## ANALYTIC SHEAVES ON KLEIN SURFACES

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## Morphisms of Klein surfaces are discussed from the sheaf-theoretic standpoint, and the cohomology of an analytic sheaf on a Klein surface is computed.

0. Let  $\mathfrak{X}$  be a Klein surface [1], [2]; that is,  $\mathfrak{X}$  consists of an underlying space X, which is a surface with boundary, and a family of equivalent dianalytic atlases on X. If  $(U_{\alpha}, z_{\alpha})$  is such an atlas, then  $z_{\alpha}: U_{\alpha} \to C^{+}$  is a homeomorphism of the open set  $U_{\alpha}$  in X onto an open subset of  $C^{+} = \{z \in C \mid \operatorname{Im}(z) \geq 0\}$ . The functions  $z_{\alpha}$  must thus be real on  $U_{\alpha} \cap \partial X$ , and it is required that  $z_{\alpha} \circ z_{\beta}^{-1}$  be dianalytic, that is, either analytic or antianalytic on each component of  $z_{\beta}(U_{\alpha} \cap U_{\beta})$ .

In this paper we define the structure sheaf of  $\mathfrak{X}$ , show that the concept of morphism given in [1], [2] coincides with the concept of a morphism of ringed spaces, and compute the cohomology of analytic sheaves on  $\mathfrak{X}$ . If  $\mathscr{F}$  is an analytic sheaf on X, and  $\widetilde{\mathscr{F}}$  is the lift of  $\mathscr{F}$  to the complex double  $\mathfrak{X}$  of  $\mathfrak{X}$ , then there is a natural isomorphism

$$H^q(\widetilde{\mathfrak{X}},\widetilde{\mathscr{F}})\cong Cigotimes_{{\scriptscriptstyle R}} H^q(\mathfrak{X},\mathscr{F})$$
 .

1. The structure sheaf  $\mathcal{O}_{\mathfrak{x}}$ . We define the structure sheaf  $\mathcal{O}_{\mathfrak{x}} = \mathcal{O}$  on  $\mathfrak{X}$  as follows. If U is open in X, let  $\mathcal{O}(U)$  be the ring of holomorphic functions on U (in the sense of [1], [2]). If  $U \supset U'$ , then the inclusion map is a morphism of Klein surfaces and we have a natural map  $\rho_{U'}^{\mathbb{V}} \colon \mathcal{O}(U) \to \mathcal{O}(U')$  (this is not quite an ordinary restriction map since the elements of  $\mathcal{O}(U)$  are not quite functions). In particular, if  $(U_{\alpha}, z_{\alpha})$  and  $(U_{\beta}, z_{\beta})$  are dianalytic charts on  $\mathfrak{X}$ ,  $U_{\alpha} \supset U_{\beta}$ , then

$$\mathscr{O}(U_{lpha})\cong egin{cases} f\colon U_{lpha} o C \,|\, f(U_{lpha} \cap \partial X) \subset R \ {
m and} \ f\circ z_{lpha}^{-1} ext{ analytic} \end{cases}$$

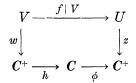
and

$$ho^{U_{lpha}}_{U_{eta}}(f) = egin{cases} f \mid U_{eta} ext{ where } z_{lpha} \circ z_{eta}^{-1} ext{ is analytic} \ egin{array}{c} eta \mid U_{eta} ext{ where } z_{lpha} \circ z_{eta}^{-1} ext{ is antianalytic} \ . \end{cases}$$

It is easily checked that this defines a sheaf of local R-algebras on  $\mathfrak{X}$ .

Let  $\mathfrak{X}, \mathfrak{Y}$  be Klein surfaces,  $f: \mathfrak{Y} \to \mathfrak{X}$  a continuous map. Then f is a morphism [1] if  $f(\partial Y) \subset \partial X$  and if for every point  $p \in Y$  there

are dianalytic charts (V, w) and (U, z) at p and f(p), and an analytic function h on w(V), such that



commutes (\$\phi\$ is the folding map,  $\phi(a + bi) = a + |b|i)$ .

Recall that a ringed space morphism  $\mathfrak{Y} \to \mathfrak{X}$  is a pair  $(f, \theta)$  where  $f: Y \to X$  is continuous and  $\theta: \mathscr{O}_x \to f_* \mathscr{O}_y$  is a morphism of sheaves of rings [4, p. 36]. Here  $f_* \mathscr{O}_y$  is the direct image sheaf:  $f_* \mathscr{O}_y(U) = \mathscr{O}_y(f^{-1}(U))$ .

