PACIFIC JOURNAL OF MATHEMATICS
Vol. 38, No. 2, 1971

THE POLYNOMIAL OF A NON-REGULAR DIGRAPH

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This paper studies the matrix equation $f(Z) = D + \lambda J$ where f is a polynomial, Z a square $(0, 1)$-matrix, D is diagonal, $\lambda \neq 0$ and J is the matrix of ones. If Z is thought of as the incidence matrix of a digraph G, the equation implies various path length properties for G. It is shown that such a graph is an amalgamation of regular subgraphs with similar path length properties. Necessary and sufficient parameter conditions on the matrix Z are given in order that it satisfy such an equation for a fixed polynomial f and all non-regular digraphs corresponding to quadratic polynomials f are found.

1. **Introduction.** The concept of the polynomial of a graph was introduced by Hoffman [3] for regular, connected, non-oriented graphs, and discussed by Hoffman and McAndrew [4] for regular directed graphs. If A is the adjacency matrix of such a graph G, the polynomial of G is taken to be the polynomial $p(x)$ of least degree with $p(A) = J$, the matrix of ones. In extending this notion to non-regular, directed graphs we are concerned with the matrix equation

$$f(Z) = D + \lambda J \tag{1.1}$$

where f is a polynomial, Z a square $(0, 1)$ matrix, D a diagonal matrix and $\lambda \neq 0$. Given (1.1) the conditions: (a) Z has constant row sums; (b) D is a scalar matrix; (c) Z has constant line sums; and (d) The graph of Z is regular; are easily seen to be equivalent. The regular case of (1.1) embraces such studies as the (v, k, λ)-problem [7], (n, k, λ)-systems on k and $k + 1$ [1], Moore graphs [4], strongly regular graphs [9, 10, 11, 12] and even the algebraic studies of central groupoids and universal algebras]2], [6].

In [8] Ryser opens the non-regular question by considering (1.1) with $f(x) = x^2$, and finding all *nonregular* solutions.

The case in which f of (1.1) has degree two is particularly inter-

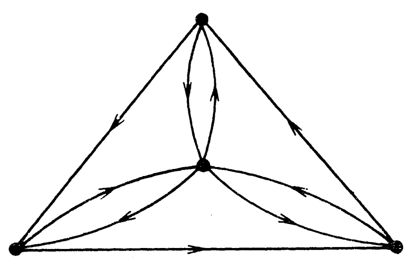

FIG. 1

esting. Here we are studying directed graphs with the feature that there are constants μ_0, μ_1 so that for distinct points p_i, p_j the number of directed paths from p_i to p_j of length two is μ_1 or μ_0 depending on whether or not there is a directed edge from p_i to p_j. For example the digraph G of figure one has this property with $\mu_1 = 1$, $\mu_0 = 2$. Its adjacency matrix is given by

$$Z = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{matrix}$$

satisfying

$$Z^2 + Z = \operatorname{diag}(1, -1, -1, -1) + 2J .$$

It is easy to see how various graph theoretic properties expressible in terms of path lengths can be reflected in (1.1) by suitably choosing the polynomial f.

In the next section we give a general structure result for matrices satisfying (1.1) showing them to be an "amalgamation" of regular solutions to similar equations. We refine this structure result by considering certain special polynomials $f(x)$. Finally we determine all non-regular quadratic graphs, i.e., those whose adjacency matrix satisfies (1.1) with f a quadratic polynomial.

Throughout J will denote a matrix of ones, I an identity matrix and subscripts on these symbols will denote their orders when necessary.

2. The structure of f-graphs.

Let G be a directed graph (loops allowed) on n vertices $\{1, \cdots, n\}$ with adjacency matrix $Z = (z_{ij})$ ($z_{ij} = 1$ if there is an edge from i to j and $z_{ij} = 0$ otherwise). Further let f be a monic polynomial with $f(0) = 0$. We say that G *is an f-graph* or that Z *carries an f-graph* if there is a diagonal matrix D and a number $\lambda \neq 0$ so that

$$f(Z) = D + \lambda J . \tag{2.1}$$

We shall sometimes say, for Z with constant line sums, that Z carries a degenerate f-graph if $f(Z)$ is a scalar matrix.

Let Z_i be a square (0, 1)-matrix of order n_i for $i = 1, 2, \cdots, t$ and $n = \sum_{i=1}^{t} n_i$. We define the *complementary direct sum* (c.d.s) of the matrices Z_i by

$$\text{c.d.s.}(Z_i \mid i = 1, \cdots, t) \equiv J_n - \sum_{i=1}^{t} \oplus (J_{n_i} - Z_i) \tag{2.2}$$

where $\sum\oplus$ denotes the usual direct sum.

Finally since evidently for P a permutation matrix with transpose P^t the matrix PZP^t will satisfy (2.1) if Z does we define A and B to be *equivalent*, $A \cong B$, if there is a permutation matrix P so that $A = PBP^t$. The relevance of these definitions will be clear from the following theorem.

THEOREM 2.1. *Let Z be a $(0, 1)$-matrix of order n. Suppose Z carries an f-graph. Then we have*

$$Z \cong \text{c.d.s.}\,(Z_i \mid i = 1, \cdots, t) \tag{2.3}$$

where the matrices Z_i of order n_i have constant line sums r_i with $r_i - n_i \neq r_j - n_j$ for $i \neq j$ and each Z_i carries an f-graph (possibly degenerate).

