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COHOMOLOGY GROUPS ASSOCIATED WITH THE $\partial\bar{\partial}$-OPERATOR

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Let M be a complex analytic manifold of complex dimension m. The manifold M, considered open, is a submanifold of a manifold M' of the same dimension, and its boundary $\partial\bar{M}$ is a smooth C^3-manifold. Let $A^{p,q}$ be the sheaf of germs of complex-valued (p, q)-forms, p and q are integers, $p \geqq 0$, $q \geqq 0$. The exterior differential of an element $u \in A^{p,q}$ can be written in a unique way as a sum $du = \partial u + \bar{\partial} u$. There is a real operator

$$d_c u = \sqrt{-1}\,(\bar{\partial} u - \partial u)$$

and the real second order operator

$$dd_c = 2\sqrt{-1}\,\partial\bar{\partial}$$

defined on $A^{p,q}$. Let $A_R^{p,q} = \{\alpha = \alpha_1 + \alpha_2 \in A^{p,q} \oplus A^{q,p} \mid \alpha_2 = \bar{\alpha}_1\}$ be the sheaf of real (p, q)-forms. Then we get two short exact sequences of sheaves

$$\begin{array}{ll} & 0 \longrightarrow \mathscr{P}_C^{q,q} \longrightarrow A^{p,q} \xrightarrow{\partial\bar{\partial}} A^{p+1,q+1} \xrightarrow{d} A^{p+2,q+1} \oplus A^{p+1,q+2} \\ (1.1) & \\ & 0 \longrightarrow \mathscr{P}_R^{p,q} \longrightarrow A_R^{p,q} \xrightarrow{dd_c} A_R^{p+1,q+1} \xrightarrow{d} A_R^{p+2,q+1} \oplus A_R^{p+1,q+2} \end{array}$$

where $\mathscr{P}_C^{p,q}$ and $\mathscr{P}_R^{p,q}$ are defined by these sequences. The purpose of this paper is to discuss the cohomology of these two sequences.

The importance of the cohomology of the first sequence,

$$(1.2) \qquad \Lambda_R^{p,q} = \frac{\operatorname{Ker} d \text{ on } \Gamma(M, A^{p,q})}{\partial\bar{\partial}\Gamma(M, A^{p-1,q-1})},$$

lies in its application to the study of strongly q-pseudoconvex manifolds—A. Andreotti, F. Norguet, B. Bigolin and others. The cohomology of the second sequence,

$$(1.3) \qquad \Lambda_R^{p,q} = \frac{\operatorname{Ker} d \text{ on } \Gamma(M, A_R^{p,q})}{dd_c\Gamma(M, A_R^{p-1,q-1})},$$

contains (for $p = q$) the refined Chern classes of complex analytic vector bundles over M. In both cases the first cohomology group $H^1(M, \cdot)$ plays the important role, therefore we restrict ourselves to this case.

As for the cohomology of the first sequence (1.1), B. Bigolin studied recently the relation of $\Lambda_C^{p,q}$ with the so called Aeppli cohomology

$$V_C^{p,q} = \frac{\operatorname{Ker} \partial\bar\partial \text{ on } \Gamma(M, A^{p,q})}{\partial\Gamma(M, A^{p-1,q}) + \bar\partial\Gamma(M, A^{p,q-1})} \tag{1.4}$$

and with $H^*(M, C)$ under certain assumptions on the manifold M (Stein, k-pseudoconvex, compact Kähler) using methods of sheaf theory. The main results of this paper are proved by direct Hilbert space methods. The cohomology of both sequences (1.1) are studied simultaneously. The statements concerning the first sequence (1.1) can be considered as another proof of some results obtained by Bigolin. It is shown that the cohomology of M with coefficients in the sheaf $\mathscr{P}_C^{p,q}$ and also in $\mathscr{P}_R^{p,q}$ is, under certain conditions on the boundary of M open, finite dimensional and isomorphic to the harmonic spaces constructed from Spencer's resolutions of the corresponding sheaves. Using the terminology of [9] we can say that the Neumann problem is solvable for the operators $\partial\bar\partial$ and dd_c, under certain pseudoconvexity conditions on the boundary of M (Theorem 3.1).

