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A STUDY OF H-SPACES VIA LEFT TRANSLATIONS

ROBERT A. NOWLAN

H-spaces are examined by studying left translations, actions and a homotopy version of left translations to be called homolations. If (F, m) is an H-space, the map $s: F \to F^F$ given by $s(x) = L_x$, i.e. $s(x)$ is left translation by x, is a homomorphism if and only if m is associative. In general, s is an A_n-map if and only if (F, m) is an A_{n+1} space.

The action $r: F^F \times F \to F$ is given by $r(\varphi, x) = \varphi(x)$. The map s respects the action only of left translations. In general, s respects the action of homolations up to higher-order homotopies. Each homolation generates a family of maps to be called a homolation family. Denoting the set of all homolation families by $H^\infty(F)$, $s: F \to F^F$ factors through $F \to H^\infty(F)$ and this latter map is a homotopy equivalence.

By a multiplication on a space F, we mean a continuous map $m: F \times F \to F$. Let m be a given multiplication on F. For any two points x and y of F, $m(x, y)$ will be denoted by xy and is called the product of x and y. For any point x of F, the assignment $x \to yx$ and $x \to xy$ determine respectively the maps

$$L_y: F \longrightarrow F\,, \qquad R_y: F \longrightarrow F$$

called the left and right translation of F by y.

This paper examines H-spaces with strict units by studying left translations and by the introduction of a homotopy version of left translations to be called homolations. One way to use left translations is as follows. If (F, m) is an H-space, the map

$$s: F \longrightarrow F^F$$

given by $s(x) = L_x$, i.e., $s(x)$ is left translation by x, is a homomorphism if and only if m is associative. Other properties of H-structures on a space F can also be interpreted in terms of properties of the map $s: F \to F^F$.

DEFINITION 1. *A* map $f: F \to Y$ is an H-map of the H-space (F, m) into the H-space (Y, w) if $w \circ (f \times f) \cong f \circ m$. (We always use "$\cong$" to denote "is homotopic to".)

In § II we prove that s is an H-map if and only if m is homotopy associative. In [2], and [3], Stasheff introduces the concepts of A_n-spaces and of A_n-maps, the former generalizes homotopy associativity and the latter generalizes H-maps. We will show that s

is an A_n-map if and only if (F, m) is an A_{n+1}-space.

In § III, H-spaces are studied in terms of actions. The action $r: F^F \times F \to F$ is given by $r(\varphi, x) = \varphi(x)$. The cross-section $s: F \to F^F$ respects the action only of left translations. The question arises: of which maps in F^F does s respect the action up to homotopy? This leads to the introduction of T-maps, that is maps $f: F \to F$ such that $f \circ m \cong m \circ (f \times 1)$. Such maps resemble left translations. Demanding a closer resemblance leads to the introduction of homolations which are maps f satisfying $f \circ m \cong m \circ (f \times 1)$ up to higher order homotopies.

If (F, m) is an associative H-space, a map $w: M \times F \to F$ is a transitive action if $w \circ (1 \times m) = m \circ (w \times 1)$. The action $r: s(F) \times F \to F$, where $s(F)$ is the set of all left translations is an example of a transitive action. A homotopy version of a transitive action is given as follows.

DEFINITION 2. Let (F, m) be an associative H-space. A map $w: M \times F \to F$ is a T-action if $w \circ (1 \times m) \cong m \circ (w \times 1)$.

If $T(F)$ is the maximal subset of F^F such that

$$r: T(F) \times F \longrightarrow F$$

is a T-action, then $T(F)$ consists of T-maps. Generalizing the notions of T-actions leads to the concept of T_n-actions and T_∞-actions, that is actions $w: M \times F \to F$ satisfying $w \circ (1 \times m) \cong m \circ (w \times 1)$ up to higher order homotopies. It is then shown that *a* T_∞-action of the set of homolations on F can be given such that $s: F \to F^F$ is a T_∞-map of actions, i.e., s respects the actions of homolations up to higher order homotopies.

Each homolation generates a family of maps to be called a homolation family. Denote by $H^\infty(F)$ the set of all homolation families. In § IV, it is proven that $s: F \to F^F$ factors through $F \to F^\infty(F)$ and that this latter map is a homotopy equivalence.

Throughout this paper, we will be working in the category of k-spaces (i.e., compactly generated spaces) as developed in [5]. The reason for this is to allow unlimited use of the "exponential law." (c.f. Theorem 5, 6 in [5]).

Some of the work included in this paper is contained in my doctoral thesis [1] completed at the University of Notre Dame. Other parts of it were suggested by Professor James D. Stasheff. I deeply appreciate his suggestions and many valuable comments during the writing of this paper.

