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GENERALIZED HAUSDORFF-YOUNG INEQUALITIES AND MIXED NORM SPACES

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We generalize the Hausdorff-Young Theorem for a locally compact connected group G by showing that if $f \in L^p(G)$, $1 < p \leqq 2$, then the Fourier transform of f is in a mixed norm space properly contained in $L^{p'}(\Gamma)$, where Γ is the dual group and $1/p + 1/p' = 1$. In the last section we apply the above theorem to obtain new results concerning sets of uniqueness for functions in $L^p(G)$, and we give new sufficient conditions which insure that the product of a continuous function and a pseudomeasure is the zero distribution.

1. Introduction.

The classical Hausdorff-Young Theorem states that if f belongs to the Lebesgue space L^p (of the unit circle), $1 < p \leqq 2$ and $\{f_n\}$ is its sequence of Fourier coefficients, then

$$\left\{\sum_{n=-\infty}^{\infty} |f_n|^{p'}\right\}^{1/p'} \leqq ||f||_p \text{ where } \frac{1}{p} + \frac{1}{p'} = 1 . \tag{1.1}$$

The companion dual result asserts that a sufficient condition for the sequence $\{f_n\}$ to be the Fourier coefficients of a function in $L^{p'}$ is

$$\left\{\sum_{n=-\infty}^{\infty} |f_n|^{p}\right\}^{1/p} < \infty . \tag{1.2}$$

In a recent paper [7] Kellogg applied a multiplier theorem of Hedlund to obtain a significant improvement of these inequalities. Precisely, he replaced (1.1) by

$$\left\{\sum_{k=-\infty}^{\infty} \left(\sum_{n \in I_k} |f_n|^{p'}\right)^{2/p'}\right\}^{1/2} \leqq A_p ||f||_p \tag{1.3}$$

and (1.2) by

$$\left\{\sum_{k=-\infty}^{\infty} \left(\sum_{n \in I_k} |f_n|^{p}\right)^{2/p}\right\}^{1/2} < \infty , \tag{1.4}$$

where A_p is a constant depending only on p and the sets I_k are the lacunary blocks defined by $I_k = \{j \in Z\colon 2^{k-1} \leqq j < 2^k\}$ if $k > 0$, $I_0 = \{0\}$ and $I_k = -I_{-k}$ if $k < 0$. These inequalities rest ultimately upon an extension of the Riesz-Thorin interpolation theorem and the following result of Paley and Hardy and Littlewood [3]: A complex number sequence $\{\lambda_n\}$ has the property $\sum_{n=0}^{\infty} \lambda_n a_n z^n \in H^2$ whenever $\sum_{n=0}^{\infty} a_n z^n \in H^1$ if and only if

$$\sup_{k \geq 0} \sum_{n \in I_k} |\lambda_n|^2 < \infty \,. \tag{1.5}$$

This suggests that the proper generalization of inequalities (1.3) and (1.4) is to the setting in which the unit circle T is replaced by a locally compact abelian (LCA) group G with ordered dual Γ-for the notion of lacunary decomposition extends naturally to ordered groups.

In this paper we carry out this extension and give some applications of these generalized inequalities. Specifically, in §3 we strengthen a theorem of Katznelson [6] and of Figà-Talamanca and Gaudry [2] on sets of uniqueness in $L^p(G)$. We also give stronger sufficient conditions than those found in Edwards [1] for the distribution fS to be zero where S is a pseudomeasure and f is an element of $C(T)$ with absolutely convergent Fourier expansion.

I wish to acknowledge the work of J. H. Wells in extending Kellogg's result to the real line and to express my appreciation for his assistance in the preparation of this paper.

2. Main results. Let G be a compact connected abelian group with dual Γ ordered by a set P of positive elements. For $1 \leqq p \leqq \infty$ define $H^p(G)$ to be the set of all $f \in L^p(G)$ such that $\hat{f}(\gamma) = 0$ for $\gamma \notin P$. (Here and subsequently $\hat{f}$ denotes the Fourier transform of f.)

DEFINITION. A lacunary decomposition of P is a countable collection $\mathscr{D} = \{D_i\}_{i=1}^\infty$ of subsets of P satisfying:
(1) $D_i \cap D_j = \phi$ if $i \neq j$, and
(2) for each i there exists an $\alpha_i \in P$ such that

$$D_i = \{\alpha \colon \alpha_i < \alpha \leqq 2\alpha_i\} \,.$$

Corresponding to such a decomposition $\mathscr{D}$ we define the mixed norm space $L^{r,s}_{\mathscr{D}}(P)$ to be the set of all $f \in \ell^\infty(P)$ with support contained in $\bigcup_{i=1}^\infty D_i$ and $\|f\|_{r,s,\mathscr{D}} < \infty$ where

$$\|f\|_{r,s;\mathscr{D}} = \begin{cases} \{\sum_{i=1}^{\infty} (\sum_{\alpha \in D_i} |f(\alpha)|^r)^{s/r}\}^{1/s}, \ 1 \leqq r < \infty, 1 \leqq s < \infty \,. \\ \sup_{1 \leqq i < \infty} (\sum_{\alpha \in D_i} |f(\alpha)|^r)^{1/r}, \ 1 \leqq r < \infty, s = \infty \,. \end{cases}$$