The technique is based on the methods developed by Hörmander as an extension of those introduced into the subject by Kohn, Morrey, and Ash. The relatively new part in this direction here is the application of Hörmander's technique to the Spencer resolution of the sheaves $\mathscr{P}_C^{p,q}$ and $\mathscr{P}_R^{p,q}$.

B. Mac Kichan told us recently that he can prove, using the δ-estimate [8], that the Neumann problem is solvable for the operator $\partial\bar\partial$ on complex-valued functions under certain boundary conditions on the open manifold M.

2. Before we start proving the main results concerning the open and compact manifolds, let us start some elementary properties of the sheaves $\mathscr{P}_C^{p,q}$ and $\mathscr{P}_R^{p,q}$ defined as the kernels of the operators $\partial\bar\partial$ and dd_c respectively—see (1.1)—and summarize the known results connected with our considerations.

PROPOSITION 2.1. *The sheaf $\mathscr{P}_C^{p,0}$ is the sheaf of germs of differential $(p, 0)$-forms $\omega = \lambda + \bar\mu$, where λ is a local holomorphic $(p, 0)$-form and μ is a $\bar\partial$-closed $(0, p)$-form.*

Proof. An element $\omega \in \mathscr{P}_C^{p,0}$ if and only if $\bar\partial\partial\omega = d\partial\omega = 0$. From the exactness of de Rham's complex, we conclude that there exists $\lambda \in A^{p,0}$ such that $d\lambda = \partial\omega$. But $\partial\omega \in A^{p+1,0}$ therefore $\partial\lambda = \partial\omega$ and $\bar\partial\lambda = 0$. Denote $\omega - \lambda = \bar\mu$. Then $\partial(\omega - \lambda) = \partial\bar\mu = 0$, therefore $\bar\partial\mu = 0$ and $\omega = \lambda + \bar\mu$ as stated above.

REMARK. If we denote by Ω^p the sheaf of germs of holomorphic

p-forms and by $\mathscr{H}^p$ the sheaf of ∂-closed $(p, 0)$-forms we see immediately that there is an exact sequence of sheaves

$$0 \longrightarrow S^p \longrightarrow \Omega^p \oplus \mathscr{H}^p \longrightarrow \mathscr{P}_C^{p,0} \longrightarrow 0 \;, \tag{2.1}$$

where S^p is the sheaf of d-closed $(p, 0)$-forms, and the corresponding exact sequence for cohomologies

$$\cdots \longrightarrow H^i(M, \Omega^p) \oplus H^i(M, \mathscr{H}^p) \longrightarrow H^i(M, P_C^{p,0}) \longrightarrow H^{i+1}(M, S^p) \longrightarrow$$
$$\longrightarrow H^{i+1}(M, \Omega^p) \oplus H^{i+1}(M, \mathscr{H}^p) \cdots .$$

PROPOSITION 2.2. *The sheaf $\mathscr{P}_R^{0,0} = \mathscr{P}_R$ is the sheaf of germs of real parts of holomorphic functions on M.*

Proof. Let $u \in A_R^{0,0}$, $dd_c u = 0$. Then $\partial\bar{\partial}u = \bar{\partial}\partial u = 0$ and $u = f + \bar{g}$, where $\bar{\partial}f = \bar{\partial}g = 0$. The function $h = f - g$ is real as u is a real function. Furthermore $\bar{\partial}h = 0$ and $\bar{\partial}h = \partial h = 0$, therefore $h =$ constant and u is the real part of the homolorphic function $2f - h$.

If β is the projection of a homolorphic function on its real part, we get immediately the exact sequence

$$0 \longrightarrow \boldsymbol{R} \xrightarrow{\alpha} \mathscr{O} \xrightarrow{\beta} \mathscr{P}_R \longrightarrow 0 \;, \tag{2.2}$$

where $\mathscr{O} = \Omega^0$. The map α gives to any $a \in R$ a constant function $0 + ia$. We claim that this sequence splits, because there is a sheaf homomorphism $b\colon \mathscr{P}_R \to \mathscr{O}$ which to each function $u \in \mathscr{P}_R$ associates a holomorphic function $u + iv$ where $u = v$ at a given point of M. We then have:

