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ON THE DENSITY OF (k, r) INTEGERS

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Let k and r be integers such that $0 < r < k$. We call a positive integer n, $a(k, r)$-integer if it is of the form $n = a^K b$, where a and b are natural numbers and b is r-free. Clearly, $a(\infty, r)$-integer is a r-free integer. Let $Q_{k,r}$ denote the set of (k, r)-integers and let $\delta(Q_{k,r})$, $D(Q_{k,r})$ respectively denote the asymptotic and Schnirelmann densities of the set $Q_{k,r}$. In this paper, we prove that $\delta(Q_{k,r}) > D(Q_{k,r}) \geqq \zeta(k)(1 - \sum_p p^{-r}) - 1/k(1 - (1/k))^{k-1}$, and deduce the known results for r-free integers.

1. Introduction and Notation. In some recent papers, ([4, 5]) we introduced a generalized class of r-free integers, which we called the (k, r)-integers. For given integers k, r with $0 < r < k$, $a(k, r)$-integer is one whose k-free part is also r-free. In the limiting case when $k = \infty$, we get the r-free integers. It is clear that $a(k, r)$-integer is an integer of the form $a^k b$, where a and b are natural numbers and b is r-free. Let $Q_{k,r}$, Q_r denote the set of all (k, r)-integers and the set of all r-free integers respectively. Also let $Q_{k,r}(x)$ denote the number of (k, r)-integers not exceeding x, with corresponding meaning for $Q_r(x)$. We write $\delta(Q_{k,r})$ for the asymptotic density of the (k, r)-integers, that is,

$$\delta(Q_{k,r}) = \lim_{x \to \infty} \frac{Q_{k,r}(x)}{x},$$

(provided this limit exists), and $D(Q_{k,r})$ for their Schnirelmann density given by

$$D(Q_{k,r}) = \inf_n \frac{Q_{k,r}(n)}{n}.$$

We define $\delta(Q_r)$ and $D(Q_r)$ analogously. Let $\psi(n)$ be the characteristic function of $Q_{k,r}$ and $\lambda(n)$ be defined by

$$\sum_{d|n} \lambda(d) = \psi(n).$$

It is easily proved (see [3]) that the function $\psi(n)$ and $\lambda(n)$ are multiplicative and for any prime p

$$\lambda(p^a) = \begin{cases} 1 & a \equiv 0 \pmod{k}, \\ -1 & a \equiv r \pmod{k}, \\ 0 & \text{otherwise.} \end{cases}$$

Further,

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(ks)}{\zeta(rs)}\ , \ Re(s) > \frac{1}{r}\ , \tag{1.1}$$

where $\zeta(s)$ is the Riemann Zeta function. In a previous paper [5], we showed that

$$Q_{k,r}(x) = \frac{x\zeta(k)}{\zeta(r)} + E(x)\ ,$$

where the error term $E(x)$ is $0(x^{\frac{1}{r}})$, for $r > 1$, uniformly in k. (We actually gave an improved estimate for the error term, but this is not required here.)

It follows that

$$\delta(Q_{k,r}) = \frac{\zeta(k)}{\zeta(r)}\ .$$

In this paper we will show that

$$\delta(Q_{k,r}) > D(_{k,r})\ .$$

The corresponding result for Q_2 was first proved by Rogers [2], and for Q_r for all $r > 1$ by Stark [6]. We also obtain a lower bound for $D(Q_{k,r})$, from which we obtain as a special case a result of Duncan [1] on a lower bound for $D(Q_r)$. The actual value of $D(Q_{k,r})$ is unknown except for the case Q_2; Rogers [3] proved that

$$D(Q_2) = \frac{53}{88}\ .$$

2. Theorem.

$$\delta(Q_{k,r}) > D(Q_{k,r}) \geq \zeta(k)(1 - \sum_p p^{-r}) - \frac{1}{k}\Big(1 - \frac{1}{k}\Big)^{k-1}\ .$$

The proof will be given in two parts, corresponding to the two results:

$$D(Q_{k,r}) \geq \zeta(k)(1 - \sum_p p^{-r}) - \frac{1}{k}\Big(1 - \frac{1}{k}\Big)^{k-1}\ ; \tag{2.1}$$

$$\delta(Q_{k,r}) > D(Q_{k,r})\ . \tag{2.2}$$

Proof of (2.1). The case $r > 1$.

It is clear that

$$Q_r(n) \geq n - \sum_p \Big[\frac{n}{p^r}\Big]\ ,$$

p ranging over all the primes.

Since

$$Q_{k,r}(n) = \sum_{a=1}^{\infty} Q_r\left(\left[\frac{n}{a^k}\right]\right),$$

it follows that

$$\begin{aligned} Q_{k,r}(n) &\geq \sum_{a=1}^{\infty}\left(\left[\frac{n}{a^k}\right] - \sum_p \left[\frac{a^{n/k}}{p^r}\right]\right) \\ &> \sum_{a=1}^{\infty}\left(\frac{n}{a^k} - \sum_p \frac{n}{a^h p^r}\right) - (n^{1/k} - 1)\,. \end{aligned}$$

Hence we have

$$\begin{aligned} \frac{Q_{k,r}(n)}{n} &> \sum_{a=1}^{\infty}\left(\frac{1}{a^k} - \sum_p \frac{1}{a^k p^r}\right) - \frac{n_{k/1} - 1}{n} \\ &= \zeta(k)\left(1 - \sum_p p^{-r}\right) + \frac{1 - n^{1/k}}{n}\,. \end{aligned}$$