Also we define $L^{r,\infty}(P)$ to be the set of all $f \in \ell^\infty(P)$ such that $\|f\|_{r,\infty} = \sup_{\alpha \in P} (\sum_{\beta \in I_\alpha} |f(\beta)|^r)^{1/r} < \infty$ where $1 \leqq r < \infty$ and $I_\alpha = \{\beta \in P \colon \alpha < \beta \leqq 2\alpha\}$. It is easy to verify that these spaces are Banach spaces and the Banach conjugate of $L^{r,s}_{\mathscr{D}}(P)$ is $L^{r',s'}_{\mathscr{D}}(P)$ if $r, s < \infty$ where $1/r + 1/r' = 1/s + 1/s' = 1$.

If r, s, u and v are real numbers in $[1, \infty]$ then $(L^{r,s}_{\mathscr{D}}(P), L^{u,v}_{\mathscr{D}}(P))$, the multipliers from $L^{r,s}_{\mathscr{D}}(P)$ into $L^{u,v}_{\mathscr{D}}(P)$, is the set of all $\lambda \in \ell^\infty(P)$

with the property that $\lambda f \in L^{u,v}_{\mathscr{D}}(P)$ whenever $f \in L^{r,s}_{\mathscr{D}}(P)$. Each such λ determines a bounded linear operator from $L^{r,s}_{\mathscr{D}}(P)$ into $L^{u,v}_{\mathscr{D}}(P)$ whose norm will be called the multiplier norm of λ.

We record the following theorem from [7].

THEOREM 1. *Let $1/p = 1/u - 1/r$ if $r > u$, $p = \infty$ if $r \leqq u$ and let $1/q = 1/v - 1/s$ if $s > v$ and $q = \infty$ if $s \leqq v$. Then $(L^{r,s}_{\mathscr{D}}(P), L^{u,v}_{\mathscr{D}}(P)) = L^{p,q}_{\mathscr{D}}(P)$ and if $\lambda \in L^{p,q}_{\mathscr{D}}(P)$, its multiplier norm is $\|\lambda\|_{p,q;\mathscr{D}}$.*

If A and B are subsets of $L^1(G)$, the multiplier space $(\hat{A}, \hat{B})$ is the space of all measurable complex-valued functions λ on Γ such that for every $f \in A$ there exists $g \in B$ with $\lambda\hat{f} = \hat{g}$. Hedlund [5, p. 54] shows that $L^{2p/2-p,\infty}(P) \subset (\hat{H}^p(G), \hat{H}^2(G))$ for $1 \leqq p \leqq 2$. This follows by showing that $L^{2p/2-p,\infty}_{\mathscr{D}}(P) \subset (\hat{H}^p(G), \hat{H}^2(G))$ for any lacunary decomposition $\mathscr{D}$ and that for $f \in H^p(G)$ and $\lambda \in L^{2p/2-p,\infty}_{\mathscr{D}}(P)$, $\|\lambda\hat{f}\|_2 \leqq K_p \|f\|_p \|\lambda\|_{2p/2-p,\infty;\mathscr{D}}$. Here K_p is a constant independent of the decomposition $\mathscr{D}$.

THEOREM 2. *If $\mathscr{D}$ is a lacunary decomposition of P and $f \in H^p(G)$, $1 \leqq p \leqq 2$, then $\hat{f} \in L^{p',2}_{\mathscr{D}}(P)$ and there exists a constant A_p independent of $\mathscr{D}$ and f so that $\|\hat{f}\|_{p',2;\mathscr{D}} \leqq A_p \|f\|_p$.*

Proof. By Hedlund's result we have $L^{2p/2-p,\infty}_{\mathscr{D}}(P) \subset (\hat{H}^p(G), \hat{H}^2(G))$ and

$$\|\lambda\hat{f}\|_2 = \|\lambda\hat{f}\|_{2,2;\mathscr{D}} \leqq K_p \|f\|_p \|\lambda\|_{2p/2-p,\infty;\mathscr{D}} \tag{2.1}$$

for $f \in H^p(G)$ and $\lambda \in L^{2p/2-p,\infty}_{\mathscr{D}}(P)$, where K_p is a constant independent of $\mathscr{D}$. Then by Theorem 1

$$\hat{H}^p(G) \subset (L^{2p/2-p,\infty}_{\mathscr{D}}(P), \hat{H}^2(G)) = (L^{2p/2-p,\infty}_{\mathscr{D}}(P), L^{2,2}_{\mathscr{D}}(P)) = L^{p',2}_{\mathscr{D}}(P) ,$$

and $\|\hat{f}\|_{p',2;\mathscr{D}}$ is the smallest number which satisfies (2.1) for all $\lambda \in L^{2p/2-p,\infty}_{\mathscr{D}}(P)$. The theorem now follows with $A_p = K_p$.