PROPOSITION 2.3. *The sequence* (2.2) *is exact and splits.*

PROPOSITION 2.4. *Let $A^{p,q}$ be the sheaf of C^∞ complex-valued (p, q)-forms and $A_R^{p,q} = \{\omega \in A^{p,q} \oplus A^{q,p} \mid \omega = \alpha + \bar{\alpha}, \alpha \in A^{p,q}\}$, then the sequences* (1.1) *are exact sequences of sheaves.*

Proof. We prove only the exactness of the second sequence (1.1) at $A_R^{p+1,q+1}$ because the proof of the first sequence is analogous. Let $u \in A_R^{p+1,q+1}$, $du = 0$. Then there exists $\omega \in A_R^{p,q+1} \oplus A_R^{p+1,q}$ such that $\omega = \alpha + \bar{\alpha} + \beta + \bar{\beta}$, $\alpha \in A^{p,q+1}$, $\beta \in A_R^{p+1,q}$, $d\omega = u$. Because $d\omega \in A_R^{p+1,q+1}$ we conclude that $\partial\bar{\alpha} = \bar{\partial}\alpha = \partial\beta = \bar{\partial}\,\bar{\beta} = 0$ as these terms belong to $A^{q+2,p}$, $A^{p,q+2}$, $A^{p+2,q}$, $A^{p,q+2}$ respectively. From $\partial\bar{\alpha} = 0$ follows that there exists $\bar{a} \in A^{p,q}$ such that $\partial\bar{a} = \bar{\alpha}$ and from $\partial\beta = 0$ we get the existence of $b \in A^{p,q}$, $\partial b = \beta$. Then $(a - \bar{a}) \in A^{p,q} \oplus A^{q,p}$ and $(\bar{b} - b) \in A^{p,q} \oplus A^{q,p}$ and $\partial\bar{\partial}(a - \bar{a} + \bar{b} - b) = \partial\alpha + \bar{\partial}\bar{\alpha} + \partial\bar{\beta} + \bar{\partial}\beta$. Put $w = -1/2\sqrt{-1}(a - \bar{a} - \bar{b} - b)$. Then we see that $\bar{w} = w \in A_R^{p,q}$ and $dd_c w = 2\sqrt{-1}\partial\bar{\partial}w = \partial\bar{\partial}(a - \bar{a} + \bar{b} - b) = d\omega = u$.

From the work of Aeppli and Bigolin we have the following information about the cohomology of M with values in the sheaves $\mathscr{P}_C^{p,q}$, $\mathscr{P}_R^{p,q}$ and the cohomology $V_C^{p,q}$ (1.4).

PROPOSITION 2.5. *Let M be a Stein manifold, then we have the following isomorphisms*:

$$V_C^{p,q} \cong H^{p+q+1}(M, \boldsymbol{C}) , \qquad p, q \geqq 0$$

$$\Lambda_C^{p,q} \cong H^1(M, \mathscr{P}_C^{p-1,q-1}) \cong H^{p+q}(M, C) , \qquad p, q \geqq 0$$

$$H^r(M, P_C^{p,q}) \cong H^{p+q+r+1}(M, \boldsymbol{C}) , \qquad r \geqq 1, p + q + 2 \geqq m = \dim_C M .$$

PROPOSITION 2.6. *If M is strongly k-pseudoconvex, then*

$$\dim_C H^r(M, \mathscr{P}_C^{p,q}) < +\infty , \qquad r\geqq 1,\ p,\ q\geqq k,\ p+q+2\geqq m=\dim_C M ,$$

$$\dim_C V_C^{p,q} < +\infty , \qquad p, q \geqq k ,$$

$$\dim_C \Lambda_C^{p,q} < +\infty , \qquad p, q \geqq k .$$

PROPOSITION 2.7. *On a compact manifold M*

$$\dim_C \Lambda_C^{p,q} < +\infty , \qquad p, q \geqq 1 ,$$

$$\dim_C H^r(M, \mathscr{P}_C^{p,q}) < +\infty , \qquad r \geqq 1, p + q + 2 \geqq m = \dim_C M ,$$

$$\dim_C V_C^{p,q} < +\infty , \qquad p + q \geqq 1 .$$

If M is a compact Kähler manifold then

$$V_C^{p,q} \cong H^q(M, \Omega^p) = H^{p,q}(M, \mathscr{O}) ,$$

$$\Lambda_C^{p,q} \cong H^q(M, \Omega^p) = H^{p,q}(M, \mathscr{O}) .$$

3. Let M be an open manifold, $M \subset M'$, a submanifold of M' such that the boundary $\partial \bar{M}$ is smooth (C^3). Let $m = \dim_C M = \dim_C M'$ as before.