We delay the proof of this elementary observation as we can say considerably more. We only state Theorem 2.1 in order to put the next theorem in proper perspective.

THEOREM 2.2. *Let Z_i be a $(0, 1)$-matrix of order n_i with constant line sums r_i for $i = 1, \cdots, t$. For $i \neq j$ suppose $r_i - n_i \neq r_j - n_j$. Put*

$$Z = \text{c.d.s.}\,(Z_i \mid i = 1, 2, \cdots, t) \tag{2.4}$$

$$r_{ij} = (r_i - n_i)\delta_{ij} + n_j \quad (i, j = 1, \cdots, t) \tag{2.5}$$

(δ_{ij} denoting Kronecker's delta) *and*

$$R = (r_{ij})\ . \tag{2.6}$$

Finally left f be a monic polynomial of degree at least two with $f(0) = 0$.

Then Z carries an f-graph if and only if there exist constants b and λ with $\lambda \neq 0$, and numbers d_i, λ_i $(i = 1, \cdots, t)$ so that

$$d_i = \lambda(r_i - n_i) + b \quad (i = 1, \cdots, t) \tag{2.7}$$

$$f(Z_i) = d_i I + \lambda_i J_{n_i} \quad (i = 1, \cdots, t) \tag{2.8}$$

and

$$f(R) = \lambda R + bI\ . \tag{2.9}$$

We proceed to prove both theorems. It follows from (2.1) that Z commutes with $D + \lambda J$ if Z carries an f-graph. Denoting $D = \operatorname{diag}(d_1, \cdots, d_n)$, the row sums of Z by ρ_i $(i = 1, \cdots, n)$ and the column sums by $\sigma_j (j = 1, \cdots, n)$ this fact may be expressed as

$$(2.10) \qquad z_{ij}(d_i - d_j) = \lambda(\rho_i - \sigma_j) .$$

We thus have $\rho_i = \sigma_i$ and if we permute the rows of Z so that rows with equal sum occur together and then perform the corresponding column permutations we obtain PZP^t with say rows $1, 2, \cdots, n_1$ of equal sum and then rows $n_1 + 1, n_1 + 2, \cdots, n_1 + n_2$ of equal sum etc. Let Z_i denote the principal submatrix of PZP^t on the lines $n_{i-1} + 1, \cdots, n_{i-1} + n_i$ $(n_0 \equiv 0)$. We assert that no entry of Z outside one of these Z_i can be a zero for then from (2.10) we have $\rho_i = \sigma_j = \rho_j$ contrary to our grouping of the rows. Thus we have established (2.3) where t is the number of distinct row sums of Z. It is immediate that Z_i of order n_i has constant line sums say $r_i (i=1, \cdots, t)$ and that $r_i - n_i \neq r_j - n_j$ for $i \neq j$ since $\rho_j = r_i + n - n_i$.

We now deal with Z in the equivalent c.d.s. form and take $t > 1$ lest Z be regular. Note from (2.10) that the diagonal matrix D has different entries in positions corresponding to different blocks Z_i. It is further clear from (2.10) that the entries in D in positions corresponding to the same block Z_i are the same. We therefore revise our notation so that $d_1, d_2, \cdots, d_t$ denote the distinct diagonal entries of D, d_i occurring n_i times. Then (2.10) says that the points $(d_i, r_i - n_i)$ for $i = 1, 2, \cdots, t$ lie on the line $y = \lambda x + b$ for some constant b. [Note it is the choice of normalization $(f(0) = 0)$ which brings the constant b into play. We could, of course, force $b = 0$ by altering the constant term of $f(x)$.]

We now investigate the powers of the matrix Z in c.d.s. form and assert that in block form, the $(i, j)^{\text{th}}$ block of size $n_i \times n_j$, we have

$$(2.12) \qquad Z^k = [B_{ij}] \;\; (i, j = 1, \cdots, t)$$

where

$$(2.13) \qquad B_{ij} = \delta_{ij} Z_i^k + g_{ij}^{(k)} J_{n_i \times n_j} .$$

The numbers $g_{ij}^{(k)}$ are given by

$$(2.14) \qquad g_{ii}^{(1)} = 0, \;\; g_{ij}^{(1)} = 1 \;\text{ for } i \neq j$$

and for $k > 1$

$$(2.15) \qquad g_{ij}^{(k)} = \sum_{e=1}^{t} n_e g_{ej}^{(k-1)} + (r_i - n_i) g_{ij}^{(k-1)} + r_j^{k-1}(1 - \delta_{ij}) .$$

This claim is easily verified inductively.

We introduce the following notational convention: If $p(x) = \sum_{i=1}^m b_i x^i$ is a polynomial in x and $\alpha^{(k)}$ is a symbol in use with superscripts then by $\hat{p}(\alpha)$ we will mean the expression $\hat{p}(\alpha) \equiv \sum_{i=1}^m b_i \alpha^{(i)}$, using the iterates of α in place of the powers of x. Then from (2.12) – (2.13) we have

$$f(Z) = [C_{ij}] \ (i, j = 1, \cdots, t) \tag{2.16}$$

C_{ij} being an $n_i \times n_j$ block given by

$$C_{ii} = f(Z_i) + \hat{f}(g_{ii})J_{n_i} \ (i = 1, \cdots, t) \tag{2.17}$$

$$C_{ij} = \hat{f}(g_{ij})J_{n_i \times n_j} \ i \neq j \ (i, j = 1, \cdots, t) . \tag{2.18}$$