Let

$$f(x) = \frac{1 - x^{1/k}}{x} = \frac{1}{x} - x^{1/k-1}\,;$$

then

$$f'(x) = \frac{1}{x^2} - \left(\frac{1}{k} - 1\right)x^{1/k-2}\,,$$

so that

$$f'(x) > 0 \text{ if } \left(1 - \frac{1}{k}\right)x^{(1/k)-2} > \frac{1}{x^2},\ \text{i.e., } \left(1 - \frac{1}{k}\right)x^{1/k} > 1\,.$$

Thus

$$f'(x)\begin{cases} > 0 \text{ when } x > \dfrac{1}{\left(1 - \dfrac{1}{k}\right)^k}\,, \\ < 0 \text{ when } x < \dfrac{1}{\left(1 - \dfrac{1}{k}\right)^k}\,. \end{cases}$$

when $x = (1 - (1/k))^{-k}$ we get the minimum value of f, which is equal to

$$f\left(\left(1 - \frac{1}{k}\right)^{-k}\right) = \frac{1 - \left(1 - \dfrac{1}{k}\right)^{-1}}{\left(1 - \dfrac{1}{k}\right)^{-k}} = -\frac{1}{k}\left(1 - \frac{1}{k}\right)^{k-1}\,.$$

Hence

$$\frac{Q_{k,r}(n)}{n} > \zeta(k)(1 - \sum_p p^{-r}) - \frac{1}{k}\left(1 - \frac{1}{k}\right)^{k-1},$$

and

$$D(Q_{k,r}) \geq \zeta(k)(1 - \sum_p p^{-r}) - \frac{1}{k}\left(1 - \frac{1}{k}\right)^{k-1}.$$

For the case $r = 1$,

$$Q_{k,1}(n) = [n^{1/k}]$$

$$\delta(Q_{k,1}) = \lim_{n\to\infty} \frac{[n^{1/k}]}{n} = 0, \text{ since } k \geqq 2 .$$

$$D(Q_{k,1}) = \inf_n \frac{[n^{1/k}]}{n} = 0 .$$

So the result still holds in this case.

REMARK 2.3. The above proof is easily seen to hold even when $k = \infty$. The corresponding result, namely,

$$D(Q_r) > 1 - \sum_p p^{-r} ,$$

is due to R. L. Duncan [1].

To prove the result in (2.2), we first obtain the following lemma.

LEMMA 2.4. *For any $\varepsilon > 0$, we have*

(i) $E(n) > n^{(1/2r)-\varepsilon}$, *for infinitely many integers n,*

(ii) $E(n) < -n^{(1/2r)-\varepsilon}$, *for infinitely many integers n.*

Proof. Let

$$\sum \left(\psi(n) - \frac{\zeta(k)}{\zeta(r)}\right)n^{-s} = R_1(s) .$$

Since

$$\sum \left(\psi(n) - \frac{\zeta(k)}{\zeta(r)}\right)n^{-s} = \frac{\zeta(ks)\zeta(s)}{\zeta(rs)} - \frac{\zeta(k)\zeta(s)}{\zeta(r)} ,$$

we have

$$\begin{aligned} R_1(s) &= \sum \left(\psi(n) - \frac{\zeta(k)}{\zeta(r)}\right)n^{-s} \\ &= \sum (E(n) - E(n-1))n^{-s} \\ &= \sum E(n)(n^{-s} - (n+1)^{-s}) . \end{aligned}$$

Also, let

$$\begin{aligned} s \sum E(n)n^{-s-1} &= R_2(s)\ , \\ \sum E(n)n^{-s-1} &= R_3(s)\ , \\ \sum n^{(1/2r)-\varepsilon} \cdot n^{-s-1} &= R_4(s)\ , \\ \sum (n^{(1/2r)-\varepsilon} - E(n))n^{-s-1} &= R_5(s)\ , \\ \sum (n^{(1/2r)-\varepsilon} + E(n))n^{-s-1} &= R_6(s)\ . \end{aligned}$$

Now suppose that for all $n \geq n_0$, $E(n) \leq n^{(1/2r)-\varepsilon}$. Then the series $R_5(s)$ converges for $a > (1/2r) - \varepsilon$ $(a = Re(s))$, and all but a finite number of coefficients of $R_5(s)$ are nonnegative. Hence the abscissa of convergence of $R_5(s)$ must be less than or equal to $(1/2r) - \varepsilon$. Let α be its abscissa of convergence, that is $\alpha \leq (1/2r) - \varepsilon$. Note that (see [2], P. 661)

$$|n^{-s} - (n+1)^{-s} - s \cdot n^{-s-1}| \leq |s|\,|s+1|\, n^{-a-2}\ .$$

This implies $R_1(s)$ also converges for $a > \alpha$. But this is false because $R_1(s)$ has singularities on $a = (1/2r)$. Thus we must have

$$E(n) > n^{(1/2r)-\varepsilon}$$

for infinitely many integers n.

Next suppose that for all $n \geq n_0$, $E(n) \geq -\, n^{(1/2r)-\varepsilon}$, then we consider the series $R_6(s)$, proceed as in (i) and arrive at the same contradiction.

Proof of the result (2.2). By the above lemma, there are infinitely many integers n for which $E(n) < 0$. For such n,

$$\frac{Q_{k,r}(n)}{n} = \frac{\zeta(k)}{\zeta(r)} + \frac{E(n)}{n} < \frac{\zeta(k)}{\zeta(r)}\ ,$$

which proves the theorem.

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