We shall construct first of all the Spencer resolution of the sheaves $\mathscr{P}_C^{p,q}$ and $\mathscr{P}_R^{p,q}$. But, because the resolution of the "real" sheaf $\mathscr{P}_R^{p,q}$ can be obtained from the "complex" one by adding certain algebraic conditions on the spaces in question, we shall consider the resolution of $\mathscr{P}_R^{p,q}$ and point out simultaneously which conditions have to be dropped in order to get the resolution of $\mathscr{P}_C^{p,q}$.

The second order operator dd_c together with its prolongations can be factored through the sheaf of germs of the jet bundle $J_l(\Lambda_R^{p,q})$, $l \geqq 2$, and thus we can define the vector bundle $R_l^{p,q} \to M'$ by the commutative diagram

$$
(3.1)_l \qquad
\begin{array}{ccccccc}
0 & \longrightarrow & R_l^{p,q} & \longrightarrow & J_l(A_R^{p,q}) & \longrightarrow & A_R^{p+1,q} \oplus A_R^{p,q+1} \\
 & & & & \uparrow{\scriptstyle j_l} & \nearrow{\scriptstyle dd_c} & \\
 & & & & A_R^{p,q} & &
\end{array}
$$

for $l \geqq 2$. Let us denote by δ the formal differential ([9]) and define the vector bundles $g_{l+1}^{p,q} \to M'$ and $P_{p,q}^i \to M'$, $0 \leqq i \leqq 2m$, by the sequences

$$
(3.2)_l \qquad 0 \longrightarrow g_{l+1}^{p,q} \longrightarrow R_{l+1}^{p,q} \xrightarrow{\pi} R_l^{p,q} \longrightarrow 0 ,
$$

where π is the ordinary jet projection. Now let

$$
(3.3) \qquad P_{p,q}^i = (\stackrel{i}{\wedge} T^* \otimes R_2^{p,q})/\delta(\stackrel{i-1}{\wedge} T^* \otimes g_3^{p,q}) ,
$$

$T^* = T^*(M')$ being the cotangent bundle of M'. It can be shown ([3]) that having chosen a splitting λ of $(3.2)_1$ we have an isomorphism

$$
(3.4) \qquad P_{p,q}^i \cong (\stackrel{i}{\wedge} T^* \otimes R_1^{p,q}) \oplus \delta(\stackrel{i}{\wedge} T^* \otimes g_2^{p,q}),\ 0 \leqq i \leqq 2m .
$$

Furthermore there is a uniquely defined 1st order differential operator D such that for any vector bundle $E \to M'$ and for the corresponding jet bundles

$$
(3.5) \qquad D\colon J_l(E) \longrightarrow T^* \otimes J_{l-1}(E) .
$$

This operator is universal for all linear differential operators on E, in the sense that for any subbundle R_l of $J_l(E)$ given by an operator in the same way as $R_l^{p,q}$ in $(3.1)_l$ was defined, D maps R_l into $T^* \otimes R_{l-1}$. Therefore

$$
(3.6)_l \qquad D\colon R_l^{p,q} \longrightarrow T^* \otimes R_{l-1}^{p,q} , \qquad p, q \geqq 0 .
$$

The restriction of D to the kernel $g_{l+1}^{p,q}$ of the jet projection π, $(3.2)_l$, is actually $(-\delta)$.

The operator D, $(3.6)_2$ and a splitting λ of $(3.6)_1$ define the 1st order differential operator $D_0 = D \cdot \lambda$,

$$
(3.7) \qquad D_0\colon R_1^{p,q} \longrightarrow T^* \otimes R_1^{p,q} .
$$

Now we are in the position to state

LEMMA 3.1. *Let $\mathscr{P}_R^{p,q} \to M'$ be the sheaf defined by the operator* dd_c (1.1). *Then the sequence*

$$(3.8)\qquad 0 \longrightarrow \mathscr{P}_R^{p,q} \longrightarrow P^0_{p,q} \xrightarrow{\ \boldsymbol{D}\ } P^1_{p,q} \xrightarrow{\ \boldsymbol{D}\ } \cdots \xrightarrow{\ \boldsymbol{D}\ } P^{2m}_{p,q} \longrightarrow 0\,,$$

where, using the isomorphism (3.4),

$$\boldsymbol{D}u = \boldsymbol{D}(\sigma, \zeta) = (D_0\sigma - \zeta,\ D_0(D_0\sigma - \zeta)),\ u \in P^i_{p,q},\ 0 \leqq i \leqq 2m\,,$$

is an exact resolution of $\mathscr{P}_R^{p,q}$ *by fine sheaves.*

Proof. It follows from the general theory—see [8].