with

$$G^{(k)} = (g_{ij}^{(k)}) \ (i, j = 1, \cdots, t) \tag{2.19}$$

we see that the structure of $f(Z)$ depends on the matrix $\hat{f}(G)$. We note, however, from (2.16), (21.7) that if Z carries an f-graph

$$f(Z_i) + \hat{f}(g_{ii})J_{n_i} = d_iI + \lambda J \text{ so that}$$
$$f(Z_i) = d_iI + (\lambda - \hat{f}(g_{ii}))J_{n_i}$$

and the Z_i carry f-graphs, degenerate should $\lambda = \hat{f}(g_{ii})$. We have thus completely proven Theorem 2.1 and continue with the necessity in Theorem 2.2 where we have already established (2.7) and (2.8). To obtain (2.9) we proceed to observe that the recursion (2.24), (2.15) can be written: $G^{(1)} = J - I$ and for $k > 1$:

$$G^{(k)} = RG^{(k-1)} + (J - I)F^{k-1} \tag{2.21}$$

where $F = \text{diag}\,(r_1, \cdots, r_t)$. We obtain an explicit formula for $G^{(k)}$ as follows. Let $E = \text{diag}\,(1/n_1, \cdots, 1/n_t)$ and consider

$$H^{(k)} \equiv [R^k - F^k]E, \ k \geqq 1 . \tag{2.22}$$

We claim that $H^{(k)}$ satisfies the recursion (2.21). For $k = 1$, $H^{(1)} = G^{(1)} = J - I$. Now for $k > 1$ we have:

$$RH^{(k-1)} + (J - I)F^{k-1} = [R^k - RF^{k-1}]E + (J - I)F^{k-1}$$

so

$$RH^{(k-1)} + (J - I)F^{k-1} = R^kE - (R - (J - I)E^{-1})F^{k-1}E .$$

But $R - (J - I)E^{-1} = F$ so we have $RH^{(k-1)} + (J - I)F^{k-1} = [R^k - F^k]E = H^{(k)}$. Thus $H^{(k)} = G^{(k)}$ and

$$\hat{f}(G) = [f(R) - f(F)]E . \tag{2.23}$$

Now if Z is carrying an f-graph the off-diagonal entries of $\hat{f}(G)$ are all λ. Thus the off-diagonal entries of $f(R)$ are given by

$$f(R)_{i,j} = \lambda n_j \ (i \neq j; i, j = 1, \cdots, t) . \tag{2.24}$$

From (2.20) we see that

$$f(r_i) = d_i + (\lambda - \hat{f}(g_{ii}))n_i$$

so that

$$\hat{f}(g_{ii}) = \frac{f(R)_{ii} - f(r_i)}{n_i} = \frac{d_i + \lambda n_i - f(r_i)}{n_i} . \tag{2.25}$$

Since $d_i = \lambda(r_i - n_i) + b$ we have

$$f(R)_{ii} = d_i + \lambda n_i = \lambda r_i + b , \tag{2.26}$$

and (2.24) and (2.26) establish (2.9).

As to the sufficiency of (2.7) – (2.9) we need only note that (2.23) is a valid expression for any polynomial f and that using (2.7), (2.8) with

$$\hat{f}(G) = [\lambda R + bI - f(F)]E$$

we see $\hat{f}(G)_{ij} = \hat{f}(g_{ij}) = \lambda$ $(i \neq j; i, j = 1, \cdots, t)$ and

$$\hat{f}(g_{ii}) = \frac{\lambda r_i + b - f(r_i)}{n_i} = \frac{d_i + \lambda n_i - f(r_i)}{n_i}$$

so that in view of $f(r_i) = d_i + n_i\lambda_i$

$$f(Z_i) + \hat{f}(g_{ii})J = d_iI + \lambda J .$$

This completes the proof of Theorems 2.1 and 2.2. An immediate corollary of these results is that we may define the notion of the polynomial of a directed graph if the graph is an f-graph for some f.

COROLLARY 2.3. *Let Z carry an f-graph for some f. Then there exists a unique monic polynomial $P_z(x)$ with $P_z(0) = 0$ of least degree so that Z carries a P_z-graph.*

Proof. Let $g(x)$ and $h(x)$ be two such polynomials. Suppose

$$g(Z) = D + \lambda J, h(Z) = H + \mu J$$

D, H diagonal $\lambda \neq 0$, $\mu \neq 0$. Then since

$$(g - h)Z = (D - H) + (\lambda - \mu)J$$

evidently Z carries a degenerate $(g - h)$-graph, i.e., $\lambda = \mu$. But for suitable constants b, c we have $d_i = \lambda(r_i - n_i) + b$ and $h_i = \mu(r_i - n_i) + c$ so that $D - H$ is a scalar matrix and since surely g and h have degree less than that of Z's minimal polynomial and $g(0) = h(0) = 0$

we have $g(x) \equiv h(x)$.