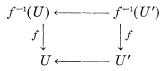
**THEOREM 1.** Let  $\mathfrak{X}, \mathfrak{Y}$  be Klein surfaces, and let  $f: Y \to X$  be a nonconstant continuous map. Then the following are equivalent:

(i) f is a morphism;

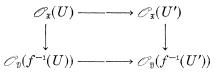
(ii) there exists a morphism  $\theta: \mathcal{O}_x \to f_* \mathcal{O}_y$  of sheaves of **R**-algebras.

Under these conditions the morphism  $\theta$  is unique, so f can be made in a unique way into a morphism of ringed spaces.

*Proof.* (i)  $\Rightarrow$  (ii). Let  $U \supset U'$  be open in X. From the commutative diagram:



of morphisms of Klein surfaces we deduce a commutative diagram



of morphisms of *R*-algebras, and this defines an *R*-algebra morphism  $\theta: \mathcal{O}_x \to f_* \mathcal{O}_y$ .

(ii)  $\Rightarrow$  (i). Let  $p \in Y$ , and let (V, w), (U, z) be dianalytic charts at p, f(p), with  $f(V) \subset U$ . Let  $z^*$  be the image of z in  $\mathscr{O}_{\mathbb{P}}(V)$  under

$$(*) \qquad \qquad \mathcal{O}_{\mathfrak{x}}(U) \to \mathcal{O}_{\mathfrak{y}}(f^{-1}(U)) \to \mathcal{O}_{\mathfrak{y}}(V)$$

Set  $h = z^* \circ w^{-1}$ . We claim  $f | V = z^{-1} \circ \phi \circ h \circ w$ , i.e. that  $z \circ (f | V) = \phi \circ z^*$ . It clearly suffices to show that  $z(f(p)) = \phi(z^*(p))$ . If this does not hold, then

$$g=rac{1}{[z-z^*(p)][z-\overline{z^*(p)}]}$$

is holomorphic at f(p), and shrinking U, V if necessary, we may assume  $g \in \mathscr{O}_{\mathfrak{x}}(U)$ . We let  $g^*$  denote its image under (\*) in  $\mathscr{O}_{\mathfrak{y}}(V)$ . But  $g^* = 1/[z^* - z^*(p)][z^* - \overline{z^*(p)}]$  which is not defined at p.

We still need to show that  $f(\partial Y) \subset \partial X$ . Let  $q \in X$ . Then  $\mathcal{O}_{x,q}$  is an *R*-algebra which contains a copy of *C* if and only if  $q \notin \partial X$ . The  $\mathcal{O}_{x,q}$  algebra  $(f_*\mathcal{O}_x)_q$  is isomorphic to

$$\prod_{f(p)=q} \mathscr{O}_{\mathfrak{Y},p}$$
 ,

so  $q \in \partial X$ , f(p) = q implies  $p \in \partial Y$ .

We now check that  $\theta$  is unique. Let U be open in  $X, g \in \mathscr{O}_{\mathfrak{x}}(U)$ ,  $p \in f^{-1}(U)$ . Let (V, w) be a dianalytic chart at p with  $V \subset f^{-1}(U)$ . Let  $g^*$  be the image of g in  $\mathscr{O}_{\mathfrak{g}}(V)$  under (\*). Then using the above arguments, either  $g^*(p) = gf(p)$  or  $g^*(p) = \overline{gf(p)}$ . If g is nonconstant, only one of these can yield an analytic function. If g is constant it can be expressed as a sum of nonconstant functions. Hence  $g^*$ , and thus  $\theta$ , are uniquely determined. The theorem is proved.

By an analytic sheaf of  $\mathfrak{X}$  we mean an  $\mathcal{O}_{\mathfrak{X}}$ -module. If  $\mathscr{F}$  is an analytic sheaf on  $\mathfrak{X}$  and  $f: \mathfrak{Y} \to \mathfrak{X}$  is a morphism then  $f^*\mathscr{F}$  is the sheaf associated to the presheaf  $V \to \mathcal{O}_{\mathfrak{Y}}(V) \bigotimes_{\mathfrak{O}_{\mathfrak{X}}(fV)} \mathscr{F}(fV)$ .

**PROPOSITION 2.** If  $\mathscr{F}$  is a coherent analytic sheaf on  $\mathfrak{X}$ , then  $f^*\mathscr{F}$  is a coherent analytic sheaf on  $\mathfrak{Y}$ .

