$. Since $\mathfrak{B}: \mathfrak{m}^\rho = \mathfrak{p}_u$ for sufficiently large ρ, we have $(\mathfrak{q}_i: \mathfrak{m}^\rho) = \mathfrak{p}_u$. There are two possibilities (I) no $\mathfrak{p}_i$ contains $\mathfrak{m}^\lambda$ for any nonnegative integer λ, or (II) some of $\mathfrak{p}_i$ contains a power of $\mathfrak{m}$. (I) leads to $\mathfrak{B} = \mathfrak{p}_u$. In case of (II), say $\mathfrak{p}_2$ contains $\mathfrak{m}^\lambda$ for some λ then $\mathfrak{m} = \mathfrak{p}_2$. We may assume that there is no other $\mathfrak{p}_j$ to contain $\mathfrak{m}^\lambda$ for any $0 \leqq \lambda \in \mathbf{Z}$. Thus for $i = 1, 3, 4, \cdots r$, $\mathfrak{q}_i: \mathfrak{m}^\lambda = \mathfrak{q}_i$ for any $0 \leqq \lambda \in \mathbf{Z}$. Since $\mathfrak{q}_2: \mathfrak{m}^\rho = k(u)[X]$ for large ρ, hence $\mathfrak{B}: \mathfrak{m}^\rho = (\mathfrak{q}_i: \mathfrak{m}^\rho) \cap (\mathfrak{q}_2: \mathfrak{m}^\rho) \cap \cdots \cap (\mathfrak{q}_\gamma: \mathfrak{m}^\rho) = \mathfrak{q}_1 \cap \mathfrak{q}_3 \cap \mathfrak{q}_4 \cap \cdots \cap \mathfrak{q}_t$ and thus $\mathfrak{p}_u \cap \mathfrak{q}_2 = (\mathfrak{p}, H_u)$.

COROLLARY 1. *If* V *is normal over* k, *then* $(\mathfrak{p}, H_u) = \mathfrak{p}_u$.

Proof. Passing to the coordinate ring of V, $k(u)[\eta]$, we have that $(u_1\eta_1 + \cdots + u_n\eta_n) \cdot k(u)[\eta]$ is unmixed. Letting $\bar{\mathfrak{p}}_u = \mathfrak{p}_u/\mathfrak{p}$, $\bar{\mathfrak{q}}_2 = \mathfrak{q}_2/\mathfrak{p}$ we have $(\sum u_i\eta_i) = \bar{\mathfrak{p}}_u \cap \bar{\mathfrak{q}}_2$ or $(\sum u_i\eta_i) = \bar{\mathfrak{p}}_u$, by Theorem 1. The unmixedness implies that $(\sum u_i\eta_i) = \bar{\mathfrak{p}}_u$, i.e., $(\mathfrak{p}, H_u) = \mathfrak{p}_u$.

COROLLARY 2. *If* V *is* k*-normal at* (0), *then* $(\mathfrak{p}, H_u) = \mathfrak{p}_u$ *i.e.,* $(\mathfrak{p}, H_u)$ *is a prime ideal.*

Proof. By Theorem 1, $(\mathfrak{p}, H_u) = \mathfrak{p}_u$ or $(\mathfrak{p}, H_u) = \mathfrak{p}_u \cap \mathfrak{q}_2$. Passing to the local ring $k(u)[\eta]_{(0)}$, of V at (0), we have $(\sum u_i\eta_i)k(u)[\eta]_{(0)} = \bar{\mathfrak{p}}_u^e$ or $\bar{\mathfrak{p}}_u^e \cap \bar{\mathfrak{q}}_2^e$ where $\bar{\mathfrak{p}}_u = \mathfrak{p}_u/\mathfrak{p}$, $\bar{\mathfrak{q}}_2 = \mathfrak{q}_2/\mathfrak{p}\bar{\mathfrak{p}}_u^e$ and $\bar{\mathfrak{q}}_2^e$, are extensions of $\bar{\mathfrak{p}}_u$ and $\bar{\mathfrak{q}}_2$ in $k(u)[\eta]_{(0)}$ respectively. Since $k(u)[\eta]_{(0)}$ is integrally closed, the unmixedness of $(\sum u_i\eta_i) \cdot k(u)[\eta]_{(0)}$ implies that $(\sum u_i\eta_i)k(u)[\eta] = \bar{\mathfrak{p}}_u$ and $(\mathfrak{p}, H_u) = \mathfrak{p}_u$.