COROLLARY.

$$(3.9)\qquad H^1(M', \mathscr{P}_R^{p,q}) \cong \frac{\operatorname{Ker} D \text{ on } \Gamma(M', P^1_{p,q})}{D\Gamma(M', P^0_{p,q})} \cong \Lambda_R^{p+1,q+1}\,.$$

In order to study this group we need an explicit description of the sheaf $P^1_{p,q}$.

Let U be a coordinate neighborhood in M' with complex analytic coordinates $(z^1, \cdots, z^m)$ related to the real coordinates $(x^1, \cdots, x^{2m})$ by the usual relations $z^j = x^{2j-1} + \sqrt{-1}x^{2j}$, $1 \leqq j \leqq m$. In order to get an expression more suitable for calculation let us introduce at this point a hermitian product $\langle , \rangle$ on the tangent bundle $T = T(M')$. This product is locally given by a hermitian matrix $h = (h_{i\bar{j}})$, $\langle \partial/\partial z^i, \partial/\partial z^j \rangle = h_{i\bar{j}}$, $\langle \partial/\partial z^i, \partial/\partial \bar{z}^j \rangle = 0$, and the matrix ${}^t h^{-1} = (h^{\bar{j}i})$ gives an inner product on the cotangent bundle T^* by the formulas $\langle dz^i, dz^j \rangle = h^{\bar{j}i}$, $\langle dz^i, d\bar{z}^j \rangle = 0$.

As the differentiation of the hermitian product involves differentials of the matrix h it turns out to be useful to intorduce a more suitable frame. Let

$$(3.10)\qquad (\omega^1, \cdots, \omega^m)$$

be C^∞ (1, 0)-forms on U such that

$$\omega^j = \sum_{k=1}^m a^j_k dz^k,\ dz^j = \sum_{k=1}^m b^j_k \omega^k$$

and $\langle \omega^i, \omega^j \rangle = \delta^{ij}$, $1 \leqq i, j \leqq m$. We denote by $(\partial/\partial\omega^1, \cdots, \partial\omega^m)$ the frame dual to $(\omega^1, \cdots, \omega^m)$.

Identifying $P^1_{p,q}$ with the direct sum in the isomorphism (3.4) we get from straightforward local considerations

PROPOSITION 3.1. *Each element* $u \in P^1_{p,q}$, $u = (\rho, \eta)$ *can be written locally in terms of the frame* (3.10) *in the form*

$$\begin{aligned}\rho &= \sum_{l=1}^{m} \rho_l \omega^l + \sum_{l=1}^{m} \rho_{\bar{l}} \bar{\omega}^l ,\\ \eta &= \sum_{l,j=1}^{m} \eta_{lj} \omega^l \wedge \omega^j + \sum_{l,j=1}^{m} \eta_{l\bar{j}} \omega^l \wedge \bar{\omega}^j + \sum_{l,j=1}^{m} \eta_{\bar{l}j} \bar{\omega}^l \wedge \omega^j \\ &\quad + \sum_{l,j=1}^{m} \eta_{\bar{l}\bar{j}} \bar{\omega}^l \wedge \bar{\omega}^j ,\\ &\eta_{lj} + \eta_{jl} = 0,\ \eta_{\bar{l}\bar{j}} + \eta_{\bar{j}\bar{l}} = 0 ,\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}\rho_l = (&\Sigma \rho_{I\bar{J},l} \omega^I \wedge \bar{\omega}^J + \sum \rho_{\bar{I}J,l} \bar{\omega}^I \wedge \omega^J \\ &+ \Sigma \sum_{k=1}^{m} \rho_{IJ;k,l} \omega^k \otimes \omega^I \wedge \bar{\omega}^J + \Sigma \sum_{k=1}^{m} \rho_{\bar{I}J;k,l} \omega^k \otimes \bar{\omega}^I \wedge \omega^J \\ &+ \Sigma \sum_{k=1}^{m} \rho_{I\bar{J};\bar{k},l} \bar{\omega}^k \otimes \omega^I \wedge \bar{\omega}^J + \Sigma \sum_{k=1}^{m} \rho_{\bar{I}J;\bar{k},l} \bar{\omega}^k \otimes \bar{\omega}^I \wedge \omega^J)\end{aligned}$$

and

$\rho_{\bar{l}} =$ exactly the same expression as for ρ_l if ρ is replaced by $\bar{\rho}$.