The Riesz Projection Theorem allows us to extend this result to $L^p(G)$.

THEOREM 3. *For $1 < p \leqq 2$ there is a constant A_p such that if $\mathscr{D}$ is a lacunary decomposition of P and $f \in L^p(G)$, then*

$$\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{\gamma \in D_k} |\hat{f}(\gamma)|^{p'}\right)^{2/p'}\right\}^{1/2} \leqq A_p \|f\|_p ,$$

where $D_k = -D_{-k}$ for $k < 0$, $D_0 = \phi$.

We now state the main result.

THEOREM 4. *Let n be a positive integer and G a compact connected abelian group with ordered dual Γ. If $f \in L^p(R^n \oplus G)$, $1 < p \leqq 2$, then*

$$\left\{\sum_{k=-\infty}^{\infty}\left(\int_{J_k \oplus R^{n-1}} \sum_{\Gamma} |\hat{f}(t, \gamma)|^{p'} dt\right)^{2/p'}\right\}^{1/2} \leqq A_p \|f\|_p ,$$

where A_p is a constant, $J_k = [2^{k-1}, 2^k]$ for $k > 0, J_0 = [-1, 1]$, and $J_k = -J_{-k}$ for $k < 0$.

Now by [8, p.40] any LCA connected group is of the form $R^n \oplus G$ where G is compact and connected and $n \geqq 0$. Hence the combination of Theorems 3 and 4 yields the desired extension of Kellogg's result.

Proof of Theorem 4. We begin with a special version of Theorem 3. Consider the compact connected group $T^n \oplus G$ which has dual $Z^n \oplus \Gamma$. We introduce an order by defining the following set $\bar{P}$ of positive elements:

$$((k_1, \cdots, k_n), \gamma) \in \bar{P} \text{ if either}$$

(1) $k_j > 0$ where $1 \leqq j \leqq n$ and $k_i = 0$ for $i < j$; or
(2) $k_j = 0$ for each j and $\gamma \in P$, the set of positive elements in Γ.
Let $\mathscr{D}$ be the lacunary decomposition determined by $\{\alpha_i\}_{i=1}^{\infty}$ where $\alpha_i = ((2^{i-1}, 0, \cdots, 0), 0) \in Z^n \oplus \Gamma$. By Theorem 3 there is a constant B_p such that if $f \in L^p(T^n \oplus G)$ then

$$\left\{\sum_{i=-\infty}^{\infty}\left(\sum_{D_i} |\hat{f}|^{p'}\right)^{2/p'}\right\}^{1/2} \leqq B_p \|f\|_p .$$

For $m \in Z$, let I_m be the lacunary block described in the introduction. If $m > 1$, $I_m \oplus Z^{n-1} \oplus \Gamma \subset D_{m-1} \cup D_m$ and if $m < -1$,

$$I_m \oplus Z^{n-1} \oplus \Gamma \subset D_m \cup D_{m+1} .$$

By the standard Hausdorff-Young Theorem

$$\left\{\left(\sum_{I_m \oplus Z^{n-1} \oplus \Gamma} |\hat{f}|^{p'}\right)^{2/p'}\right\}^{1/2} \leqq \|f\|_p, \quad m = -1, 0, 1.$$

Then by the triangular inequality

$$\left\{\sum_{m=-\infty}^{\infty}\left(\sum_{I_m \oplus Z^{n-1} \oplus \Gamma} |\hat{f}|^{p'}\right)^{2/p'}\right\}^{1/2} \leqq C_p \|f\|_p \tag{2.2}$$

for all $f \in L^p(T^n \oplus G)$, where $C_p = 2B_p + 3$.

Now let f be a continuous function on $R^n \oplus G$ which has compact support and choose a positive number L such that $f(t, x) = 0$ for

$(t, x) \in R^n \oplus G$ and $\|t\|_\infty > L$. Choose positive integers m and ℓ so that $m < \ell$ and $2^{m-\ell}L < \pi$. Let $h(s, x) = f(s_1 2^{\ell-m}, \cdots, s_n 2^{\ell-m}, x)$ $(2^{\ell-m})^n(2\pi)^n$, $(s, x) \in T^n \oplus G$, where we identify T with $R/2\pi$. Then $h \in L^p(T^n \oplus G)$ and for $(k_1, \cdots, k_n, \gamma) \in Z^n \oplus \Gamma$

$$\begin{aligned}
&\hat{h}(k_1, \cdots, k_n, \gamma) \\
&= \int_{-2^{m-\ell}L}^{2^{m-\ell}L} \cdots \int_{-2^{m-\ell}L}^{2^{m-\ell}L} \int_G e^{-ik_1s_1} \cdots e^{-ik_ns_n}(\gamma, -x) f(s2^{\ell-m}, x)(2^{\ell-m})^n \, dxds \\
&= \int_{-L}^{L} \cdots \int_{-L}^{L} \int_G e^{-ik_1 2^{m-\ell}t_1} \cdots e^{-ik_n 2^{m-\ell}t_n}(\gamma, -x) f(t, x) \, dxdt \\
&= \hat{f}(k_1 2^{m-\ell}, \cdots, k_n 2^{m-\ell}, \gamma) .
\end{aligned}$$