Recall that V/k is a quasi-absolutely irreducible variety if k is quasi-algebraically closed in the field $k(\xi_1, \cdots, \xi_n)$ of rational functions on V/k; a prime ideal $\mathfrak{A}$ in $k[X_1, \cdots, X_n]$ is quasi-absolutely irreducible if $\bar{k}[X_1, \cdots, X_n]\mathfrak{A}$ is primary, where $\bar{k}$ is the algebraic closure of k. By [11; Th. 10, p. 371], $\mathfrak{p}$ is quasi-absolutely irreducible if and only if V/k is quasi-absolutely irreducible. V/k is absolutely irreducible if k is algebraically closed in $k(\xi)$ and $k(\xi)$ is separable over k. A prime ideal $\mathfrak{A}$ in $k[X_1, \cdots, X_n]$ is absolutely irreducible if $\bar{k}[X_1, \cdots, X_n]$. $\mathfrak{A}$ is a prime ideal. It is well known that the prime ideal $\mathfrak{p}$ of V/k is absolutely irreducible if and only if V/k is.

THEOREM 2. *If* V/k *is quasi-absolutely irreducible of dimension*

$r \geqq 3$ and if k is infinite, then $V \cap H_u/k(u)$ is quasi-absolutely irreducible.

Proof. Let (η) be a generic point of $V \cap H_u$ over

$$k(u) = k(u_1, \cdots, u_n) .$$

By Lemma 2, (η) is a generic point of V over k. Let η_1, η_2, and η_n be algebraically independent over k. By Lemma 3, (η) is a generic point of V over $k(u_2, \cdots, u_n)$. By [11; Lemma 5, p. 368], $k(u_2, \cdots, u_n)$ is quasi-algebraically closed in $k(u_2, \cdots, u_n)(\eta)$. Let $\Sigma = k(u_2, \cdots, u_{n-1})(\eta)$, u_n is algebraically independent over Σ. Viewing $k(u_2, \cdots, u_{n-1})$ as the field k and u_n as the u in [11; corollary, p. 369], we have $\Sigma(u_n) = k(u_2, \cdots, u_{n-1})(u_n)(\eta) = k(u)(\xi)$. Let ξ_1 and ξ_2 in [11; corollary, p. 369] be replaced by $-(u_2\eta_2 + \cdots + u_{n-1}\eta_{n-1})/\eta_1$ and $-\eta_{n/n_1}$ respectively, one sees that $-(u_2\eta_2 + \cdots + u_{n-1}\eta_{n-1})/\eta_1$ and η_n/η_1 are algebraically independent over $k(u_2, \cdots, u_{n-1})$. Hence by the same corollary we have that

$$\begin{aligned} &k(u_2, \cdots, u_{n-1})(u_n)(-(u_2\eta_2 + \cdots + u_{n-1})/\eta_1 - u_n\eta_n/\eta_1) \\ &= k(u_2, \cdots, u_{n-1})(u_n)(u_1) = k(u) \end{aligned}$$

quasi-algebraically closed in $\Sigma(u_n) = k(u)(\eta)$.

LEMMA 4. *Let K be a regular finitely generated extension of an infinite field k with* $\operatorname{tr.deg}_k K \geqq 3$. *Let x, y, z be three elements of K algebraically independent over k, and $z/x \notin K^p k$, where p is the characteristic of k. Then for all but a finite number of constants $c \in k$, K is a regular extension of $k(y + cz/x)$. Moreover, let τ be an indeterminate $K(\tau)$ is regular over $k(\tau)(y + \tau z/x)$.*

Proof. [5; Lemma 3].