Professor Hoffman has observed that the condition that Z carry an f-graph for some f can be expressed in terms of the spectra of the matrices Z_i of (2.4) and that of the *parameter* matrix R of (2.6) as follows. For $n_i > 1$ delete from the eigenvalue set of Z_i the line sum r_i *if* Z_i is irreducible and call the resultant set Λ_i. Let Λ_0 be the set of eigenvalues of the matrix R. Then Z carries an f-graph for some f if and only if

$$\Lambda_i \cap \Lambda_j = \phi \text{ for all } n_i, n_j > 1,\ i \neq j,\ i, j = 0, 1 \cdots, t\ .$$

To see this one need only observe that the congruences

$$f(x) \equiv d_i \bmod \dot{m}_i(x)$$
$$f(x) \equiv \lambda x + b \bmod q(x)$$

(where $\dot{m}_i(x)$ is the minimal polynomial of Z_i with $(x - r_i)$ divided out in case Z_i is irreducible and $q(x)$ is the minimal polynomial of R) are satisfied by f so that the $\dot{m}_i(x)$ and $q(x)$ cannot share roots. Note that for $\dot{m}_i(x)$ and $q(x)$ one needs the fact that $r_i - n_i = (d_i - b)/\lambda$ is *not* a root of $q(x)$. The converse statement is also quite immediate in view of Theorem 2.2.

We shall call the matrices Z_i (or the obviously associated subgraphs G_i) in the c.d.s. form of Z the *regular constituents of* Z. So Z is regular if it has one constituent and we will call Z *near-regular* if it has precisely two regular-constituents.

Now the parameter matrix R (2.6) is readily seen to be similar to the symmetric matrix

$$S = \operatorname{diag}(r_1 - n_i, \cdots, r_t - n_t) + (\sqrt{n_i n_j})\ . \tag{2.27}$$

Indeed

$$R = E^{1/2} S E^{-1/2}$$

where $E = \operatorname{diag}(1/n_1, \cdots\cdot, 1/n_t)$. Easily for the $(r_i - n_i)$ distinct the matrix R has t distinct real characteristic roots, and that precisely one of these roots is positive. Now if Z carries an f-graph we have $f(R) = \lambda R + bI$ so that the minimal polynomial (= characteristic polynomial) of R divides $f(x) - \lambda x - b$. We thus establish

COROLLARY 2.4. *Let Z carry an f-graph with t regular constituents. Then $t \leqq$ degree f, and, if equality holds, $f(x) - \lambda x - b$ is the characteristic polynomial of R.*

The case of equality may occur. For example the matrix

$$Z = \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix}$$

has 3 regular constituents and its polynomial is $x^3 - x^2 - 4x$. However with various restrictions on $f(x)$ we can make stronger statements about the number of constituents of an f-graph.

COROLLARY 2.5. *Suppose $f(x)$ is an even polynomial with non-negative coefficients. Then a non-regular f-graph is near-regular.*

Proof. For suitable $\lambda > 0$ and b we have that $q(x)$, the characteristic polynomial of the parameter matrix R, devides $g(x) = f(x) - \lambda x - b$. But $g(-x) = f(x) + \lambda x - b$ so that g has at most one negative root. However $q(x)$ has $t - 1$ negative roots. Thus $t \leqq 2$.

An interesting class of polynomial graphs are the x^r-graphs investigated by Ryser [7] for $r = 2$. These graphs have the feature that the number of paths of length r joining any two distinct points is constant. The preceding corollary shows that for r even such graphs, if not regular, are near regular. This property extends to all x^r-graphs.

COROLLARY 2.6. *A non-regular x^r-graph ($r \geqq 2$) is near regular.*

Proof. Again $g(x) = f(x) - \lambda x - b = x^r - \lambda x - b$ must have at least t distinct real roots, where t is the number of regular constituents of the graph. But $g'(x)$ has at most two real roots so that $t \leqq 3$. The following argument suggested by Professor Hoffman shows that $t \neq 3$. Suppose $t = 3$. Then $q(x) = x^3 - ax^2 + hx - \Delta$ where $a = r_1 + r_2 + r_3$, $h = \sum_{i<j} (r_i r_j - n_i n_j)$ and $\Delta = \det R$, is the characteristic polynomial of R. Easily $a > 0$, $h < 0$, $\Delta > 0$. Since $q(x)$ divides $f(x) - \lambda x - b$ we may write

$$(2.28) \quad (x^r - \lambda x - b) = (x^3 - ax^2 + hx - \Delta)(x^{r-3} + a_4 x^{r-4} + \cdots + a_{r-1}x + a_r)\,.$$

Equating coefficients in this identity we obtain

$$(2.29) \quad \begin{aligned} & a_4 - a = 0 \\ & a_5 - aa_4 + h = 0 \\ & a_6 - aa_5 + ha_4 - \Delta = 0 \\ & a_{j+3} - aa_{j+2} + ha_{j+1} - \Delta a_j = 0 \ \ (4 \leqq j \leqq r-3)\,. \end{aligned}$$

The relations (2.29) imply for $i = 4, \cdots, r$ that $a_i > 0$. Now equating the coefficients of x^2 we obtain the contradiction

$$-aa_r + ha_{r-1} - \Delta a_{r-2} = 0 .$$

Of course $r = 3$ implies $a = 0$.

We remark that the existence of a *non-regular* x^r-graph on more than two points $r > 2$ remains in question.

Suppose one constituent of an f-graph is a (looped or unlooped) point. Then we shall refer to the graph as a (looped or unlooped) *cone*. For example let K_a denote the full (looped) directed clique on a vertices. Take some number say $m > 1$ of copies of K_a and form the looped cone over this graph with respect to an additional point. The resulting graph is a non-regular $(x^2 - ax)$-graph. We shall in fact see in the next section that with one exception all non-regular quadratic graphs are cones.

3. Non-regular quadratic graphs. In this section we determine all non-regular f-graphs for $f(x) = x^2 - ax$. We characterized these graphs in the introduction in terms of path lengths. From Corollary 2.4 we know such graphs are near regular. Our first observation is that there is precisely one such graph which is not a cone.