II. A_n-maps and A_n-spaces We first study H-spaces in relation to cross-sections to evaluation maps. Let F be any space. Let the evaluation map $v: F^F \to F$ be defined by $v(\varphi) = \varphi(e)$, where φ is in F^F for some e in F. The map v has a cross-section $s: F \to F^F$ if and only if F admits a multiplication with right unit e. Given such a cross-section s we can define

$$m(x, y) = s(x)(y) \qquad \text{for } x, y \text{ in } F$$

so that m has e as a right unit. Since

$$s(x)(e) = v(s(x)) = x ,$$

this multiplication has a two-sided unit if s is a base point preserving map, that is $s(e) =$ identity. We will make this assumption throughout this paper.

If F has a multiplication m with e as right unit, we define $s(x) = L_x$, where L_x is left translation by x. It follows that s is a homomorphism if and only if m is associative.

Thus certain properties of H-structures on a space F can be interpreted in terms of properties of the map $s: F \to F^F$. As an example we have the following proposition.

PROPOSITION 1. *The map $s: F \to F^F$ is an H-map if and only if m is homotopy associative.*

Proof. If s is an H-map of (F, m) into (F^F, c) (where c is composition of maps), there exists a homotopy

$$G: I \times F^2 \longrightarrow F^F$$

such that

$$G(0, x, y) = c \circ (s \times s)(x, y) = L_x \circ L_y$$

and

$$G(1, x, y) = s \circ m(x, y) = L_{xy} .$$

Then m can be shown to be homotopy associative by defining a homotopy

$$G': I \times F^3 \longrightarrow F$$

by

$$(1) \qquad G'(t, x, y, z) = G(t, x, y)(z)$$

Conversely, if m is homotopy associative, a homotopy G' exists such that

$$G'(0, x, y, z) = x(yz)$$

and

$$G'(1, x, y, z) = (xy)z$$

and the homotopy G can be defined as in (1).

In seeking to generalize this proposition, we first need generalizations of the concepts of homotopy associativity and of H-map. In [2] and [3], Stasheff introduces the concepts of A_n-spaces and of A_n-maps; the former generalizes homotopy associativity and the latter generalizes H-maps. A space which is an A_n-space for all n is said to be an A_∞-space. Any associative H-space is an A_∞-space. A_∞-spaces are homotopy equivalent to associative H-spaces.

DEFINITION 3. An A_n-structure on a space X consists of an n-tuple of maps

$$\begin{array}{ccccccc} X = E_1 & \subset & E_2 & \subset & \cdots & \subset & E_n \\ \downarrow p_1 & & \downarrow p_2 & & & & \downarrow p_n \\ * = B_1 & \subset & B_2 & \subset & \cdots & \subset & B_n \end{array}$$

such that $p_{i_*}: \pi_q(E_i, X) \to \pi_q(B_i)$ is an isomorphism for all q, together with a contracting homotopy $h: CE_{n-1} \to E_n$ of the cone of E_{n-1}, CE_{n-1} such that $h(CE_{i-1}) \subset E_i$. Such an A_n-structure will be denoted by $(p_1, \cdots, p_n)$. If there exists an infinite collection $p_1, p_2, \cdots$ such that for each n, $(p_1, \cdots, p_n)$ is an A_n-structure, then we call $(p_1, p_2, \cdots)$ an A_∞-structure.

Theorem 5 of [2] asserts that an A_n-structure on a space X is equivalent to an "A_n-form", that is a family of maps $\{M_2, \cdots, M_n\}$ where each

$$M_i: I^{i-2} \times X^i \longrightarrow X$$

is suitably defined on the boundary I^{i-1} in terms of M_j for $j < i$.

DEFINITION 4. A space X together with an A_n-form will be called an A_n-space.

In this paper, we are more interested in A_n-forms than A_n-structures, so we introduce the former in some detail. It is first necessary to become acquainted with a special cell-complex K_i which is homeomorphic to I^{i-2} for $i \geqq 2$. The standard cells K_i are objects similar to standard simplices Δ^i and standard cubes I^i, having faces and degeneracies. The difference between the K_i and the simplices and the cubes is that:

(1) The index i does not refer to the dimension of the cell but rather to the number of factors X with which K_i is to be associated.

(2) K_i has degeneracy operators $s_1, \cdots, s_i$ defined on it.
and

(3) K_i has $(i(i-1)/2) - 1$ faces.