THEOREM 3. *If V/k is an absolutely irreducible variety of dimension $r \geqq 3$ defined over an infinite field k, then $V \cap H_u/k(u)$ is an absolutely irreducible variety.*

Proof. $V \cap H_u/k(u)$ is irreducible. Let (ξ) be a generic point of $V \cap H_u$ over $k(u)$. By Lemma 3, (ξ) is a generic point of V over k, hence $\operatorname{tr.deg}_k k(\xi) \geqq 3$ and $k(\xi)$ is a regular extension over k by [12; Proposition 1, p. 69]. Let ξ_1, ξ_2 and ξ_n be three elements in a separable transendental basis of $k(\xi)$ over k. Let $K = k(u_2, \cdots, u_{n-1})(\xi)$, u_n is algebraically independent over K. Viewing $k(u_2, \cdots, u_{n-1})$ as the field k and u_n as the τ in Lemma 4, we have $K(u_n) = k(u)(\xi)$. Let $y = -(u_2\xi_2 + \cdots + u_{n-1}\xi_{n-1})$, $z = \xi_n$ and $x = \xi_1$, then x, y and z are algebraically over $k(u_2, \cdots, u_{n-1})$. By [6, Proposition 1, p. 185] and [6; corollary to Proposition 2, p. 186], $z/x = -\xi_n/\xi_1 \notin K^p k(u_2, \cdots, u_{n-1})$, we have that $K(u_n)$ is a regular extension over

$$k(u_2, \cdots, u_{n-1})(u_n)\left(\frac{y - u_n z}{k}\right) = k(u) .$$

Therefore $k(u)(\xi)$ is a regular extension over $k(u)$, hence $V \cap H_n/k(u)$ is an absolutely irreducible variety.

Let $\{F_1, \cdots, F_s\}$ be a set of generators of $\mathfrak{p}$ in $k[x]$. Let P be a point on V. According to [14], P is k-simple on V if and only if the mixed Jacobian of $\{F_1, \cdots, F_s\}$ is of rank $n - r$ at P. When $k(P)$ is separable over k, P is k-simple on V if and only if the classical Jacobian of $\{F_1, \cdots, F_s\}$ is of rank $n - r$ at P.

Following Theorem 1, we denote $\mathfrak{p}_u$ as the sole isolated component of $(\mathfrak{p}, H_u)$ and $W_u/k(u)$ as its variety in the sequel.

THEOREM 4. *Let V/k be of dimension $r \geqq 2$. Then $P \in W_u$ is $k(u)$-simple if and only if P is k-simple on V.*

Proof. Let $P \in W_u$ be k-simple on V. By Theorem 1, $(\mathfrak{p}, H_u) = \mathfrak{p}_u \cap \mathfrak{A}$, where $\mathfrak{A}$ is the embedded component with $(X_1, \cdots, X_n)$ as radical. Let (η) be a generic point of V over $k(u)$, and let (ξ) be a generic point of W_u over $k(u)$. Let $k(u)[\eta]_p$ and $k(u)[\xi]_p$ be the local rings of V and W_u at P respectively. $k(u)[\eta]_p$ is regular and

$$k(u)[\xi]_p \cong k(u)[\eta]_p/\bar{\mathfrak{p}}_u \cdot k(u)[\eta]_p ,$$

where $\bar{\mathfrak{p}}_u$ is the residue of $\mathfrak{p}_u$ modulo $\mathfrak{p}$. If $P \neq (0)$[1], let $\mathfrak{A}$ be the residue of $\mathfrak{A}$ modulo $\mathfrak{p}$ and let $\mathfrak{m}_p$ be the maximal ideal of $k(u)[\eta]_p$, then $\mathfrak{A}k(u)[\eta] \not\subset \mathfrak{m}_p$. For otherwise $(\eta_1, \cdots, \eta_n)^\rho \subset \mathfrak{m}_p$ for some integer $\rho > 0$, as $(X_1, \cdots, X_n)^\rho \subset \mathfrak{A}$. Thus $P = (0)$, a contradiction. Therefore, when $P \neq (0)$, $(\Sigma u_i\eta_i) \cdot k(u)[\eta]_p = \bar{\mathfrak{p}}_u \cdot k(u)[\eta]_p$, and $k(u)[\xi]_p \cong k(u)[\eta]_p/(\Sigma u_i\eta_i)k(u)[\eta]_p$. By [16; Th. 26, p. 303], to show that $k(u)[\xi]_p$ is regular it is sufficient to show that $\sum u_i\eta_i \notin \mathfrak{m}_p^2$. But this is the case, for if $\sum u_i\eta_i \in \mathfrak{m}_p^2$, taking partial derivatives with respect to u_i for $i = 1, 2, \cdots, n$, we have $\eta_i \in \mathfrak{m}_p$ for $i = 1, 2, \cdots, n$, i.e., $P = (0)$ a contradiction. Therefore $k(u)[\xi]_p$ is regular. If $P = (0)$, then (0) is k-normal on V. By Corollary 2 to Theorem 1, $(\mathfrak{p}, H_u) = \mathfrak{p}_u$. In viewing [14, Th. 7, p. 28], we let $F_1, \cdots, F_s$ be a basis of $\mathfrak{p}$, and let F_i's and X_i's be so arranged that $(\det(\partial F_i/\partial X_j))_{(0)} \neq 0$, where $i, j = 1, 2, \cdots, n - r$, and the subscript (0) means that we replace (X) by (0) after the determinant of the Jacobian is formed, as the rank of