$$\begin{aligned}\eta_{lj} &= \Sigma \sum_{k=1}^{m} \eta_{I\bar{J};k,lj} \omega^k \otimes \omega^I \wedge \bar{\omega}^J + \Sigma \sum_{k=1}^{m} \eta_{\bar{I}J;k,lj} \omega^k \otimes \bar{\omega}^I \wedge \omega^J ,\\ \eta_{l\bar{j}} &= \Sigma \sum_{k=1}^{m} \eta_{I\bar{J};k,l\bar{j}} \omega^k \otimes \omega^I \wedge \bar{\omega}^J + \Sigma \sum_{k=1}^{m} \eta_{\bar{I}J;k,l\bar{j}} \omega^k \otimes \bar{\omega}^I \wedge \omega^J ,\\ \eta_{\bar{l}j} &= \Sigma \sum_{k=1}^{m} \eta_{I\bar{J};\bar{k},\bar{l}j} \bar{\omega}^k \otimes \omega^I \wedge \bar{\omega}^J + \Sigma \sum_{k=1}^{m} \eta_{\bar{I}J;\bar{k},\bar{l}j} \bar{\omega}^k \otimes \bar{\omega}^I \wedge \omega^J ,\\ \eta_{\bar{l}\bar{j}} &= \Sigma \sum_{k=1}^{m} \eta_{I\bar{J};\bar{k},\bar{l}\bar{j}} \bar{\omega}^k \otimes \omega^I \wedge \bar{\omega}^J + \Sigma \sum_{k=1}^{m} \eta_{\bar{I}J;\bar{k},\bar{l}\bar{j}} \bar{\omega}^k \otimes \bar{\omega}^I \wedge \omega^J ,\end{aligned}$$

stands for $|I| = p, |J| = q$,

$$\begin{aligned}\omega^I &= \omega^{i_1} \wedge \cdots \wedge \omega^{i_p},\\ \omega^J &= \omega^{j_1} \wedge \cdots \wedge \omega^{j_q},\ i_1 < i_2 < \cdots < i_p,\ j_1 < j_2 < \cdots < j_q .\end{aligned}$$

All these components satisfy the conditions

$$\rho_{\bar{l}} = \overline{\rho_l} , \qquad \eta_{\bar{l}\bar{j}} = \overline{\eta_{lj}} , \qquad \eta_{\bar{l}j} = \overline{\eta_{l\bar{j}}} . \tag{3.12}$$

REMARK. The Spencer resolution of the sheaf $\mathscr{P}_C^{p,q}$ is an exact sequence

$$0 \longrightarrow \mathscr{P}_C^{p,q} \longrightarrow P^0_{Cp,q} \xrightarrow{D} P^1_{Cp,q} \xrightarrow{D} \cdots \xrightarrow{D} P^{2m}_{Cp,q} \longrightarrow 0$$

where the vector bundles $P^i_{Cp,q}$ are defined in an obvious way by an expression similar to (3.3). Each element $u \in P^1_{Cp,q}$, $u = (\rho, \eta)$ has the local form given by the previous Proposition 3.1, but the conditions "for reality" (3.12) are not satisfied.

The complex tangent bundle $T = V \oplus \bar{V}$ splits into holomorphic and antiholomorphic parts. Let V^* and $\bar{V}^*$ be their duals.

PROPOSITION 3.2.

$$P^1_{p,q} \cong (V^* \otimes A_R^{p,q}) \oplus T^* \otimes (V^* \otimes A_R^{p,q}) \oplus T^* \wedge V^* \otimes (V^* \otimes A_R^{p,q}) . \tag{3.13}$$

Proof. It is easily seen directly or from previous Proposition 3.1.