The following description of the indexing of the faces of K_i is due to Stasheff. Consider a word with i letters, and all meaningful ways of inserting one set of parentheses. To each such insertion except for $(x_1, \cdots, x_i)$, there corresponds a cell of L_i, the boundary of K_i. If the parentheses enclose x_k through x_{k+s-1}, we regard this cell as the homeomorphic image of $K_r \times K_s$ $(r + s = i + 1)$ under a map which we denote by $\partial_k(r, s)$. Two such cells intersect only on their boundaries and the "edges" so formed correspond to inserting two sets of parentheses in the word. We obtain K_i by induction, starting with $K_2 = *$ (a point), supposing K_2 through K_{i-1} have been constructed. Then construct L_i by fitting together copies of $K_r \times K_s$ subject to certain conditions given in § 2 of [2], that is the fitting together of copies of $K_r \times K_s$ as dictated by the above description of the indexing. Finally, take K_i to be the cone on L_i.

The following is part of Theorem 5 of [2].

THEOREM 2. *A space X admits an A_n-structure if and only if there exist maps $M_i\colon K_i \times X^i \to X$ for $2 \leqq i \leqq n$ such that*

(1) *$M_2(*, e, x) = M_2(*, x, e) = x$ for x in X, $* = K_2$ and*

(2) *For $\rho \in K_r$, $\sigma \in K_s$, $r + s = i + 1$, we have*

$$\begin{aligned} &M_i(\partial_k(r, s)(\rho, \sigma), x_1, \cdots, x_i) \\ &= M_r(\rho, x, \cdots, x_{k-1}, M_s(\sigma, x_k, \cdots, x_{k+s-1}), \cdots, x_i)\ . \end{aligned}$$

We note that an A_2-space is just an H-space. In the case $i = 3$, K_3 is homeomorphic to I and (2) asserts that M_3 is a homotopy between $M_2 \circ (M_2 \times 1)$ and $M_2 \circ (1 \times M_2)$, to be imprecise between $(xy)z$ and $x(yz)$. Thus M_3 is an associating homotopy and M_2 is a homotopy associative action.

In the case $i = 4$, we consider the five ways of associating a product of four factors. If the multiplication M_2 is a homotopy associative multiplication, the five products are then related by the following string of homotopies:

$$x(y(zw)) \cong x((yz)w) \cong (x(yz))w \cong ((xy)z)w \cong (xy)(zw) \cong x(y(zw))\ .$$

Thus we have defined a map of $S^1 \times X^4 \to X$ and the map M_4 can be regarded as an extension of the map to $I^2 \times X^4$.

If X is an associative H-space, it admits A_∞-forms; it is only necessary to define

$$M_i(\tau, x_1, \cdots, x_i) = x_1 x_2 \cdots x_i \text{ for } \tau \text{ in } K_i \text{ and } 1 \leqq i .$$

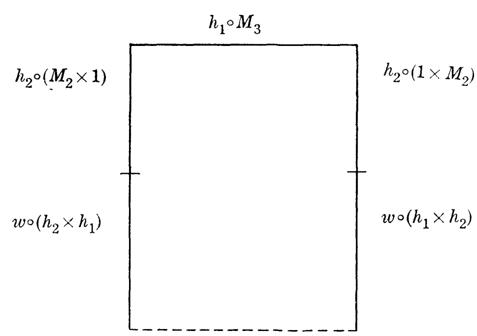
This will be called a trivial A_∞-form. If X is an A_∞-space then there is an associative H-space Y of the homotopy type of X.

DEFINITION 5. Let $(X, \{M_i\})$ be an A_n-space and (Y, w) be an associative H-space. A map $f: X \to Y$ is an A_n-map if there exists maps $h_i: K_{i+1} \times X^i \to Y$, $1 \leqq i \leqq n$, called sputnik homotopies, such that $h_1 = f$ and for ρ in K_r, σ in $K_s (r + s = i + 1)$, we have

$$\begin{aligned} &h_i(\partial_k(r, s)(\rho, \sigma), x_1, \cdots, x_i) \\ &= h_{r-1}(\rho, x_1, \cdots, x_{k+1}, M_s(\sigma, x_k, \cdots, x_{k+s-1}), \cdots, x_i) \text{ if } k \neq r \\ &= h_{r-1}(\rho, x_1, \cdots, x_{r-1}) h_{s-1}(\sigma, x_r, \cdots, x_i) \text{ if } k = r . \end{aligned}$$

Note that when $n = 2$, f is just an H-map, as h_2 is a homotopy between $f \circ M_2$ and $w \circ (f \times f)$. In the case $n = 3$, since K_4 is homeomorphic to I^2, we have a map of $S^1 \times X^3 \to Y$ and $h_3: K_4 \times X^3 \to Y$ can be thought of as an extension of this map to $I^2 \times X^4$.

Consider the following cross-section of $I^2 \times X^4$ showing a typical I^2. Assign to the "faces" of I^2 the homotopies $h_2 \circ (M_2 \times 1)$, $w \circ (h_2 \times h_1)$, $w \circ (h_1 \times h_2)$, $h_2 \circ (1 \times M_2)$ and $h_1 \circ M_3$ as indicated



The broken line represents a point. The map h_3 then appropriately fills in the figure.