From (2.2) we obtain

$$\sum_{j=-\infty}^{\infty} \Big(\sum_{I_j \oplus Z^{n-1} \oplus \Gamma} |\hat{h}(k, \gamma)|^{p'} \Big)^{2/p'} \leqq C_p^2 \|h\|_p^2 . \tag{2.3}$$

Since $\|h\|_p = (2^{\ell-m})^{n/p'}(2\pi)^{n/p'} \|f\|_p$ we may rewrite (2.3) in the form

$$\sum_{j=-\infty}^{\infty} \Big(\sum_{I_j \oplus Z^{n-1} \oplus \Gamma} |\hat{f}(k_1 2^{m-\ell}, \cdots, k_n 2^{m-\ell}, \gamma)|^{p'} (2^{\ell-m})^{-n} \Big)^{2/p'} \leqq C_p^2 (2\pi)^{2n/p'} \|f\|_p^2$$

so that, in particular,

$$\sum_{j=-\ell}^{\ell} \Big(\sum_{I_j \oplus Z^{n-1} \oplus \Gamma} |\hat{f}(k_1 2^{m-\ell}, \cdots, k_n 2^{m-\ell}, \gamma)|^{p'} (2^{m-\ell})^{n} \Big)^{2/p'} \leqq A_p^2 \|f\|_p^2 \tag{2.4}$$

where $A_p = C_p (2\pi)^{n/p'}$.

Now for $r = 0, \cdots, m-1$ it is evident that certain partial sums of

$$\begin{aligned}
&\sum_{I_{\ell-r} \oplus Z^{n-1} \oplus \Gamma} |\hat{f}(k_1 2^{m-\ell}, \cdots, k_n 2^{m-\ell}, \gamma)|^{p'} (2^{m-\ell})^n \\
&\qquad = \sum_{k_1 = 2^{\ell-r-1}}^{2^{\ell-r}-1} \sum_{k_2, \cdots, k_n \in Z} \sum_{\gamma \in \Gamma} |\hat{f}(k_1 2^{m-\ell}, \cdots, k_n 2^{m-\ell}, \gamma)|^{p'} (2^{m-\ell})^n
\end{aligned}$$

will converge to $\int_{2^{m-r-1}}^{2^{m-r}} \int_{R^{n-1}} \sum_\Gamma |\hat{f}(t, \gamma)|^{p'} dt$ as ℓ tends to infinity. Similarly, for $r = 0, -1, \cdots, -m+1$, the limit superior of the sum

$$\sum_{I_{-(\ell+r)} \oplus Z^{n-1} \oplus \Gamma} |\hat{f}(k_1 2^{m-\ell}, \cdots, k_n 2^{m-\ell}, \gamma)|^{p'} (2^{m-\ell})^n$$

is not less than $\int_{-2^{m+r}}^{-2^{m+r-1}} \int_{R^{n-1}} \sum_\Gamma |\hat{f}(t, \gamma)|^{p'} \, dt$. Also since $2/p' \leqq 1$,

$$\begin{aligned}
&\sum_{j=-(\ell-m)}^{\ell-m} \Big(\sum_{I_j \oplus Z^{n-1} \oplus \Gamma} |\hat{f}(k_1 2^{m-\ell}, \cdots, k_n 2^{m-\ell}, \gamma)|^{p'} (2^{m-\ell})^n \Big)^{2/p'} \\
&\qquad \geqq \Big\{ \sum_{k_1 = -2^{\ell-m}+1}^{2^{\ell-m}-1} \sum_{Z^{n-1} \oplus \Gamma} |\hat{f}(k_1 2^{m-\ell}, \cdots, k_n 2^{m-\ell}, \gamma)|^{p'} (2^{m-\ell})^n \Big\}^{2/p'} ,
\end{aligned}$$

and the limit superior of the second sum dominates the integral

$$\left(\int_{-1}^{1}\int_{R^{n-1}}\sum_{\Gamma}|\hat{f}(t,\gamma)|^{p'}\,dt\right)^{2/p'}.$$

Thus, by letting ℓ tend to infinity in (2.4), we obtain

$$\left\{\sum_{k=-m}^{m}\left(\int_{J_k\oplus R^{n-1}}\sum_{\Gamma}|\hat{f}(t,\gamma)|^{p'}\,dt\right)^{2/p'}\right\}^{1/2}\leqq A_p\,||f||_p\,,$$

and since m is arbitrary, our proof is complete for continuous functions with compact support. The general result now follows since such functions are dense in $L^p(R^n\oplus G)$.