[1] If $P \neq 0$, and if P is k-simple on V, then P remains simple on $W_u/k(u)$ follows also from [**13**; the theorem of Bertini, p. 138].

$$J(F_1, \cdots, F_s, X_1, \cdots, X_n)_{(0)} = n - r .$$

Consider

$$\Delta_j = \det \begin{bmatrix} \partial F_1/\partial X_1 & \cdots & \partial F_1/\partial X_{n-r} & \partial F_1/\partial X_j \\ \vdots & & & \\ \partial F_{n-r}/\partial X_1 & \cdots & \partial F_{n-r}/\partial X_{n-r} & \partial F_{n-r}/\partial X_j \\ u_1 & \cdots & u_{n-r} & u_j \end{bmatrix}_{(0)}$$

where $\eta - r + 1 \leqq j < \eta$. If $\Delta_j = 0$ for some j then $u_1, \cdots, u_{n-r}, u_j$ are algebraically dependent over k. This is a contradiction, hence (0) is k-simple on W_u. Conversely, assume that $P \in W_u$ is $k(u)$-simple on W_u. If $P \neq (0)$, we have $k(u)[\xi]_p \cong k(u)[\eta]_p/(\Sigma u_i\eta_i) \cdot k(u)[\eta]_p$ from the above. If $P = (0)$, then P is $k(u)$-normal on W_u. By Theorem 6 in the following V/k is normal at (0), therefore $(\mathfrak{p}, H_u) = \mathfrak{p}_u$ and $k(u)[\xi]_{(0)} \cong k(u)[\eta]_{(0)}/(\Sigma u_i\eta_i) \cdot k(u)[\eta]_{(0)}$. Therefore $k(u)[\xi]_p \cong k(u)[\eta]_p/(\Sigma u_i\eta_i) \cdot k(u)[\eta]_p$ if P is $k(u)$-simple on W_u. Since $ht((\Sigma u_i\eta_i) \cdot k(u)[\eta]_p) = 1$, it follows from [8; (9; 11), p. 28] that $k(u)[\eta]_p$ is a regular local ring. Hence P is k-simple on V.

By an argument similar to the proof of Lemma 2, we have the following.

COROLLARY. *If V/k is of dimension $r \geqq 3$ and if V/k is locally free of $(r-1)$-dimensional singularities, then $V \cap H_u/k(u)$ is locally free of $(r-2)$-dimensional singularities.*

Note. If $r = 2$, the corollary is clearly false as one sees by taking V to be a cone with vertex at (0).

THEOREM 5. *If V/k is a complete intersection of dimension $\geqq 3$ and if V is k-normal at* (0), *then the generic hyperplane section $V \cap H_u$ is also $k(u)$-normal at* (0).

Proof. $V/k(u)$ is $k(u)$-normal at (0), by Lemma 1. By corollary to Theorem 1, $(\mathfrak{p}, H_u) = \mathfrak{p}_u$ is prime. For any polynomial $F \neq 0$ in $k(u)[X]$, by [7; Th. p. 49] or [16; Th. 26, p. 203], $(\mathfrak{p}_u, F) = (\mathfrak{p}, H_u, F)$ is unmixed. Hence, passing to the quotient modulo $\mathfrak{p}_u$, we have that every nonzero principal ideal in the coordinate ring $k(u)[\xi]$ of $V \cap H_u$ is unmixed. It follows that every nonzero principal ideal in the local ring of $V \cap H_u$ at (0), $k(u)[\xi]_{(0)}$, is also unmixed. Since V/k is k-normal at (0), therefore V/k is locally free of $(r-1)$-dimensional singularities at (0). By the above corollary, $V \cap H_u$ is locally free of $(r-2)$-dimensional singularities at (0). It follows from Proposition 1 that $V \cap H_u$ is $k(u)$-normal at (0).