A map which is an A_n-map for all n will be called an A_∞-map.

We are now in a position to prove the following generalization of proposition 1.

THEOREM 3. (A) *Let* $(F, \{M_i\})$ *be an* A_n*-space; then* $s: F \to F^F$ *is an* A_{n-1} *map.*

(B) *s can be shown to be an* A_n*-map if and only if* $(F, \{M_i\})$ *can* *be given the structure of an* A_{n+1} *space.*

Proof. (A) *Given that* $(F, \{M_i\})$ *is an* A_n*-space, all that is necessary to show that* s *is an* A_{n-1} *map is to define* $h_1 = s$ *and* $h_i\colon K_{i+1} \times F^i \to F^F$ $1 \leqq i \leqq n-1$ by

$$h_i(\partial_k(r, t)(\rho, \sigma), x_1, \cdots, x_i)(y) = M_{i+1}(\partial_k(r, t)(\rho, \sigma), x_1, \cdots, x_i, y) .$$

(B) *It is clear that* $(F, \{M_i\})$ *can be extended to an* A_{n+1}*-space (that is there exists a map* $M_{n+1}\colon K_{n+1} \times F^{n+1} \to F$*) if and only if there exists a map* $h_n\colon K_{n+1} \times F^n \to F^F$ *given by*

$$h_n(\partial_k(r, t)(\rho, \sigma), x_1, \cdots, x_n)(y) = M_{n+1}(\partial_k(r, t)(\rho, \sigma), x_1, \cdots, x_n, y) .$$

COROLLARY 4. *An* A_∞*-form on* F *is equivalent to the existence of sputnik homotopies* $h_i\colon K_{i+1} \times F^i \to F^F$ *for all* i *making* s *an* A_∞*-map.*

III. T_n-maps and Homolations. We assume throughout this section that (F, m) is an associative H-space with a strict unit. In that case, the map

$$s\colon F \longrightarrow F^F$$

given by

$$s(f)(y) = m(f, y)$$

is a homomorphism.

We now study left translations via actions. The space F^F acts on F by

$$r\colon F^F \times F \longrightarrow F$$
$$r(\varphi, f) = \varphi(f) .$$

The cross-section s respects the action only of left translations, for consider the diagram:

(1)
$$\begin{array}{ccc} F^F \times F & \xrightarrow{\quad 1 \times s \quad} & F^F \times F^F \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle c} \\ F & \xrightarrow{\quad s \quad} & F^F . \end{array}$$

Suppose

$$s(\varphi(f)) = \varphi \circ s(f) .$$

Since s is left translation, we have $\varphi(fy) = \varphi(f)y$, that is the following diagram is commutative.

$$(2)\qquad \begin{array}{ccc} F \times F & \xrightarrow{\ m\ } & F \\ {\scriptstyle \varphi \times 1}\downarrow & & \downarrow{\scriptstyle \varphi} \\ F \times F & \xrightarrow{\ m\ } & F\,. \end{array}$$

In particular,

$$\varphi(y) = \varphi(ey) = \varphi(e)y$$

and φ is left translation by $\varphi(e)$. So diagram (1) commutes only on $s(F) \times F \subset F^F \times F$ where $s(F)$ is the set of left translations. Thus s is a map of spaces on which $s(F)$ acts.

The result tells us something about the action

$$r\colon s(F) \times F \longrightarrow F$$

namely, it is transitive.

Note that the following diagram is commutative

$$(3)\qquad \begin{array}{ccc} s(F) \times F \times F & \xrightarrow{\ 1 \times m\ } & s(F) \times F \\ {\scriptstyle r \times 1}\downarrow & & \downarrow{\scriptstyle r} \\ F \times F & \xrightarrow{\ m\ } & F \end{array}\,.$$

Let us consider the following question: what is the nature of the action r when diagram (1) is only required to be homotopy commutative. Denote by $T_2(F)$ the maximal subset of maps φ in F^F such that

$$s[\varphi(f)] \cong \varphi_0 s(f)$$

in the sense that there exists a homotopy

$$\theta_2\colon I \times T_2(F) \times F \longrightarrow F^F$$

such that

$$\theta_2(0, \varphi, f) = \varphi \circ s(f)$$

and

$$\theta_2(1, \varphi, f) = s[\varphi(f)]\,.$$

In this case, it follows that for each φ in $T_2(F)$ there exists a homotopy

$$\varphi_2\colon I \times F^2 \longrightarrow F$$

depending continuously on φ such that

$$\varphi_2(0, f, y) = \varphi(fy)$$

and

$$\varphi_2(1, f, y) = \varphi(fy)y \text{ .}$$

DEFINITION 6. Let (F, m) be an associative H-space. A map $f: F \rightarrow F$ is a T-map if there exists a homotopy $I \times F^2 \rightarrow F$ such that $f \circ m \cong m \circ (f \times 1)$.