There is also a dual to this generalized Hausdorff-Young Theorem in case G is a compact group. Suppose $\{\Gamma_k\}_{k=-\infty}^{\infty}$ is a collection of subsets of Γ for which a generalized Hausdorff-Young Theorem holds (e.g., Theorem 3). For p and q in $[1,\infty)$ let $L^{p,q}(\Gamma)$ be the space of all measurable complexvalued functions λ on Γ satisfying

$$||\lambda||_{p,q}=\left\{\sum_{k=-\infty}^{\infty}\left(\int_{\Gamma_k}|\lambda(\gamma)|^p d\gamma\right)^{q/p}\right\}^{1/q}<\infty\,.$$

THEOREM 5. *If $1<p\leqq 2$ and $\lambda\in L^{p,2}(\Gamma)$ then there exists $f\in L^{p'}(G)$ with $\hat{f}=\lambda$ and $||f||_{p'}\leqq B_p\,||\lambda||_{p,2}$, where B_p is a constant depending only on p.*

Proof. Since $L^1(G)*L^p(G)=L^p(G)$ ($*$ denotes convolution) we have $\hat{L}^1(G)\subset(\hat{L}^p(G),\hat{L}^p(G))$. By assumption a generalized Hausdorff-Young Theorem holds in $L^p(G)$. Thus $\hat{L}^p(G)\subset L^{p',2}(\Gamma)$ and the previous inclusion implies that $\hat{L}^1(G)\subset(\hat{L}^p(G),L^{p',2}(\Gamma))$. But $(\hat{L}^p(G), L^{p',2}(\Gamma))=(L^{p,2}(\Gamma),\hat{L}^{p'}(G))$ by duality; hence $\hat{L}^1(G)\subset(L^{p,2}(\Gamma),\hat{L}^{p'}(G))$ or, equivalently, $L^{p,2}(\Gamma)\subset(\hat{L}^1(G),\hat{L}^{p'}(G))$. However $(\hat{L}^1(G),\hat{L}^{p'}(G))$ is known to be $\hat{L}^{p'}(G)$ [1, p.255] so that $L^{p,2}(\Gamma)\subset\hat{L}^{p'}(G)$. An application of the closed graph theorem establishes existence of the constant B_p.

3. Applications. Let A denote the space of all functions f continuous on T which have absolutely convergent trigonometric expansions and norm given by

$$||f||_A=\sum_{n=-\infty}^{\infty}|\hat{f}(n)|\,.$$

With multiplication defined as the pointwise product of functions, A becomes a Banach algebra. Each element of the dual of A may be identified with a pseudomeasure S, that is, a distribution on $C^\infty(T)$ whose sequence of Fourier coefficients $\hat{S}(n)=S(e^{-in\theta})$ $(n\in Z)$ belong to ℓ^∞. Given a pseudomeasure S and an f in A the distribution fS

is defined by $fS(u) = S(fu)$ for $u \in C^\infty(T)$. In [1] Edwards shows that fS is the zero distribution when f and S satisfy the following conditions:

(a) $S \in \ell^q, 1 \leqq q \leqq \infty$;

(b) f vanishes on supp $(S) = E$; and

(c) there exists a sequence of positive numbers ε_j tending to zero such that $f(x) = O(\varepsilon_j^{1/2-1/q})$ for all x such that dist $(x, E) \leqq \varepsilon_j$.

THEOREM 6. *Suppose S is a pseudomeasure and $f \in A$. If conditions* (b) *and* (c) *above hold and either*

$$(a_1)\ \hat{S} \in \ell^{q,2} \text{ if } 1 < q \leqq 2$$

or

$$(a_2)\ \hat{S} \in \ell^{q,\infty} \text{ if } 2 < q < \infty$$

then $fS = 0$.

Proof. If $1 < q \leqq 2$ it follows from Theorem 5 that $\hat{S} = \hat{g}$ for some $g \in L^{q'}(T)$. Therefore $S(u) = \int_{-\pi}^{\pi} u(t)g(t)\ dt/2\pi$, $u \in C^\infty(T)$, and supp $(S) =$ supp (g). But $fS(u) = S(fu) = \int_{-\pi}^{\pi} f(t)u(t)g(t)\ dt/2\pi$, $u \in C^\infty(T)$, and supp $(f) \cap$ supp $(g) = \phi$; hence $fS = 0$.

Now $\ell^q \subsetneqq \ell^{q,2}$ when $q < 2$, so we have a stronger result in this case.

For the case $q > 2$, we need only apply our generalized Hausdorff-Young theorem in 13.5.5 and the appropriate version of Hölders inequality in 13.5.9 of the argument in Edwards [1, pp.101-102] to obtain $fS = 0$ when $\hat{S} \in \ell^{q,\infty}$ and f satisfies (b) and (c).

The condition $\hat{S} \in \ell^{q,\infty}$ when $2 < q < \infty$ is a significant weakening of condition (a) since it does not require that $\lim_{|n|\to\infty} |\hat{S}(n)| = 0$.