THEOREM 6. *If $V \cap H_u$ is $k(u)$-normal at* (0), *then V/k is normal at* (0).

Proof. This theorem is really a consequence of [3; Lemma 4, p. 360] ([8; (36.9), p. 134]). Indeed, let (η) be a generic point of V over $k(u)$. Passing to $k(u)[\eta]$, by Theorem 1, we have $(u_1\eta_1 + \cdots + u_n\eta_n) \cdot k(u)[\eta] = \bar{\mathfrak{p}}_u \cap \bar{\mathfrak{q}}$, where $\bar{\mathfrak{p}}_u$ and $\bar{\mathfrak{q}}$ are residues of $\mathfrak{p}_u$ and $\mathfrak{q}$ modulo $\mathfrak{p}$ respectively. It is clear that (1) $(u_1\eta_1 + \cdots + u_n\eta_n) \cdot k(u)[\eta]_{(0)} = \bar{\mathfrak{p}}_u \cdot k(u)[\eta]_{(0)} \cap \bar{\mathfrak{q}}k(u)[\eta]_{(0)}$, $u_1\eta_1 + \cdots + u_n\eta_n$ is in the Jacobson radical of $k(u)[\eta]_{(0)}$, (2) $(u_1\eta_1 + \cdots + u_n\eta_n) \cdot (k(u)[\eta]_{(0)})_{\bar{\mathfrak{p}}_u} = \bar{\mathfrak{p}}_u \cdot (k(u)[\eta]_{(0)})_{\bar{\mathfrak{p}}_u}$, and (3) let (ξ) be a generic point of $V \cap H_u$ over $k(u)$, then

$$\frac{k(u)[\eta]_{(0)}}{\bar{\mathfrak{p}}_u k(u)[\eta]_{(0)}} \cong k(u)[\xi]_{(0)} ,$$

which is integrally closed as $V \cap H_u$ is $k(u)$-normal at (0). Moreover, let $k(u)[\eta]^*_{(0)}$ be the integral closure of $k(u)[\eta]_{(0)}$ in $k(u)(\eta)$, and let $\mathfrak{p}'$ be a minimal prime divisor of $(u_1\eta_1 + \cdots + u_n\eta_n) \cdot k(u)[\eta]^*_{(0)}$. It follows from [2; Th. 2, p. 253] and [2; Th. 3; p. 254] that $ht(\mathfrak{p}' \cap k(u)[\eta]_{(0)}) = ht\mathfrak{p} = 1$. Therefore $\mathfrak{p}' \cap k(u)[\eta]_{(0)} = \bar{\mathfrak{p}}_u$, i.e., every minimal prime divisor of $(u_1\eta_1 + \cdots + u_n\eta_n) \cdot k(u)[\eta)^*_{(0)}$ lies over $\mathfrak{p}_u$. The above verify the conditions of [3; Lemma 4, p. 360], therefore $k(u)[\eta]_{(0)}$ is integrally closed.

3. The local normal problem. Throughout this section let V/k be a variety of dimension $r \geqq 3$, passing through (0) with (ξ) as a generic point over k and let H_u: $u_1X_1 + \cdots + u_nX_n = 0$ be a generic hyperplane through (0). If V/k is normal at (0), is it true that $H_u \cap V$ $k(u)$-normal at (0)? If V/k is a complete intersection then by Theorem 5, the answer to the question is yes. However we shall prove the answer to the question is negative in general.

DEFINITION 4. (a) Let R be a Noetherian ring. Subset $\{a_1, \cdots, a_q\}$ of R is a prime sequence if for each $i = 1, 2, \cdots, q$, a_i is not a zero divisor in the ring $R/(a_1, \cdots, a_{i-1}) \cdot R$.