THEOREM 3.1. *Let Z carry a non-regular quadratic graph which is not a cone. Then*

$$(3.1) \qquad Z \cong \begin{matrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{matrix}$$

and carries an $(x^2 + x)$-graph.

Proof. Let Z_1, Z_2 of orders n_1, n_2 and line sums r_1 and r_2 be the regular constituents of Z and suppose $n_i > 1$ $(i = 1, 2)$. We note from Corollary 2.4 that

$$(3.2) \qquad r_1 + r_2 = \lambda + a .$$

We first remark that neither constituent can be $J - I$. For suppose $r_1 = n_1 - 1$. Then from (3.2) $r_2 = \lambda + a - n_1 + 1$ and we have

$$(3.3) \qquad f(Z) = \left.\frac{f(Z_1) + n_2 J}{\lambda J}\right| \frac{\lambda J}{f(Z_2) + n_1 J} .$$

So with $Z_1 = J - I$ we have

$$n_1 + n_2 - a - 2 = \lambda = r_2 - a + n_1 - 1$$

which forces $n_2 - r_2 = 1 = n_1 - r_1$ implying Z is regular. One can similarly eliminate $Z_i = J$ for $n_i > 1$. Consider next the possibility $Z_1 = I$. From (3.2) and (3.3) we have

$$\lambda = n_2 = 1 + r_2 - a$$

so the constituent Z_2 satisfies

$$Z_2^2 + (n_2 - r_2 - 1)Z_2 = [(n_2 - 1)(r_2 - \lambda_2) - \lambda_2]I + \lambda_2 J$$

where $\lambda_2 = n_2 - n_1$ and the coefficient of I is determined by equating line sums. Since the elements of Z_2^2 cannot exceed r_2 we have

$$(n_2 - 1)(r_2 - \lambda_2) \leqq (n_2 - 1) \text{ with } r_2 \geqq \lambda_2 .$$

If $r_2 = \lambda_2 + 1$ we have $r_2 = n_2 - n_1 + 1$ whence $r_2 - n_2 = 1 - n_1 = r_1 - n_1$ and Z is regular. Thus $r_2 = \lambda_2$ and trace $Z_2 = 0$ with

$$Z_2^2 + (n_2 - r_2 - 1)Z_2 = r_2(J - I) . \tag{3.4}$$

Now row i and column i of Z_2 are different while $z_{ij} = 0$ forces row i equal column j. Hence there is at most one off diagonal zero in any row of Z_2 and $r_2 = n_2 - 2$, $a = -1$. It is then almost immediate from (3.4) that $r_2 = 1$, $n_2 = 3$ and

$$Z_2 \cong \begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} .$$

Then $\lambda_2 = n_2 - n_1 = 1$ so $n_1 = 2$ and we obtain (3.1). We now suppose neither constituent is a point, $J - I$ or I and assert:

$$\begin{aligned} \lambda + n + a - 2(r_1 + n_2) \geqq 0 \\ \lambda + n + a - 2(r_2 + n_1) \geqq 0 \end{aligned} . \tag{3.5}$$

This can be seen by considering $(0, -1)$-matrix $Z - J$. The quantities (3.5) are off diagonal entries in $f(Z - J)$ and for $Z_i \neq I$ these entries are nonnegative. But in view of (3.2) adding the entries in (3.5) gives zero.
Thus

$$\begin{aligned} r_1 = \frac{\lambda + n_1 - n_2 + a}{2} = \frac{\lambda_1 + n_1 + a}{2} \\ r_2 = \frac{\lambda + n_2 - n_1 + a}{2} = \frac{\lambda_2 + n_2 + a}{2} \end{aligned} \tag{3.6}$$

where $\lambda_i = \lambda + n_i - n$, $f(Z_i) = d_i I + \lambda_i J$ (see 3.3). Now $f(r_i) = d_i +$

$n_i\lambda_i$ gives $d_i = (r_i - \lambda_i)(n_i - r_i)$ and if Z_i has a zero in diagonal position we can assert that $d_i + \lambda_i \leqq r_i$. Whence

$$(r_i - \lambda_i)(n_i - r_i) \leqq (r_i - \lambda_i) ,$$

forcing $r_i = \lambda_i$ and $a = \lambda_i - n_i = r_i - n_i$. Thus Z is regular unless trace $(Z_i) = n_i$. But in that event we have $d_i + \lambda_i + a \leqq r_i$ which, viewed with (3.6) implies $d_i \leqq n_i - r_i$ or

$$(3.7) \qquad (r_i - \lambda_i)(n_i - r_i) \leqq (n_i - r_i) .$$

Hence $r_i = \lambda_i$, $\lambda_i + 1$. From (3.6) if we are to avoid $r_1 - n_1 = r_2 - n_2$ we conclude $r_1 = \lambda_1$, $r_2 = \lambda_2 + 1$ with $a = r_1 - n_1 = r_2 - n_2 + 1$. Now (3.3) and (3.2) imply $\lambda = \lambda_1 + n_2 = \lambda_2 + n_1 = r_1 + r_2 - a$ and hence $a = r_1 - n_1 + 1 = r_1 - n_1$ which contradiction completes the proof.

As all remaining non-regular quadratic graphs are cones they fall naturally into two classes: looped and unlooped. We determine these classes separately in the next two theorems.