Our second application concerns sets of uniqueness for $L^p(G)$, $1 < p < 2$. In a recent paper which generalized earlier work of Katznelson [6, p.101-103], Figa-Talamanca and Gaudry proved the existence of sets $E \subset G$ of positive Haar measure such that if $f \in L^1(G)$, supp $(f) \subset E$ and $\hat{f} \in L^p(G)$ for some $p, 1 \leqq p < 2$, then $f = 0$. Here G is a nondiscrete LCA group with dual Γ.

For our generalization we shall only need to assume that there is a decomposition $\{\Gamma_k\}_{k=-\infty}^{\infty}$ of Γ for which a generalized Hausdorff-Young Theorem holds and that when f is measurable on Γ and $1 < p < 2$ then

$$\sum_{k=-\infty}^{\infty} \left(\int_{\Gamma_k} |f(\gamma)|^p d\gamma \right)^{2/p} < \infty$$

always implies

$$\sum_{k=-\infty}^{\infty}\left(\int_{\Gamma_k} |f(\beta-\gamma)|^p d\gamma\right)^{2/p} < \infty \text{ for each } \beta \in \Gamma .$$

For example, these conditions hold relative to the decompositions in the statement and proof of Theorem 4. Subject to this restriction on the regularity of the lacunary decomposition $\{\Gamma_k\}_{k=-\infty}^{\infty}$ we have the following result.

THEOREM 7. *There is a subset E of G of positive Haar measure such that if $f \in L^1(G)$, $\operatorname{supp}(f) \subset E$ and*

$$\|\hat{f}\|_{p,2} = \left\{\sum_{k=-\infty}^{\infty}\left(\int_{\Gamma_k} |\hat{f}(\gamma)|^p d\gamma\right)^{2/p}\right\}^{1/2} < \infty$$

for some p, $1 < p < 2$, then $f = 0$.

Essentially the proof consists of establishing the following lemma.

LEMMA. *If $1 < p < 2, 1 > \varepsilon > 0$, and M is a subset of G of measure 1, then there exists a subset $E_{\varepsilon,p}$ of M of measure greater than or equal $1-\varepsilon$ and a function $\Phi \in L^1(G) \cap L^\infty(G)$ such that $\Phi = 1$ on $E_{\varepsilon,p}$ and $\|\hat{\Phi}\|_{p',2} < \varepsilon$.*

Proof of Theorem 7. Let $\varepsilon_n = 1/4^n$, $p_n = 2 - \varepsilon_n$ and choose M a subset of G of Haar measure 1. Let $E = \bigcap_{n=1}^{\infty} E_{\varepsilon_n,p_n}$, where E_{ε_n,p_n} and Φ_n are as in the lemma.

Suppose $f \in L^1(G)$, $\operatorname{supp} f \subset E$, and $\|\hat{f}\|_{p,2} < \infty$ for some p, $1 < p < 2$. Since $\|\hat{f}\|_\infty < \infty$ and $p/2 < 1$ we have $\|\hat{f}\|_{2,2} < \infty$. Let $\gamma \in \Gamma$. It follows from our assumption on the regularity of $\{\Gamma_k\}$ that $\|\hat{f}_\gamma\|_{p,2} < \infty$ and $\|\hat{f}_\gamma\|_{2,2} < \infty$ where $\hat{f}_\gamma(\beta) = \hat{f}(\gamma - \beta)$. Choose N so that $n > N$ implies $p_n > p$. By the interpolation theorem [4, p. 1069] there is a constant K_γ such that $\|\hat{f}_\gamma\|_{p_n,2} \leqq K_\gamma$ for $n > N$. By Parseval's identity and Hölder's inequality we have

$$\begin{aligned}|\hat{f}(\gamma)| &= \left|\int_G (\gamma, -x) f(x)\Phi_n(x)dx\right| \\ &= \left|\int_\Gamma \hat{f}(\gamma-\beta)\hat{\Phi}_n(\beta)d\beta\right| \leqq \|\hat{f}_\gamma\|_{p_n,2}\|\hat{\Phi}_n\|_{p_n',2} \leqq \varepsilon_n K_\gamma .\end{aligned}$$

Thus $\hat{f}(\gamma) = 0$ for $\gamma \in \Gamma$ and hence $f = 0$.

Proof of the Lemma. Since our proof closely parallels that in [2] we omit similar details and computations.

Define a sequence $\{\pi_n\}_{n=0}^{\infty}$ of partitions of M so that $\pi_0 = \{M\}$

and π_n is obtained from π_{n-1} by dividing each set of π_{n-1} into two sets of equal measure. Define a sequence $\{r_n\}_{n=0}^{\infty}$ of Rademacher functions on M with respect to these partitions. (i.e., $r_0 = \chi_M$, r_n is constant on each set of π_n with the value ± 1 and $\int_A r_n(x)dx = 0$ for each $A \in \pi_{n-1}$, $n > 0$.) Forming all possible finite products of the r_n yields an orthogonal system $W = \{w_i\}$ of Walsh functions whose linear span contains all functions supported on M and constant on the sets of π_n for some $n \geqq 0$.