(b) Let R be a local ring, the number of elements of a maximal prime sequence in R is called the homological co-dimension of R, and is denoted by $\operatorname{cod} h(A)$. If $\operatorname{cod} h(A) = \dim A$, we say that A is a Cohen-Macaulay ring.

For a general commutative ring R and a multiplicative system S which does not contain 0, it is well known [15, p. 219] that $(\mathfrak{A}: \mathfrak{B})^e \subset \mathfrak{A}^e: \mathfrak{B}^e$ and $(\mathfrak{X}: \mathfrak{Y})^c \subset \mathfrak{X}^c: \mathfrak{Y}^c$, where $(*)^e = (*) \cdot R_S$, $(*)^c = f^{-1}(*)$, f is the canonical homomorphism of R into R_S and where $\mathfrak{A}, \mathfrak{B}$ are two ideals in R, and $\mathfrak{X}, \mathfrak{Y}$ are two ideals in R_s.

PROPOSITION 5. *Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{X}$ and $\mathfrak{Y}$ be the same as above. Then* (a) $(\mathfrak{A}: \mathfrak{B})^e = \mathfrak{A}^e: \mathfrak{B}^e$; *if* $\mathfrak{A} \supset \operatorname{Ker} f$ *and* $\mathfrak{B}$ *is finitely generated, also* (b) $(\mathfrak{X}: \mathfrak{Y})^c = \mathfrak{X}^c: \mathfrak{Y}^c$ *if* $\mathfrak{Y}$ *is finitely generated.*

Proof. Let $\mathfrak{B} = (b_1, \cdots, b_t)R$, we have $\mathfrak{B}^e = (f(b_1), \cdots, f(b_t)) \cdot R_s$. Let $x \in \mathfrak{A}^e: \mathfrak{B}^e$. Then $x\mathfrak{B}^e \subset \mathfrak{A}^e$ and $xf(b_i) = f(a_i)/f(s_i)$ for some $a_i \in \mathfrak{A}$ and $s_i \in S$. Therefore $f(\pi_i s_i)xf(b_i) \in f(\mathfrak{A})$. For each $b \in f(\mathfrak{B})$, $b = \sum_j f(r_j)f(b_j)$ for some $r_j \in R$. Now $f(\pi_i s_i)xb = \sum_j f(\pi_i s_i)xf(r_j)f(b_j) \in f(\mathfrak{A})$, which implies that $f(\pi_i s_i)x \in f(\mathfrak{A}): f(\mathfrak{B})$. Hence $x \in (f(\mathfrak{A}): f(\mathfrak{B}))R_s$. Since $\mathfrak{A} \supset \operatorname{Ker} f$, by [15; (15), p. 148], $f(\mathfrak{A}): f(\mathfrak{B}) = f(\mathfrak{A}: \mathfrak{B})$. Therefore $x \in (\mathfrak{A}: \mathfrak{B})^e$ and $\mathfrak{A}^e: \mathfrak{B}^e = (\mathfrak{A}: \mathfrak{B})^e$. The proof of (b) is similar.

LEMMA 5. *$k(u)[\xi]_{(0)}$ is Cohen-Macaulay if and only if $k[\xi]_{(0)}$ is Cohen-Macaulay, where $k[\xi]$ is the coordinate ring of V/k, and u is an indeterminate over $k(\xi)$.*

Proof. If $k[\xi]_{(0)}$ is Cohen-Macaulay, then there exist $\ell_1, \cdots \ell_r$ such that $\{\ell_1, \cdots \ell_r\}$ forms a maximal prime sequence, where $r = \dim V$. Thus $(\ell_1, \cdots, \ell_i)k[\xi]_{(0)}: (\ell_{i+1}) \cdot k[\xi]_{(0)} = (\ell_1, \cdots, \ell_i) \cdot k[\xi]_{(0)}$ for $i = 1, 2, \cdots r$. By [15; (1), p. 227], [15; (15), (21), p. 148] Proposition 5 and [16; (3), p. 221] one has $(\ell_1 \cdots, \ell_i)k(u)[\xi]_{(0)}: (\ell_{i+1})k(u)[\xi]_{(0)} = (\ell_1, \cdots, \ell_i)k(u)[\xi]_{(0)}$, for $i = 1, 2, \cdots, r$. Therefore $\{\ell_1, \cdots, \ell_r\}$ remains as a maximal prime sequence of $k(u)[\xi]_{(0)}$. Thus $k(u)[\xi]_{(0)}$ is Cohen-Macaulay.