Thus we see that the maps in $T_2(F)$ are T-maps. The homotopy is given by

$$\varphi_2(t, f, y) = \theta_2(t, \varphi, f)(y) \text{ .}$$

In particular, we note that for each φ in $T_2(F)$

$$\varphi(y) = \varphi(ey) \cong \varphi(e)y$$

indicating that up to homotopy φ acts like left translation by $\varphi(e)$. Thus the maps in $T_2(F)$ in this sense resemble left translations. We will investigate this resemblance further.

Our results show that the action

$$r: T_2(F) \times F \longrightarrow F^F$$

is a T-action in the sense that there exists a homotopy

$$\lambda_2: I \times T_2(F) \times F^2 \longrightarrow F$$

such that

$$\lambda_2: r \circ (1 \times m) \cong m \circ (r \times 1) \text{ .}$$

In fact, we can take λ_2 to be adjoint to θ_2:

$$\lambda_2(t, \varphi, f, y) = \theta_2(t, \varphi, f)(y) \text{ .}$$

If φ is a true left translation, it follows that

$$\varphi(xyz) = \varphi(xy)z = \varphi(x)yz \qquad \text{for } x, y, z \text{ in } F$$

however for a map φ in $T_2(F)$, the most we can claim using a rather loose notation is that:

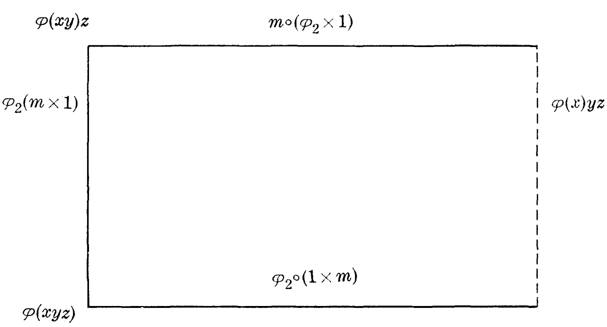
$$\varphi(xyz) \cong \varphi(xy)z \cong \varphi(x)yz \cong \varphi(xyz) \text{ .}$$

This string of homotopies defines a map

$$\dot{I}^2 \times F \longrightarrow F$$

where $\dot{I}^2$ is the boundary of I^2.

This can be illustrated in the following diagram, representing $\dot{I}^2 \times F^3$ showing only $\dot{I}^2$ with "faces" labeled by the homotopies connecting the maps given above. Note that the edge of $\dot{I}^2$ represented by the broken line is just a point. (This is because F is an associative H-space. If F were only homotopy associative, this face would be labeled by the associating homotopy applied to $\varphi(x)$, y, z. The following discussion could be carried out for A_n-spaces but the details are bad enough in the associative case, which is the case of interest for applications [1].)



The problem of making a map φ in $T_2(F)$ more closely "resemble" a left translation, requires that we be able to extend the map

$$\dot{I}^2 \times F^3 \longrightarrow F$$

to a map

$$I^2 \times F^3 \longrightarrow F\,.$$

Thus we will need higher homotopy conditions on the maps φ in $T_2(F)$. Suppose for the moment that there exists a map

$$\varphi_3\colon I^2 \times F^3 \longrightarrow F$$

such that

$$\varphi_3(0, t_2, x, y, z) = \varphi_2(t_2, xy, z)$$
$$\varphi_3(t_1, 0, x, y, z) = \varphi_2(t_1, x, yz)$$
$$\varphi_3(1, t_2, x, y, z) = \varphi(x)yz$$

and

$$\varphi_3(t_1, 1, x, y, z) = \varphi_2(t_1, x, y)\cdot z\,.$$

Let $T_3(F)$ denote the maximal subset of $T_2(F)$ such that for each φ in $T_2(F)$, there exists φ_2 and φ_3 depending continuously on φ and φ_2 subject to the conditions already mentioned. In this case, the action $r\colon T_3(F) \times F \to F$ is such that there exist maps

$$\lambda_2: I \times T_3(F) \times F^2 \longrightarrow F$$

such that

$$\lambda_2: r(1 \times m) \cong m(r \times 1)$$

and

$$\lambda_3: I^2 \times T_3(F) \times F^3 \longrightarrow F$$

such that

$$\begin{aligned} \lambda_3(0, t_2, \varphi, x, y, z) &= \lambda_2(t_2, \varphi, xy, z) \\ \lambda_3(t_1, 0, \varphi, x, y, z) &= \lambda_2(t_1, \varphi, x, yz) \\ \lambda_3(1, t_2, \varphi, x, y, z) &= r(\varphi, x)\cdot yz \end{aligned}$$