*Proof.* The proof given in [5, p. 47] for Riemann surfaces carries over to the Klein surface case.

2. The complex double. Let  $\mathfrak{X}$  be a Klein surface,  $\pi: \mathfrak{X} \to \mathfrak{X}$ its complex double. Recall that if  $(U_{\alpha}, z_{\alpha})$  is a dianalytic atlas on  $\mathfrak{X}$ , then  $(\widetilde{U}_{\alpha}, \widetilde{z}_{\alpha})$  is a dianalytic atlas on  $\widetilde{\mathfrak{X}}$ , where  $\widetilde{U}_{\alpha} = \pi^{-1}(U_{\alpha}) = U'_{\alpha} \cup U''_{\alpha}$ ,  $U'_{\alpha} \cap U''_{\alpha} = \pi^{-1}(U_{\alpha} \cap \partial X)$ , and  $\pi$  maps  $U'_{\alpha}$  and  $U''_{\alpha}$  each homeomorphically onto  $U_{\alpha}$ . The function  $\widetilde{z}_{\alpha}$  is defined by

$$\widetilde{z}_lpha(p) = egin{cases} z_lpha(p) & p \in U'_lpha \ \overline{z_lpha(p)} & p \in U''_lpha \ p \in U''_lpha \end{cases}$$

 $U'_{\alpha}$  is identified with  $U'_{\beta}$  where  $z_{\alpha} \circ z_{\beta}^{-1}$  is analytic, and with  $U''_{\beta}$  where  $z_{\alpha} \circ z_{\beta}^{-1}$  is anti-analytic. This construction yields the Riemann surface (without boundary)  $\tilde{\mathfrak{X}}$  as a double cover of  $\mathfrak{X}$ , folded along  $\partial X$ .

If U is open in X, let  $\widetilde{U} = \pi^{-1}(U)$ . We denote the structure sheaf of  $\widetilde{\mathfrak{X}}$  by  $\widetilde{\mathscr{O}}$ .

**PROPOSITION 3.** There is a canonical isomorphism

$$(\dagger) \qquad \qquad C \bigotimes_{R} \mathscr{O}(U) \cong \widetilde{\mathscr{O}}(\widetilde{U})$$

for every open set  $U \subset X$ .

*Proof.* We may cover U by dianalytic charts  $(U_{\alpha}, z_{\alpha})$ . It then suffices to verify (†) for  $U_{\alpha}$ , since  $\mathscr{O}(U)$  is the difference kernel of  $\prod_{\alpha} \widetilde{\mathscr{O}}(\widetilde{U}_{\alpha}) \rightrightarrows \prod_{\alpha,\beta} \widetilde{\mathscr{O}}(\widetilde{U}_{\alpha} \cap \widetilde{U}_{\beta})$  and  $C \bigotimes_{\mathbf{R}}$  is exact.

Let  $\sigma$  be the canonical anti-involution of  $\widetilde{\mathfrak{X}}$  which commutes with  $\pi$ , and let  $\kappa$  denote complex conjugation. If we identify  $\mathscr{O}(U_{\alpha})$  with its image in  $\widetilde{\mathscr{O}}(\widetilde{U}_{\alpha})$  then we see

$$\mathscr{O}(U_{\scriptscriptstylelpha})=\{g\in\widetilde{\mathscr{O}}(\widetilde{U}_{\scriptscriptstylelpha})\,|\,g=\kappa g\sigma\}$$
 .

But any  $g \in \mathscr{O}(U_{\alpha})$  can be written as

$$g = \frac{1}{2}(g + \kappa g\sigma) + \frac{1}{2}(g - \kappa g\sigma)$$

and hence the canonical map

$$C \bigotimes_{R} \mathscr{O}(U_{\alpha}) \to \widetilde{\mathscr{O}}(\widetilde{U}_{\alpha})$$

is surjective. This map is easily seen to be injective, completing the proof.

If  $\mathscr{F}$  is an analytic sheaf on  $\mathfrak{X}$ , let  $\widetilde{\mathscr{F}} = \pi^* \mathscr{F}$ .