Conversely, let $k(u)[\xi]_{(0)}$ be Cohen-Macaulay, let $\{\ell_1(u; \xi), \cdots \ell_r(u; \xi)\}$ be a maximal prime sequence of $k(u)[\xi]_{(0)}$. Then, for $i = 1, 2, \cdots, r$, we have $(\ell_1(u; \xi), \cdots, \ell_i(u; \xi)) \cdot k(u)[\xi]_{(0)}: (\ell_{i+1}(u; \xi)) \cdot k(u)[\xi]_{(0)} = (\ell_1(u; \xi), \cdots, \ell_i(u; \xi)) \cdot k(u)[\xi]_{(0)}$. By [15; (21), p. 148], going back to the polynomial ring $k(u)[x]$, we have $(\ell_1(u; x), \cdots, \ell_i(u; x), \mathfrak{p})k(u)[x]_{(0)}: (\ell_{i+1}(u; x), \mathfrak{p})k(u)[x]_{(0)} = (\ell_1(u; x), \cdots, \ell_i(u; x), \mathfrak{p})k(u)[x]_{(0)}$. In viewing [4; Satz 3, p. 59], one sees that

$$\overline{(\ell_1(u; x), \cdots, \ell_i(u; x), \mathfrak{p})k(u)[x]_{(0)}}: \overline{(\ell_{i+1}(u; x), \mathfrak{p})k(u)[x]_{(0)}} = \overline{(\ell_1(u; x), \cdots, \ell_i(u; x), \mathfrak{p})k(u)[x]_{(0)}}$$

almost always for $i = 1, 2, \cdots r$, where the bar means specialization of u to elements in k. Passing to the local ring of $V/k(u)$ at (0), by [15; (15), p. 148], we have $\overline{\ell_1(u; \xi, \cdots, \ell_i(u; \xi))k(u)[\xi]_{(0)}: \ell_{i+1}(u; \xi)k(u)[\xi]_{(0)}} = \overline{(\ell_1(u; \xi), \cdots, \ell_i(u; \xi))k(u)[\xi]_{(0)}}$ almost always for $i = 1, 2, \cdots r$. Let $a \in k$ be such that the above holds and $\ell_i(a; \xi) \neq 0$, for $i = 1, 2, \cdots, r$, then $(\ell_1(a; \xi), \cdots, \ell_i(a; \xi))k[\xi]_{(0)}: (\ell_{i+1}(a; \xi))k[\xi]_{(0)} = (\ell_1(a; \xi), \cdots, \ell_i(a; \xi)) \cdot k[\xi]_{(0)}$ for $i = 1, 2, \cdots r$. Therefore $\{\ell_1(a, \xi), \cdots, \ell_r(a, \xi)\}$ forms a system of prime sequence of $k[\xi]_{(0)}$. Hence $k[\xi]_{(0)}$ is Cohen-Macaulay.

THEOREM 7. *Let V/k and H_u be the same as the above. It is not* *true in general that if V/k is k-normal at* (0), *then* $V \cap H_u/k(u)$ *is* $k(u)$*-normal at* (0).

Proof. Suppose that if V/k is k-normal at (0), then $V \cap H_u/k(u)$ is $k(u)$-normal at (0). Let (ξ) be a generic point of V over k and let (η) be that of $V \cap H_u$ over $k(u)$. Applying the supposition to $V \cap H_u/k(u)$, we get $(V \cap H_u) \cap H_{u(2)}k(u, u(2))$-normal at (0), where

$$H_{u(2)}: u_{21}X_1 + \cdots + u_{2n}X_n = 0$$

is a generic hyperplane through (0) on

$$V \cap H_u/k(u) \quad \text{and} \quad u(2) = \{u_{21}, \cdots, u_{2n}\}$$

are algebraically independent over $k(u)(\xi, \eta)$. Repeating the supposition and Corollary 2 to Theorem 1 in this way until dimension r of V is cut down to 2, we have then

$$V \cap H_u \cap H_{u(2)} \cap \cdots \cap H_{u(r-2)}k(u, u(2), \cdots, u(\gamma - 2))\text{-normal}$$