and

$$\lambda_3(t_1, 1, \varphi, x, y, z) = \lambda_2(t_1, \varphi, x, y)\cdot z .$$

This latter map is given by

$$\lambda_3(t_1, t_2, \varphi, x, y, z) = \varphi_3(t_1, t_2, x, y, z) .$$

On the other hand, there exist maps

$$\theta_2: I \times T_3(F) \times F \longrightarrow F^F$$

such that

$$\theta_2: \varphi\circ s(f) \cong s[\varphi(f)]$$

and

$$\theta_3: I^2 \times T_3(F) \times F^2 \longrightarrow F^F$$

such that

$$\theta_3: (t_1, t_2, \varphi, x, y)(z) = \lambda_3(t_1, t_2, \varphi, x, y, z) .$$

Parallel to every demand that a map $\varphi: F \to F$ more closely resemble a left translation by satisfying higher homotopy conditions will be the requirement of higher homotopy conditions on the action r and similar higher homotopy conditions on the map s.

DEFINITION 7. Let (X, m) be an associative H-space. A map $\varphi: X \to X$ is a T_n-map of X into itself if there exists a family of maps

$$\varphi_i: I^{i-1} \times X^i \longrightarrow X \qquad 1 \leqq i \leqq n$$

such that $\varphi_1 = \varphi$ and

$$\begin{aligned} &\varphi_i(t_1, \cdots, t_{i-1}, x_1, \cdots, x_i) \\ &= \varphi_{i-1}(t_1, \cdots, \hat{t}_k, \cdots, t_{i-1}, x_1, \cdots, x_k x_{k+1}, \cdots, x_i) \qquad \text{if } t_k = 0 \\ &= \varphi_k(t_1, \cdots, t_{k-1}, x_1, \cdots, x_k)\cdot(x_{k+1}, x_{k+2} \cdots x_i) \qquad \text{if } t_k = 1 . \end{aligned}$$

In case φ_i exists for all i, we call φ a homolation, that is, a homotopy translation. Denote the set of all homolations by $T_\infty(F)$.

DEFINITION 8. Let (F, m) be an associative H-space. A homolation family on F is a collection of maps $\{\varphi_i: I^{i-1} \times F^i \to F, \forall_i \geqq 1\}$ where φ_1 is a homolation and $\varphi_1: F \to F$ is a homotopy equivalence. We will denote by $H^\infty(F)$, the set of all homolation families. $H^\infty(F)$ is a subspace of $C(F; F) \times C(I \times F^2; F) \times \cdots$ where $C(I^j \times F^{j+1}; F)$ is the set of all continuous maps $f: I^j \times F^{j+1} \to F$ (with the k-topology derived from the compact-open topology).

DEFINITION 9. Let (X, m) be an associative H-space. A map

$$w: M \times X \longrightarrow X$$

of M on X is said to be a T_n-action if there exist maps

$$w_i: I^{i-1} \times M \times X^i \longrightarrow X \qquad 1 \leqq i \leqq n$$

such that $w_1 = w$ and

$$\begin{aligned} &w_i(t_1, \cdots, t_{i-1}, g, x_1, \cdots, x_i) \\ &= w_{i-1}(t_1, \cdots, \hat{t}_k, \cdots, t_{i-1}, g, \cdots x_k x_{k+1}, \cdots, x_i) \qquad \text{if } t_k = 0 \\ &= w_k(t_1, \cdots, t_{k-1}, g, x_1, \cdots, x_k)\cdot(x_{k+1}x_{k+2} \cdots x_i) \qquad \text{if } t_k = 1 . \end{aligned}$$

If a map $w: M \times X \to X$ is a T_n-action for all n, then w is said to be a T_∞-action.

THEOREM 5. *Let $T_n(F)$ denote the maximal subset of F^F such that there exist maps $\lambda_i: I^{i-1} \times T_n(F) \times F^i \to F$ for $1 \leqq i \leqq n$ making $r: T_n(F) \times F \to F$ a T_n-action; then $T_n(F)$ consists of T_n-maps.*

Proof. We may define the maps

$$\varphi_i: I^{i-1} \times F^i \longrightarrow F \qquad 1 \leqq i \leqq n$$

by

$$\varphi_i(t_1, \cdots, t_{i-1}, f_1, \cdots, f_i) = \lambda_i(t_1, \cdots, t_{i-1}, \varphi, f_1, \cdots, f_i) .$$