THEOREM 3.2. *Let G be a non-regular $(x^2 - ax)$-graph which is the looped cone over G_1 carried by the matrix Z_1. Then one of the following holds:*

(i) *a is a positive integer and*

$$(3.8) \qquad Z_1 \cong \sum \oplus J_a .$$

(ii) $a = 1$ *and*

$$(3.9) \qquad Z_1 \cong \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}$$

(3.10) (iii) $a = 0$, $Z_1 = 0$.

Proof. The matrix Z_1 with line sums r_1 must satisfy

$$(3.11) \qquad Z_1^2 - aZ_1 = d_1 I + (r_1 - a)J$$

with $d_1 = (r_1 - n_1)(r_1 - a)$ [See (3.3)]. Note here that if $r_1 = n_1 - 1$ then $Z_1 = J - P$ for P a permutation. Now (3.11) will force $n_1 = 3$ and P carrying either of the cycles (123) or (132) yielding (3.9) of the theorem or $n_1 = 2$, $a = 1$ of case (i) $-$ (3.8). Hence we take $1 \leqq r_1 \leqq n_1 - 2$. We must have $r_1 \geqq a$ since $r_1 < a$ implies $d_1 + (a_1 - a) > 0$ and the entries of Z_1^2 do not exceed r_1. We further assert that $r_1 = a$ only for the family (i) of the theorem for here $Z_1^2 = r_1 Z_1$.

Quite generally from (3.11) we see that $z_{ij} = 1$, $i \neq j$, implies row i equal to column j. Further if some $z_{ii} = 0$ we have

$$d_1 + (r_1 - a) = (r_1 - n_1 + 1)(r_1 - a) \geqq 0\,,$$

and since $r_1 \geqq a$ we must conclude that $r_1 = a$ and obtain the family (i). The remaining candidates for Z_1 have trace n_1. If for some i, row i and column i are equal then evidently $d_1 = 0$ and $r_1 = a$ again. But we still have that row i and column j are indentical if $z_{ij} = 1$ for $i \neq j$. To avoid an occurrence of row i equals column i easily $r_1 = 1, 2$. In the former instance $Z_1 = I$ (of family (i)) in the latter since $i \neq j$ and $z_{ij} = 0$ forces row i and column j to meet in a 2-a positions we deduce that $a = 1$, $n_1 = 3 = r_1 + 1$ and we have case (ii).

Finally we treat the case of a non-looped cone. There are several such graphs as the next theorem shows.

THEOREM 3.3. *Let G be a non-regular $(x^2 - ax)$-graph which is a non-looped cone over G_1 carried by Z_1. Then one of the following holds.*

(i) $a = 0$ *and Z_1 is a symmetric permutation matrix or* $Z_1 = 0$.

(ii) $a = -1$ *and* $Z_1 \cong \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix}$

(iii) $a = 2$ *and*

$$Z_1 \cong \begin{pmatrix} 1&1&1&1&0&0&0&0 \\ 1&1&0&0&0&0&1&1 \\ 1&1&1&1&0&0&0&0 \\ 0&0&1&1&1&1&0&0 \\ 0&0&0&0&1&1&1&1 \\ 0&0&1&1&1&1&0&0 \\ 1&1&0&0&0&0&1&1 \\ 0&0&0&0&1&1&1&1 \end{pmatrix}$$

(iv) $a = +1$ *and Z_1 is equivalent to one of the following six matrices:*

$$\text{(a)}\ \begin{matrix} 1&0&1&1&0 \\ 0&1&0&1&1 \\ 1&0&1&0&1 \\ 1&1&0&1&0 \\ 0&1&1&0&1 \end{matrix} \qquad \text{(b)}\ \begin{matrix} 0&1&1&1&0&0 \\ 1&1&0&0&0&1 \\ 0&1&1&1&0&0 \\ 0&0&1&1&1&0 \\ 1&0&0&0&1&1 \\ 1&0&0&0&1&1 \end{matrix} \qquad \text{(c)}\ \begin{matrix} 0&1&1&1&0&0&0 \\ 0&1&0&1&1&0&0 \\ 0&1&1&0&0&0&1 \\ 0&0&1&1&0&1&0 \\ 1&0&0&0&1&0&1 \\ 1&0&0&0&1&1&0 \\ 1&0&0&0&0&1&1 \end{matrix}$$

(d) $\begin{matrix} 1&1&1&0&0&0\\ 1&1&0&0&0&1\\ 0&0&1&1&1&0\\ 1&0&0&1&0&1\\ 0&1&1&0&1&0\\ 0&0&0&1&1&1 \end{matrix}$ (e) $\begin{matrix} 1&1&1&1&0&0&0\\ 0&1&1&0&0&1&1\\ 0&0&1&1&1&1&0\\ 0&1&0&1&1&0&1\\ 1&1&0&0&1&1&0\\ 1&0&0&1&0&1&1\\ 1&0&1&0&1&0&1 \end{matrix}$ (f) $\begin{matrix} 1&1&1&1&0&0\\ 1&1&0&0&1&1\\ 1&1&1&1&0&0\\ 0&0&1&1&1&1\\ 1&1&0&0&1&1\\ 0&0&1&1&1&1 \end{matrix}$

(v) *$a + 2$ is a positive integer and*

(a) $Z_1 \cong \sum \oplus (J - I)_{a+2}$ *or* (b) $Z_1 \cong \left[\begin{array}{c|c|c} 0 & & \\ \cdot & & \\ \cdot & J_{a+2} & 0_{a+2,a+1} \\ \cdot & & \\ 0 & & \\ \hline 1 & & \\ \cdot & & \\ \cdot & 0_{a+2} & J_{a+2,a+1} \\ \cdot & & \\ 1 & & \end{array}\right]$