Before we proceed any further with the general situation ($p, q \geqq 0$) let us make an observation about $\mathscr{P}_R = \mathscr{P}_R^{0,0}$. From the general theory it follows that for any kth order, involutive, linear differential operator $\mathscr{D}$, with constant coefficients, from a vector bundle $E \to M$ into a vector bundle $F \to M$ there is in a certain sense a unique exact Spencer resolution $\mathscr{R}_k$ of the sheaf $\mathscr{S}$ of germs of solutions to the homogeneous system $\mathscr{D}_S = 0$. The resolution $\mathscr{R}_{k+l}$ of the sheaf $\mathscr{S}$ corresponding to the lth prolongation $j^l \cdot \mathscr{D}$ of the operator $\mathscr{D}$ is also exact and has the same cohomology as $\mathscr{R}_k$ for any $l \geqq 0$. Let us look in particular at the resolution of the sheaf of germs of holomorphic functions $\mathscr{O}$ corrresponding to the first order operator $\bar{\partial}$:

$$0 \longrightarrow \mathscr{O} \longrightarrow C_1^0 \xrightarrow{D} C_1^1 \xrightarrow{D} \cdots \xrightarrow{D} C_1^n \longrightarrow 0, \; n = 2m . \tag{3.14}$$

This resolution is defined in a way analogous to (3.8) and C_1^i is the vector bundle such that $u \in C_1^i$ is a pair $u = (\sigma, \xi)$, where σ is a complex-valued i-form and ξ is a complex-valued $(i+1)$-form which belongs to the ideal generated by the dz's (in the coordinates in $U \subset M'$). $Du = D(\sigma, \xi) = (d\sigma - \xi, -d\xi)$.

To the first prolongation $j^1 \cdot \bar{\partial}$ of $\bar{\partial}$ corresponds an exact resolution

$$0 \longrightarrow \mathscr{O} \longrightarrow C_2^0 \xrightarrow{D} C_2^1 \xrightarrow{D} \cdots \xrightarrow{D} C_2^n \longrightarrow 0 \tag{3.15}$$

where the C_2^i's and D are defined using the general principle [8]. Let us call (3.15) a prolongation of the Dolbeault resolution of $\mathscr{O}$. It is not difficult to prove

PROPOSITION 3.3. *The resolution* (3.8) *of* $\mathscr{P}_R^{0,0} = \mathscr{P}_R$ *is the quotient of de Rham's resolution for* $\boldsymbol{R}$ *and the prolongation* (3.15) *of Dolbeault's resolution for* $\mathscr{O}$. *In other words the following diagram is exact and commutative* (writing $P^i = P^i_{0,0}$):

$$
(3.16)\qquad
\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \boldsymbol{R} & \longrightarrow & A^0 & \xrightarrow{d} & A^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & A^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & C_2^0 & \xrightarrow{D} & C_2^1 & \xrightarrow{D} & \cdots & \xrightarrow{D} & C_2^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathscr{P} & \longrightarrow & P^0 & \xrightarrow{D} & P^1 & \xrightarrow{D} & \cdots & \xrightarrow{D} & P^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & & & 0\,. & &
\end{array}
$$

It turns out that the resolution of $\mathscr{P}_R$ can be somewhat simplified. Let us define the following vector bundles over M'; for i odd:

$V^{p,q}$ is the bundle of complex-valued (p, q)-forms $p > q$,

U^{i+1} is the bundle of complex-valued $(i+1)$-forms which belong to the ideal generated by the dz's;

for i even:

$V^{p,q}$ is the bundle of complex-valued (p, q)-forms $p > q$,

$W^{i/2,i/2}$ is the bundle of $(i/2, i/2)$-forms of type $\alpha + \bar{\alpha}$,

U^{i+1} is the bundle of complex-valued $(i+1)-$forms which belong to the ideal generated by the dz's.