DEFINITION 10. Let (X, m) and (M, v) be associative H-spaces and $w: M \times X \to X$ be a T_n-action. A homomorphism $f: X \to M$ is said to be a T_n-map of actions if there exist maps

$$\theta_i: I^{i-1} \times M \times X^{i-1} \longrightarrow M$$

such that $\theta_1 = 1_M$ and

$$\begin{aligned} &\theta_i(t_1, \cdots, t_{i-1}, g, x_1, \cdots, x_{i-1}) \\ &= \theta_{i-1}(t_1, \cdots, \hat{t}_k, \cdots, t_{i-1}, g, \cdots, x_k x_{k+1}, \cdots, x_{i-1}) \\ &\hspace{16em} \text{if } t_k = 0,\ k \neq i-1 \\ &= v[\theta_{i-1}(t_1, \cdots, t_{i-2}, g, x_1, \cdots, x_{i-2}), f(x_{i-1})] \qquad \text{if } t_{i-1} = 0 \\ &= f[m(w_k(t_1, \cdots, t_{k-1}, g, x_1, \cdots, x_k), x_{k+1}x_{k+2} \cdots x_i)] \quad \text{if } t_k = 1\,. \end{aligned}$$

If θ_i exists for all i, then f is said to be a T_∞-map of actions.

COROLLARY 6. *The map $r: T_\infty(F) \times F$ is a T_∞-action and s is then a T_∞-map of actions.*

Proof. Define $\lambda_i: I^{i-1} \times T_\infty(F) \times F^i \to F$ by

$$\lambda_i(t_1, \cdots, t_{i-1}, \varphi, f_1, \cdots, f_i) = \varphi_i(t_1, \cdots, t_{i-1}, f_1, \cdots, f_i)$$

and

$$\theta_i: I^{i-1} \times T^\infty(F) \times F^{i-1} \longrightarrow F^F$$

by

$$\theta_i(t_1, \cdots, t_{i-1}, \varphi, f_1, \cdots f_{i-1})(f_i) = \lambda_i(t_1, \cdots, t_{i-1}, \varphi, f_1, \cdots, f_i)\ .$$

IV. The homotopy equivalence of F and $H^\infty(F)$. As we have seen, we can identify an associative H-space with the set of left translations of that space. We note that this identification of F in F^F as left translation is not homotopy invariant: $\varphi(fx) = \varphi(f)x$ is not a homotopy statement. Our definition of homolation is homotopy invariant and it characterizes $F \to F^F$ from a homotopy point of view.

We are now in a position to prove the following theorem. Recall that $H^\infty(F)$ is the set of all homolation families.

THEOREM 7. *If (F, m) is a connected associative H-space, the map $s: F \to F^F$ factors through $H^\infty(F)$, and the factor $F \to H^\infty(F)$ is a homotopy equivalence.*

Proof. Define a map

$$\tau: F \longrightarrow H^\infty(F)$$

as follows:

$$\tau(f) = \Phi_f = \{\varphi_1^f, \varphi_2^f, \cdots\}$$

where

$$\varphi_1^f\colon F \longrightarrow F$$

is given by

$$\varphi_1^f(g) = fg$$

that is left translation of F. φ_1^f is a homotopy equivalence since F is connected (see [4]).

The remaining maps are given by

$$\varphi_k^f(t_1, \cdots, t_{k-1}, f_1, \cdots, f_k) = ff_1 \cdots f_k \qquad \text{for all } k .$$

The map τ is continuous, since the composition of maps

$$F \xrightarrow{\tau} C(F; F) \times C(I \times F^2; F) \cdots \xrightarrow{p^{(k)}} C(I^{k-1} \times F^k; F)$$

is continuous for each k and $p^{(k)}$ is projection onto the corresponding factor.

On the other hand, define the map

$$\mu\colon H^\infty(F) \longrightarrow F$$

by

$$\mu(\Gamma) = \gamma_1(e)$$

where $\Gamma = \{\gamma_1, \gamma_2, \cdots\}$ is in $H^\infty(F)$ and e is the unit of F.

The map μ is continuous, since it is the composition of maps

$$H^\infty(F) \xrightarrow{p_1} H^\infty(F)_1 = T_\infty(F) \xrightarrow{w_e} F$$

where p_1 is projection of $H^\infty(F)$ on that part of $H^\infty(F)$ contained in F^F, namely the set of homolations, here denoted by $H^\infty(F)_1$, and the map w_e is the evaluation map at e (continuous in the k-topology).

Note that $\mu(\tau(f)) = \mu(\Phi_f) = \varphi_1^f(e) = fe = f$ so that $\mu\circ\tau = 1_F$.

On the other hand

$$\tau\circ\mu(\Gamma) = \tau(\gamma_1(e)) = \Phi_{\gamma_1(e)} = \{\varphi_1^{\gamma_1(e)}, \varphi_2^{\gamma_1(e)} \cdots\} .$$

We claim that $\tau\circ\mu \cong 1_{H^\infty(F)}$, that is there exists a map

$$H_t\colon H^\infty(F) \longrightarrow H^\infty(F)$$

such that $H_0 = 1_{H^\infty(F)}$ and $H_1 = \tau\circ\mu$.