Proof. We have

$$Z_1^2 - aZ_1 = d_1 I + (r_1 - a - 1)J \tag{3.12}$$

with

$$d_1 = (r_1 - a)(r_1 - n_1) + n_1 = (r_1 - a - 1)(r_1 - n_1) + r_1 \tag{3.13}$$

using (3.3) and $f(r_1) = d_1 + n_1(r_1 - a - 1)$. We note that $r_1 = 1$ easily gives $a = 0$ with Z_1 a symmetric permutation or $a = -1$ with Z_1 carrying either of the three cycles (123) or (132). We also note that the preceding Theorm 3.2 finds all candidates here with trace zero. For then $W = Z + I$ carries a non-regular $(x^2 - (a + 2)x)$-graph with a looped point as one constituent. These observations give cases (ii) and (*va*) of the present theorem as the only Z_1's with trace $Z_1 = 0$. We further remark that $Z_1 \neq J$ and that the examples (*va*) are characterized by $r_1 = a + 1$. For in this case $d_1 = r_1$ and $Z_1^2 - aZ_1 = (a + 1)I$, so that if Z_1 has a diagonal entry equal to one a cannot be positive since the entries of Z_1^2 do not exceed $a + 1$. Thus $a = 0$, $r_1 = 1$ discussed above, or $a = -1$, $Z_1 = 0$, case *v* with $a = -1$. The remaining possibilities have trace $Z_1 = 0$ and are also discussed above.

We have to consider then trace Z_1 positive and easily here:

$$a + 1 < r_1 < n_1 - 1 \,. \tag{3.14}$$

We first suppose some $Z_{ii} = 0$. With $Z_1^2 = (Q_{ij})$ we have

(3.15) $$Q_{ii} = (r_1 - a - 1)(r_1 - n_1 + 1) + r_1 .$$

But surely $r_1 - Q_{ii} \leqq n_1 - r_1 - 1$ so that (3.15) implies

(3.16) $$(n_1 - r_1 - 1)(r_1 - a - 1) \leqq (n_1 - r_1 - 1) .$$

Thus $r_1 = a + 2$ and we may suppose $a \geqq 0$. Now (3.12) will become

(3.17) $$Z_1^2 - aZ_1 = [2(a + 2) - n_1]I + J .$$

Take

(3.18) $$Z_1 \cong \begin{array}{c|c|c}0 & \overbrace{1 \text{---} 1}^{a+2} & 0 \text{---} 0 \\ \hline & B & * \\ \hline 1 & & \\ \vdots & 0 & * \\ 1 & & \end{array} \quad \left.\vphantom{B}\right\} a + 2 .$$

The matrix B of (3.18) has column sums $(a + 1)$ and so contains $(a + 2)$ zeros. Thus B has a row with at most one zero. This row will meet any column of B in at least a positions. If this row has a zero in an off-diagonal position we have $a = 0, 1$. If this row has its single zero on the diagonal, say $b_{11} = Z_{22} = 0$ then $Z_{21} = 1$ so that row 2 of Z_1 can meet column 3 of Z_1 in the proper number of positions ($z_{23} = 1 \Rightarrow Q_{23} = a + 1$). But then $Q_{21} = a + 1 = 2a + 4 - n_1$ so $n_1 = a + 3$ and Z would be regular with $r_i - n_i = -1$. Thus unless B has a row of all ones $a = 0, 1$. We consider the case that B indeed has a row of all ones.

Placing this row initially in B and maintaining equivalence a look at row two and column one of Z_1 gives $n_1 = 2(a+2)$, $Z_1^2 - aZ_1 = J$. Recall we are avoiding a row in B with precisely one zero, so that B has a row with at least two zeros. This row in Z_1 has a one in its first position and by checking the row and column through its off-diagonal zero we see this row in B has at most one nonzero entry. This row accounts for at least $a + 1$ of the zeros of B. Were the remaining zero not in this row we would have a row in B with precisely one zero. Thus B has a zero row and all other rows are full. This gives Z_1 the form of case v.

We are left with the cases $a = 0, 1$ and note from (3.17) that $4 \leqq n_1 \leqq 7$. With $a = 0$, $n_1 = 4, 5$ and $n_1 = 5$ gives trace $(Z_1) = 0$ while $n_1 = 4$ gives (*vb*) with $a = 0$. The choice $a = 1$, $n_1 = 5$ is easily eliminated and, for $n_1 = 6$, one obtains (*ivb*) and (*vb*). Finally with $n_1 = 7$ the matrix (*ivc*) is obtained after some work.