Since $1 < p < 2$ we can choose t, $0 < t < 1$, so that $1/p = 1 - t/2$. Now let N denote a positive integer such that

$$32A_p\varepsilon^{-t/2}N^{t/2-1/2} < \varepsilon/2 \tag{3.1}$$

where A_p is the constant in the generalized Hausdorff-Young Theorem for G. We will show the existence of disjoint compact sets $K_1, \cdots, K_n$ in Γ and orthogonal linear combinations of Walsh functions $\phi_1, \cdots, \phi_n$ with supp $(\phi_j) \subset M, 1 \leqq j \leqq N$, satisfying the following conditions:

$$\int_G |\phi_j(x)|\, dx \leqq 2 \text{ and } \int_G |\phi_j(x)|^2 dx \leqq 2^k + 1, 1 \leqq j \leqq N\,, \tag{3.2}$$

where $1/2^k < \varepsilon/N \leqq 1/2^{k-1}$;

(3.3) the measure of $\{x \in M\colon\ \phi_j(x) = 1\}$ is greater than or equal $1 - \varepsilon/N$, $1 \leqq j \leqq N$; and

$$\left\{\sum_{k=-\infty}^{\infty}\left(\int_{\Gamma_k \cap K_n}\left|\sum_{j=1}^{N}\hat{\phi}_j(\beta)\right|^{p'} d\beta\right)^{2/p'}\right\}^{1/2} \leqq 4\,||\hat{\phi}_n||_{p',2} \tag{3.4}$$

and

$$\left\{\sum_{k=-\infty}^{\infty}\left(\int_{\Gamma_k \backslash K_n}|\hat{\phi}_n(\beta)|^{p'} d\beta\right)^{2/p'}\right\}^{1/2} \leqq \varepsilon/2N,\ 1 \leqq n \leqq N\,.$$

Denote the sums in (3.4) by $||\sum_{j=1}^{N}\hat{\phi}_j||_{p',2;K_n}$ and $||\hat{\phi}_n||_{p',2;\Gamma\backslash K_n}$, respectively.

Let $\phi_1 = r_0$ and choose a compact set K_1 so that

$$||\hat{\phi}_1||_{p',2;\Gamma\backslash K_1} < \varepsilon/2N\,.$$

Assume we have n functions and compact sets satisfying the above with (3.4) replaced by

$$\left\|\sum_{j=1}^{n}\hat{\phi}_j\right\|_{p',2;K_m} \leqq \left(2 + \frac{2n}{N}\right)||\hat{\phi}_m||_{p',2} \tag{3.5}$$

and

$$||\hat{\phi}_m||_{p',2;\Gamma\backslash K_m} \leqq \varepsilon/2N,\ 1 \leqq m \leqq n\,.$$

By use of Bessel's inequality and Dini's Theorem we obtain the existence of a finite set $F \subset \mathrm{W}$ such that

$$\sum_{w_i \notin F} |(\gamma, w_i)|^2 < \left(\varepsilon/2N \sum_{j=1}^{n} |K_j|\right)^2 (1/(2^k + 1))$$

uniformly for γ in the compact set $K_1 \cup K_2 \cup \cdots \cup K_n$, where $|K_j|$ denotes the measure of K_j. We have assumed $|K_j| \neq 0, 1 \leqq j \leqq n$, for otherwise ϕ_j is the desired function with $E_{\varepsilon,p} = \{x\colon\ \phi_j(x) = 1\}$. Suppose further that F is chosen so as to contain all Walsh functions appearing in the expansions of $\phi_1, \cdots, \phi_n$. Let m be a positive integer so that the elements of F (and hence $\phi_1, \cdots, \phi_n$) are constant on the sets $E_1, \cdots, E_{2^m}$ of π_m. For each j, $1 \leqq j \leqq 2^m$, consider the partition of E_j into 2^k subsets $E'_{1j}, \cdots, E'_{2^k j}$ determined by π_{m+k}. Define ϕ_{n+1} to be zero off M and on M, to be as follows

$$\phi_{n+1}(x) = \begin{cases} \dfrac{2^k}{1 - 2^k} & \text{if } x \in E'_{1j} \\ 1 & \text{if } x \in E_j \backslash E'_{1j} . \end{cases}$$

Then for $1 \leqq j \leqq 2^m$, $\int_{E_j} \phi_{n+1}(x)dx = 0$ and therefore

$$\int_M \phi_{n+1}(x) w(x) dx = 0$$

for each $w \in F$. Thus $\phi_{n+1} = \sum_{w_j \notin F} a_j w_j$ and ϕ_{n+1} is easily seen to satisfy (3.2) and (3.3). For $\gamma \in \cup_{j=1}^{n} K_j$,

$$\begin{aligned} |\hat{\phi}_{n+1}(\gamma)| &\leqq \sum_{w_j \notin F} \left|a_j \int w_j(x)\gamma(-x)dx\right| \leqq \sum_{w_j \notin F} |a_j|\ |(w_j, \gamma)| \\ &\leqq ||\phi_{n+1}||_2 \left(\sum_{w_j \notin F} |(w_j, \gamma)|^2\right)^{1/2} \leqq \varepsilon/2N \sum_{j=1}^{n} |K_j| . \end{aligned}$$