To see this, let $H^\infty(F)_k$ be the subspace of $H^\infty(F)$ which is contained in $C(I^{k-1} \times F^k; F)$. The map $H_t = \{H_t^1, H_t^2, \cdots\}$ will consist of homotopies

$$\{H_t^k\}\colon H^\infty(F) \longrightarrow H^\infty(F)_k \qquad \text{for each } k$$

such that $H_0^k = 1_{H^\infty(F)_k}$ and $H_1^k = \tau\circ\mu \mid H^\infty(F)_k$ and the H_t^k are compatible.

Define $H_t^k\colon H^\infty(F) \to H^\infty(F)_k$ as follows:

$$H_t^k(\Gamma)(t_1, \cdots t_{k-1}, f_1, \cdots, f_k) = \gamma_{k+1}(t, t_1, \cdots, t_{k-1}, e, f_1, \cdots, f_k)\ .$$

The map is continuous as each γ_{k+1} in Γ is continuous and $\Gamma \to \gamma_{k+1}$ is continuous being projection.

Note if $t_j = 0$

$$\begin{aligned} &H_t^k(\Gamma)(t_1, \cdots, t_{k-1}, f_1, \cdots, f_k) \\ &= \gamma_k(t, t_1, \cdots \hat{t}_j, \cdots t_{k-1}, e, f_1, \cdots f_jf_{j+1}, \cdots, f_k) \\ &= H_t^{k-1}(\Gamma)(t_1, \cdots, \hat{t}_j, \cdots, t_{k-1}, f_1, \cdots, f_jf_{j+1}, \cdots f_k) \end{aligned}$$

while if $t_j = 1$

$$\begin{aligned} &H_t^k(\Gamma)(t_1, \cdots, t_{k-1}, f_1, \cdots, f_k) \\ &= \gamma_{j+1}(t, t_1, \cdots, t_{j-1}, e, f_1, \cdots f_j)(f_{j+1}, \cdots, f_k) \\ &= H_t^j(\Gamma)(t_1, \cdots t_{j-1}, f_1, \cdots, f_j)(f_{j+1}, \cdots, f_k)\ . \end{aligned}$$

Thus $\{H_t^k\}$ is in $H^\infty(F)$. Further

$$\begin{aligned} H_0^k(\Gamma)(t_1, \cdots, t_{k-1}, f_1, \cdots, f_k) &= \gamma_k(t_1, \cdots, t_{k-1}, ef_1, \cdots f_k) \\ &= \gamma_k(t_1, \cdots, t_{k-1}, f_1, \cdots, f_k)\ . \end{aligned}$$

Thus $H_0^k = 1_{H^\infty(F)_k}$ $\{H_0^k(\Gamma)\} = \Gamma$ and

$$\begin{aligned} &H_1^k(\Gamma)(t_1, \cdots, t_{k-1}, f_1, \cdots, f_k) \\ &= \gamma_1(e)f_1 \cdots f_k \\ &= \varphi_1^{\gamma_1(e)}(t_1, \cdots, t_{k-1}, f_1, \cdots, f_k) \\ &= \tau\circ\mu(\Gamma)(t_1, \cdots, t_{k-1}, f_1, \cdots, f_k)\ . \end{aligned}$$

Thus $H_1^k = \tau\circ\mu \mid H^\infty(F)_k$, $\{H_1^k(\Gamma)\} = \tau\circ\mu(\Gamma)$. This completes the proof that F and $H^\infty(F)$ are homotopy equivalent.

Now $H^\infty(F)$ is itself an H-space; we can define composition of families as well as just maps $F \to F$ (see [1]). The map $F \to H^\infty(F)$ is an A_∞-map and hence induces $B_F \to B_{H^\infty(F)}$ which is again a homotopy equivalence if F is a CW-complex.

In my thesis [1], I show that $B_{H^\infty(F)}$ is a classifying space for fibrations with A_∞-actions of F on the total space. The above homotopy equivalence then shows a fibre space admits such an A_∞-action if and only if it admits an associative action.

REFERENCES

1. R. A. Nowlan, *A_n-actions on Fibre Spaces,* Thesis, University of Notre Dame, 1969.
2. J. D. Stasheff, *Homotopy Associativity of H-spaces (I)*, Trans. Amer. Math. Soc., **108** (1963), 275-292.
3. ———, *Homotopy Associativity of H-spaces (II)*, Trans. Amer. Math. Soc., **108** (1963), 293-312.
4. ———, *On Homotopy Abelian H-spaces,* Proc. Cambridge Phil. Soc., **57** (1961), 734-745.
5. N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J,. (1967), 133-152.

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SOUTHERN CONNECTICUT STATE COLLEGE