As this discussion was on the assumption of some $z_{ii} = 0$ we are left to investigate those Z_1 satisfying (3.12), (3.13) and trace $Z_1 = n_1$. To that end we investigate $W = Z_1 - I$. Evidently

$$W^2 + (2 - a)W = (d_1 + a - 1)I + (r_1 - a - 1)J \; . \tag{3.19}$$

Let $S = r_1 - 1$, the line sum of W. Consider the principal submatrix B of W through the columns with ones in row one. This $S \times S$ matrix B has column sums $S - 2$ and trace zero. So B has a row with sum at least $S-2$ and this row will meet the corresponding column of B in at least $S - 3$ positions. Thus $d_1 + S - 1 \geqq S - 3$ or $d_1 \geqq -2$. Since $d_1 > 1$ would force corresponding hits in W to exceed S we deduce

$$-2 \leqq d_1 \leqq 1 \; . \tag{3.20}$$

Now with $d_1 = -2$ we have that B has line sums $S - 2$ and row i of B hits column i of Z for $2 \leqq i \leqq S + 1$ in a minimum of $S - 3$ positions. Since now $d_1 + S - 1 = S - 3$ we conclude that column one of Z has zeros in positions 2 through $S + 1$ forcing row 1 to miss column 1 and $d_1 + s - 1 = 0$ so $S = 3$. One then easily eliminates $a = 0$ and $a = 3$ and $a = 2$ forces $W^2 = J - I$ of order 10. But $J - I$ of order 10 has a negative determinant, so we are left with $a = 1$, $n_1 = 7$ and the matrix IVe pops up unique to within equivalence.

We proceed with the cases on d_1 according to (3.20). If $d_1 = -1$ (3.19) becomes

$$W^2 + (2 - a)W = (a - 2)I + (S - a)J \tag{3.21}$$

with

$$(n_1 - S - 1)(S - a) = S + 2 \; . \tag{3.22}$$

The eigenvalues of W are then S of multiplicity one ($S \neq a$ so W is irreducible) and then the roots of $x^2 + (2 - a)x + (2 - a) = 0$. These roots are $(a - 2) \pm \sqrt{a^2 - 4})/2$. For $a \neq 2$ these are irrational or imaginary so that trace $W = 0$ means n_1 is odd and

$$S + \frac{n_1 - 1}{2}(a - 2) = 0 \; .$$

Then $a = 0, 1$ with $S = n_1 - 1$, $n_1 - 1/2$ respectively. The former gives $Z_1 = J$ earlier eliminated and the latter is incompatible with (3.22). In case $\mathrm{r} = 2$ we have $W^2 = (S - 2)J$. The eigenvalues of W are then S and 0 denying trace $W = 0$.

The case $d_1 = +1$ is similarly eliminated as follows. The eigenvalues of W other than S are

$$\frac{a-2\pm\sqrt{a^2+4}}{2}\,.$$

For $a\neq 0$ these are not rational. Hence n_1 is odd and $S+(n_1-1)/2(a-2)=0$, yielding $a=1$, $S=(n_1-1)/2$. This is incompatible with $S^2+(2-a)S=d_1+S-1+(n_1-1)(S-a)$ unless $S=2$, $n_1=5$, $W^2+W=(I+J)_5$ yielding the matrix (iva). In case $a=0$ we have $W^2+2W=SJ$, $S^2+2S=n_1S$ so $S=n_1-2$, $r_1=n_1-1$ and Z is regular.

The final case is $d_1=0$. Here the above techniques fail as the spectrum of W is $\{a-1,-1,S\}$ with appropriate multiplicities. So consider the structure of W:

$$W\cong\left[\begin{array}{c|ccc|c} 0 & \overbrace{1\ \cdots\ 1}^{S} & & & 0\ \cdots\ 0 \\ \hline 1 & 0 & & & \\ \vdots & B & \ddots & \alpha & C \\ 1 & & & & \\ 0 & & & 0 & \\ \hline 1 & 1\cdots 1\,0\,0 & & & 0\,1\,0\cdots 0 \\ 0 & & & \varepsilon & \ddots \\ \vdots & & & & \\ 0 & & & & 0 \end{array}\right]. \tag{3.23}$$

It is not difficult to see that W has the structure of (3.23). The $(S+2)$rd row being obtained by considering its inner product with column one. Now investigating this row and column $S+1$ recalling that B has column sums $S-2$ we have

$$1+\tau+\varepsilon=S-a\,. \tag{3.24}$$

Where $\varepsilon=0,1$, $\tau=S-2$, $S-3$ depending on whether α is zero or one. In any case (3.24) shows $a=0,1,2$. From our remaks about the spectrum of W it follows that if $a=0$ $n_1=S+1$ and $Z_1=J$[1]. If $a=1$ we have $W^2+W=(S-1)T$ so $S^2+S=n_1(S-1)$ yielding

$$S=\frac{(n_1-1)\pm\sqrt{(n_1-3)^2-8}}{2}\,. \tag{3.25}$$

From (3.25) and the fact that S is a nonnegative integer with $S<n_1-1$ we conclude that $n_1=6$, $S=2,3$. These parameters give cases (ivd) and (ivf).

[1] $\Lambda(W)=\{S,-1,a-1\}$, if m is the multiplicity of $a-1=-1$ we have $S-m-(n_1-m-1)=0$ or $S=n_1-1$.

Finally for $a = 2$ we have

$$W^2 = I + (S - 2)J \text{ with } S^2 = 1 + n_1(S - 2)$$

or

$$S = \frac{n_1 \pm \sqrt{(n_1 - 4)^2 - 12}}{2}. \tag{3.26}$$

Then (3.26) forces $n_1 = 8$, $S = 3, 5$. In case $S = 3$ one obtains the matrix (iii) and the case $S = 5$ violates $a \leqq n_1 - S - 2$ easily seen to be necessary from (3.23) since the matrix C has column $S - a$.

We remark in conclusion that the various matrices in Theorem 3.3 are easily seen to be non-equivalent by considerations of parameters and trace.

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Received October 6, 1970 and in revised form March 16, 1971. This research was supported in part by ARO contract DA 131-124-ARO(D)-336.

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