THEOREM 4. There is a canonical isomorphism

$$C \bigotimes_{R} \mathscr{F}(\mathfrak{X}) \cong \widetilde{\mathscr{F}}(\widetilde{\mathfrak{X}})$$

*Proof.* We may choose a base for the topology of X consisting of sets of the form  $U_{\alpha}$ , where  $(U_{\alpha}, z_{\alpha})$  is a dianalytic atlas on X. Then sets of the form  $U'_{\alpha}, U''_{\alpha}$  (where  $U_{\alpha} \cap \partial X = \emptyset$ ) and of the form  $\widetilde{U}_{\alpha}$  (where  $U_{\alpha} \cap \partial X \neq \emptyset$ ) form a base B for the topology of  $\widetilde{\mathfrak{X}}$ . Since  $\widetilde{\mathscr{O}}(\widetilde{U}) \bigotimes_{\mathscr{O}(U)} \mathscr{F}(U) \cong C \bigotimes_{\mathbb{R}} \mathscr{F}(U)$ , it suffices to show that the sequence

$$(\uparrow\uparrow) \qquad \begin{array}{l} 0 \to \widetilde{\mathscr{O}}(\widetilde{\mathfrak{X}}) \bigotimes_{\mathscr{O}(\mathfrak{X})} \mathscr{F}(\mathfrak{X}) \to \prod_{V \in B} \widetilde{\mathscr{O}}(V) \bigotimes_{\mathscr{O}(\pi V)} \mathscr{F}(\pi V) \\ \\ \rightrightarrows \prod_{V, W \in B} \widetilde{\mathscr{O}}(V \cap W) \bigotimes_{\mathscr{O}} (\pi(V \cap W)) \mathscr{F}(\pi(V \cap W)) \ . \end{array}$$

is exact. When  $U'_{\alpha}$  and  $U''_{\alpha}$  are disjoint then  $\widetilde{\mathscr{O}}(\widetilde{U}_{\alpha}) = \widetilde{\mathscr{O}}(U'_{\alpha}) \times \widetilde{\mathscr{O}}(U''_{\alpha})$ so (††) may be replaced by

$$\begin{split} 0 &\to \widetilde{\mathscr{O}}(\mathfrak{X}) \bigotimes_{\mathscr{O}(X)} \mathscr{F}(\mathfrak{X}) \to \prod_{\alpha} \widetilde{\mathscr{O}}(\widetilde{U}_{\alpha}) \bigotimes_{\mathscr{O}(U_{\alpha})} \mathscr{F}(U_{\alpha}) \\ & \rightrightarrows \prod_{\alpha,\beta} \widetilde{\mathscr{O}}(\widetilde{U}_{\alpha\beta}) \bigotimes_{\mathscr{O}(U_{\alpha\beta})} \mathscr{F}(U_{\alpha\beta}) \end{split}$$

and this last is exact because of Proposition 3 and the fact that  $\mathcal{F}$  is a sheaf.

Since the functors  $\mathscr{F} \to C \bigotimes_{\mathbb{R}} \mathscr{F}(\mathfrak{X})$  and  $\mathscr{F} \to \widetilde{\mathscr{F}}(\mathfrak{X})$  are canonically isomorphic, so are their derived functors [3], and we have

THEOREM 5. Let  $\mathscr{F}$  be an analytic sheaf on the Klein surface  $\mathfrak{X}$ . Then there is a canonical isomorphism

$$H^{q}(\widetilde{\mathfrak{X}},\widetilde{\mathscr{F}})\cong C\bigotimes_{R}H^{q}(\mathfrak{X},\mathscr{F})$$

for all  $q \geq 0$ .

COROLLARY. (Cartan Theorem B) Let  $\mathfrak{X}$  be a non-compact Klein surface,  $\mathscr{F}$  a coherent analytic sheaf on  $\mathfrak{X}$ . Then  $H^q(\mathfrak{X}, \mathscr{F}) = 0$  for all  $q \geq 1$ 

*Proof.* Use Theorem 5 and Proposition 2 to reduce to the case of a non-compact Riemann surface [6, p. 270].

## References

1. N. Alling and N. Greenleaf, *Klein Surfaces and Real Algebraic Function Fields*, Bull. Amer. Math. Soc. **75** (1969), 869-872.

2. \_\_\_\_\_, Klein Surfaces, Springer Lecture Notes in Mathematics. (to appear).

3. A. Grothendieck, Sur Quelques Points d' Algebre Homologique, Tohoku Math. J., 9 (1957), 119-221.

4. A. Grothendieck and J. Dieudonne, *Elements de Geometrie Algebrique I*, Pub. I.H.E.S., 1969.

5. R. Gunning, Lectures on Vector Bundles over Riemann Surfaces, Princeton University Press, Princeton, 1967.

6. R. Gunning and H. Rossi, Analytic Functions of Several Complex Variables, Prentice Hall, Englewood Cliffs, N.J., 1965.

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