at (0), where $u(i) = \{u_{i1}, \cdots, u_{in}\}$, and $\{u_{i1}, \cdots, u_{in}\}$ are indeterminates over $k(u, u(2), \cdots, u(i-1)(\xi, \eta, \eta_2, \eta_{i-1})$ with $\eta_j = (\eta_{j1}, \cdots, \eta_{jn})$ being a generic point of $V \cap H_u \cap H_{u(2)} \cap \cdots \cap H_{u(j)}$ over $k(u, u(2), \cdots u(j))$. Let $U = \{u, u(2), \cdots, u(\gamma - 2)\}$, then $k(U) = k(u, u(2), \cdots, u(\gamma - 2))$. Consider $V/k(U)$, (ξ) is a generic point of V over $k(U)$ · Correspondingly in the coordinate ring $k(U)[\xi]$ of V over $k(U)$ we have then $r - 2$ quantities $\ell_i = u_{i1}\xi_1 + \cdots + u_{in}\xi_n$, $i = 1, 2, \cdots r - 2$, such that $(\ell_1, \cdots, \ell_i)$ is a prime ideal in $k(U)[\xi]_{(0)}$ and $\ell_{i+1} \notin (\ell_1, \cdots, \ell_i)k(U)[\xi]_{(0)}$. Thus $\{\ell_1, \cdots, \ell_{r-2}\}$ is a prime sequence in the local ring $k(U)[\xi]_{(0)}$. Let R be $k(U)[\xi]_{(0)}/(\ell_1, \cdots, \ell_{r-2}) \cdot k(U)[\xi]_{(0)}$, then R is integrally closed of dimension 2. By [16; (3), p. 397], R is Cohen-Macaulay. Let $a, b \in k(U)[\xi]_{(0)}$ be such that their residues modulo $(\ell_1, \cdots, \ell_{r-2}) \cdot k(U)[\xi]_{(0)}$ form a maximal prime sequence of R, then $\{\ell_1, \cdots, \ell_{r-2}, a, b\}$ is a prime sequence of $k(U)[\xi]_{(0)}$. Therefore $\dim k(U)[\xi]_{(0)} = \operatorname{cod} hk(U)[\xi]_{(0)}$ and hence $k(U)[\xi]_{(0)}$, is a Cohen-Macaulay ring. It follows from Lemma 5 that $k[\xi]_{(0)}$ is a Cohen-Macaulay ring. So under the supposition, we conclude that $k[\xi]_{(0)}$ is integrally closed implies that $k[\xi]_{(0)}$ is Cohen-Macaulay. But on the other hand, [1; Proposition, p. 655] and [1; Th. 5, p. 653] yield an example of a local ring of an algebraic variety at a rational point which is a factorial local ring (hence normal), but not a Cohen-Macaulay local ring. Hence the above supposition yields a contradiction.

THEOREM 8. *If V/k is normal at* (0), *and the local ring $k[\xi]_{(0)}$ is a Cohen-Macaulay ring, then $V \cap H_u/k(u)$ is normal at* (0).

Proof. By the corollary to Theorem 4, $(\mathfrak{p}, H_u)$ is free of $(\gamma - 2)$-

dimensional singularities. By Lemma 5, $k(u)[\xi]_{(0)}$ is Cohen-Macaulay. For any nonzero $a(u;\xi)$ in $k(u)[\xi]_{(0)}$ not in the prime ideal

$$(u_1\xi_1 + \cdots + u_n\xi_n) \cdot k(u)[\xi]_{(0)}, \{a(u, \xi), u_1\xi_1 + \cdots + u_n\xi_n\}$$

forms a prime sequence of $k(u)[\xi]_{(0)}$, therefore by [16; Lemma 5, p. 401], $(a(u, \xi), u_1\xi_1 + \cdots + u_n\xi_n) \cdot k(u)[\xi]_{(0)}$, is unmixed. Hence every nonzero principal ideal of $k(u)[\xi]_{(0)}/(u_1\xi_1 + \cdots + u_n\xi_n) \cdot k(u)[\xi]_{(0)}$, is unmixed. It follows from Proposition 1 that $V \cap H_u$ is $k(u)$-normal at (0).

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