If $1 \leqq m_0 \leqq n$ and $||\hat{\phi}_{m_0}||_{p',2} < \varepsilon/2$ we are done; otherwise

$$\varepsilon/N \sum_{j=1}^{n} |K_j| \leqq 2\, ||\hat{\phi}_{m_0}||_{p',2}/N \sum_{j=1}^{n} |K_j|$$

and therefore

$$||\hat{\phi}_{n+1}||_{p',2;K_{m_0}} \leqq |K_{m_0}|\ ||\hat{\phi}_{n+1}||_{\infty;K_{m_0}} \leqq 2\, ||\hat{\phi}_{m_0}||_{p',2}/N .$$

Hence

$$\left|\left|\sum_{j=1}^{n+1} \hat{\phi}_j\right|\right|_{p',2;K_{m_0}} \leqq \left(2 + \frac{2(n+1)}{N}\right) ||\hat{\phi}_{m_0}||_{p',2}$$

by the triangular inequality. Also since $||\hat{\phi}_{n+1}||_{p',2;\bigcup_{j=1}^{n} K_j} < \varepsilon/2N$, there exists a compact set K_{n+1} disjoint from $\cup_{j=1}^{n} K_j$ such that

$$||\hat{\phi}_{n+1}||_{p',2;\Gamma \backslash K_{n+1}} < \varepsilon/2N .$$

Then

$$\left\|\sum_{j=1}^{n+1} \hat{\phi}_j\right\|_{p',2;K_{n+1}} \leqq \|\hat{\phi}_{n+1}\|_{p',2} + \sum_{j=1}^{n} \|\hat{\phi}_j\|_{p',2;\Gamma\setminus K_j}$$
$$\leqq \|\hat{\phi}_{n+1}\|_{p',2} + \varepsilon/2 \leqq \left(2 + \frac{2(n+1)}{N}\right) \|\hat{\phi}_{n+1}\|_{p',2} ,$$

where again we have assumed that $\|\hat{\phi}_{n+1}\|_{p',2;K_{n+1}} \geqq \varepsilon/2$. The existence of the functions now follows by induction.

Let $\Phi = (1/N) \sum_{j=1}^{N} \phi_j$ and $E_{\varepsilon,p} = \{x \in M\colon \Phi(x) = 1\}$. Clearly $|E_{\varepsilon,p}| \geqq 1 - \varepsilon$ and

$$\|\hat{\Phi}\|_{p',2} \leqq \|\hat{\Phi}\|_{p',2;\bigcup_{j=1}^{N} K_j} + \|\hat{\Phi}\|_{p',2;\Gamma\setminus\bigcup_{j=1}^{N} K_j} \leqq \|\hat{\Phi}\|_{p',2;\bigcup_{j=1}^{N} K_j} + \varepsilon/2.$$

In order to complete the proof it suffices to show that

$$\|\hat{\Phi}\|_{p',2;\bigcup_{j=1}^{N} K_j} < \varepsilon/2 .$$

But

$$\|\hat{\Phi}\|^2_{p',2;\bigcup_{m=1}^{N} K_m} = \sum_{\ell=-\infty}^{\infty} \left(\int_{\Gamma_\ell \cap (\bigcup_{m=1}^{N} K_m)} \left|\frac{1}{N}\sum_{j=1}^{N} \hat{\phi}_j(\gamma)\right|^{p'} d\gamma\right)^{2/p'}$$
$$= \frac{1}{N^2} \sum_{\ell=-\infty}^{\infty} \left(\sum_{m=1}^{N} \int_{\Gamma_\ell \cap K_m} \left|\sum_{j=1}^{N} \hat{\phi}_j(\gamma)\right|^{p'} d\gamma\right)^{2/p'}$$
$$\leqq \frac{1}{N^2} \sum_{m=1}^{N} \left\|\sum_{j=1}^{N} \hat{\phi}_j\right\|^2_{p',2;K_m} \leqq \frac{16}{N^2} \sum_{m=1}^{N} \|\hat{\phi}_m\|^2_{p',2}$$
$$\leqq \frac{16A_p^2}{N^2} \sum_{m=1}^{N} \|\phi_m\|_p^2 \leqq \frac{16A_p^2}{N^2} \sum_{m=1}^{N} (\|\phi_m\|_1^{1-t} \|\phi_m\|_2^t)^2$$
$$\leqq \frac{16A_p^2}{N} 2^{2(1-t)} (2^k + 1)^t \leqq [32A_p\varepsilon^{-t/2} N^{t/2-1/2}]^2 < \left(\frac{\varepsilon}{2}\right)^2 .$$

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