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ON SOLVABLE O*-GROUPS

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The purpose of this paper is to prove the existence of O*-groups of arbitrary solvable length, as well as of non-solvable O*-groups.

By a *partial order* for a group G we mean a reflexive, antisymmetric and transitive relation, $\leqq$, on G such that if g and h are elements of G and $g \leqq h$, then $xgy \leqq xhy$ for all x and y in G. If also any two elements g and h of G are *comparable* (i.e., either $g \leqq h$ or $h \leqq g$), then the partial order for G is called a *total* (or *full*, or *linear*) *order*. The group G is an *O^*-group* if any partial order for G is included in some total order for G.

A group G is *solvable of length n*, where n is a positive integer, if the derived chain of G reaches the unit subgroup, E, in n steps:

$$G = G^1 \supsetneqq G^2 \supsetneqq \cdots \supsetneqq G^n \supsetneqq G^{n+1} = E,$$

where G^{i+1} is the derived group of G^i (denoted below by $G^{i+1} = [G^i, G^i]$).

It has been shown that non-abelian free groups are not O^*-groups ([1], [2], [3], [4], [6]). Further, Kargapolov [5] and Kargapolov, Kokorin and Kopytov [6] have produced solvable groups which are not O^*-groups even though they admit a full order: these are the free r-step solvable groups on k generators for $r \geqq 3$ and $k \geqq 2$. In view of these results one may ask if there exist solvable O^*-groups of arbitrary length, and nonsolvable O^*-groups. The answers are affirmative.

THEOREM. *For every positive integer m there exists an O^*-group G that is solvable of length m.*

Proof. Let F be the free group on k generators for some fixed $k \geqq 2$. Let F_i be the ith term in the lower central series for F, where $F_1 = F$, and let F^i be the ith derived group for F, where $F^1 = F$. Consider F/F_i, the free nilpotent group of class i with k generators. By varying i we shall obtain the desired groups G of the theorem.

We first claim that F/F_i is torsion-free for every positive integer i. If not, then for some i there exists an element $a \in F$ and a positive integer p such that $a \notin F_i$, but $a^p \in F_i$. Now $a \in F_h - F_{h+1}$ for some positive integer $h \leqq i - 1$. Thus $a^p \in F_i \subseteqq F_{h+1}$, and so F_h/F_{h+1} is not torsion-free. On the other hand, Witt's theorem (see, e.g., [8, p. 41]) states that F_h/F_{h+1} is a free abelian group (and hence torsion-free), a contradiction. Thus F/F_i is torsion-free, as claimed.

Malcev [9] has shown that a torsion-free nilpotent group is an O*-group. Hence F/F_i is an O*-group for every positive integer i. Now for every such i the solvable length of F/F_i is finite, since F/F_i is nilpotent. Thus we shall complete our proof by establishing the following lemma.

LEMMA. *For every positive integer m, there exists an integer n such that solvable length of F/F_n is m.*

Proof. We first note that for every positive integer i, there exists an integer j such that $F_j \not\supseteq F^i$. This follows from the fact that $F^i \neq E$ for each i (hence F is not solvable), together with the theorem of Magnus (cf. [8, p. 38]) which asserts that $\bigcap_{i=1}^{\infty} F_i = E$. We next show that for each i and j,

$$(F/F_j)^i = F^i F_j/F_j. \tag{1}$$

Indeed, it is readily seen that if A and B are subgroups of a group G and if B is invariant under conjugation by elements of A, then $(AB/B)^2 = A^2B/B$. From this, an induction on i shows that for a normal subgroup N of a group G it is true that $(G/N)^i = G^iN/N$ for all i, which implies the desired result.

Note that for each i there exists J such that for $j \geqq J$, the solvable length of F/F_j exceeds i. This follows from (1) and the fact that, by our first assertion, we can choose J such that $F_J \not\supseteq F^i$. In particular, then, the solvable length of F/F_j is unbounded with increasing j. Note also that the solvable length of F/F_{j+1} exceeds the solvable length of F/F_j by at most 1. For if F/F_j is solvable of length $r - 1$, then $(F/F_j)^r = E$. Thus, by (1) we have $F^rF_j/F_j = E$, which implies $F^r \subseteqq F_j$. On the other hand, F/F_{j+1} has solvable length $\leqq r$ since (again using (1))

$$\begin{aligned}(F/F_{j+1})^{r+1} &= [(F/F_{j+1})^r, (F/F_{j+1})^r] \\ &= [F^rF_{j+1}/F_{j+1}, F^rF_{j+1}/F_{j+1}] \subseteqq [F_j/F_{j+1}, F_j/F_{j+1}] = E,\end{aligned}$$

where $\subseteqq$ holds since both F^r and F_{j+1} are subsets of F_j, and the final equality derives from the fact that F_j/F_{j+1} is abelian by Witt's theorem (above). The lemma follows at once from these results and the fact that $F/F_2 = F_1/F_2$ has solvable length 1 by Witt's theorem.

The proof of the theorem is now complete.

COROLLARY. *There exist nonsolvable O*-groups.*

Proof. Kargapolov [5] and Kokorin [7] have shown that the restricted direct product of O*-groups is an O*-group. Thus the restricted direct product, $G = \prod_{i=1}^{\infty} F/F_i$, of the groups F/F_i is an O*-group. If G were solvable of length m, then each F/F_i would have solvable length $\leqq m$; for if a subgroup $H \subseteqq G$, then $H^k \subseteqq G^k$ for every k. Since this contradicts the fact noted above that the solvable length of F/F_j is unbounded with increasing j, G is a non-solvable O*-group.

Note. The mapping φ of F into the *unrestricted* (or *complete*) direct product, $\prod_{i=1}^{\infty} F/F_i$, of the groups F/F_i given by

$$\varphi(a) = (aF_1, \cdots, aF_n, \cdots) \text{ for every } a \in F$$

is a monomorphism by Magnus' theorem, above. Since F is not an O*-group (see [1], [4], or [6]), we have an immediate example of a subdirect product of O*-groups which is not itself an O*-group. (In [5], Kargapolov uses some of the groups F/F_i to show that the class of O*-groups is not closed under formation of unrestricted direct products.)

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