Now let us define

$$
\begin{aligned}
&W^0 = W^{0,0} \oplus U^1\\
&W^1 = V^{1,0} \oplus U^2\\
&W^2 = V^{2,0} \oplus W^{1,1} \oplus U^3\\
&W^3 = V^{3,0} \oplus V^{2,1} \oplus U^4\\
&\vdots\\
&W^{2i} = \oplus\,(V^{p,q}) \oplus W^{i,i} \oplus U^{2i+1}\,, \qquad p + q = 2i,\ 0 \leqq i \leqq m\,,\\
&W^{2i-1} = \oplus\,(V^{p,q}) \oplus U^{2i}\,, \qquad p + q = 2i - 1,\ 1 \leqq i \leqq m\,.
\end{aligned}
$$

PROPOSITION 3.4. *The following diagram is exact and commutative*

$$
(3.17)\qquad
\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \boldsymbol{R} & \longrightarrow & A_R^0 & \xrightarrow{d} & A_R^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & A_R^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & C_1^0 & \xrightarrow{D} & C_1^1 & \xrightarrow{D} & \cdots & \xrightarrow{D} & C_1^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathscr{P}_R & \longrightarrow & W^0 & \xrightarrow{D'} & W^1 & \xrightarrow{D'} & \cdots & \xrightarrow{D'} & W^n & \longrightarrow & 0\,. \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & & & 0 & &
\end{array}
$$

The operator 'D is defined by D.

Now let us turn our attention to the open submanifold M of

M'. Let the boundary $\partial\bar{M}$ be a smooth (C^3) submanifold of codimension 1 in M'. A function r on M' is said to define the boundary of M if $r < 0$ on M, $r > 0$ on $M' - \bar{M}$, and $r = 0$ on $\partial\bar{M}$, with grad $r \neq 0$ on $\partial\bar{M}$. Let $U \subset M'$ be a coordinate neighborhood, $U \cap \partial\bar{M} \neq \emptyset$, with the coordinates $(x^1, \cdots, x^n)$, $n = 2m$. Having chosen the hermitian metric on M' it can be shown (see for example C. B. Morrey, Jr. "Multiple Integrals in the Calculus of Variations.") that the coordinate system can be chosen in such a way that on $\partial\bar{M}$

$$\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^n} \right\rangle = 0,\ i < n\ , \tag{3.18}$$

and $x^n = r = 0$ defines $\partial\bar{M}$. This done, assume that

$$\omega^m = \frac{1}{|dr|} \sum_{l=1}^{m} \frac{\partial r}{\partial z^l} dz^l \text{ in } U \tag{3.19}$$

and

$$\left\langle \frac{\partial}{\partial \omega^j}, \frac{\partial}{\partial r} \right\rangle = 0\ , \qquad j < m, \text{on } \partial\bar{M}. \tag{3.20}$$

Notice that $\langle \omega^m, \omega^m \rangle = 1$ and $(\omega^1, \cdots \omega^m)$ is an orthonormal frame (which can be obtained by the Grame-Schmidt orthogonalization process).

Because $\partial/\partial\omega^j = \sum_{k=1}^m b_j^k \partial/\partial z^k$, we get from (3.18), (3.20) that

$$b_j^m = 0, \qquad 1 \leqq j < m\ .$$

Therefore on $\partial\bar{M} \cap U$ we have $\partial b_j^m/\partial x^i = 0$, $i < m$, $1 \leqq j < m$. Finally, on $\partial\bar{M} \cap U$, we have the identities

$$\frac{\partial r}{\partial \omega^j} = 0\ , \qquad \frac{\partial^2 r}{\partial \omega^j \partial \omega^k} = 0\ , \qquad j, k < m\ . \tag{3.21}$$

Let $*$ denote the usual star operator, $*: A^{p,q} \to A^{m-q,m-p}$. This operator can be defined by the formula

$$\langle \phi, \psi \rangle\ \gamma = \phi \wedge *\bar{\psi}\ , \qquad \phi, \psi \in A^{p,q}$$

where $\gamma = *(1)$ is the volume element on M'. The volume element has the local form $*(1) = \det(h)\,(\sqrt{-1})^m dz^1 \wedge \cdots \wedge dz^m \wedge d\bar{z}^1 \wedge \cdots d\bar{z}^m$, or, in our special frame,

$$*(1) = (\sqrt{-1})^m \omega^1 \wedge \cdots \wedge \omega^m \wedge \bar{\omega}^1 \wedge \cdots \wedge \bar{\omega}^m\ . \tag{3.22}$$

For any C^2-function ϕ on M let $L^2(P^i_{p,q}, \phi)$ be the space of all smooth sections u *of* $P^i_{p,q}$ such that

$$||u||_\phi^2 = \int_M |u|^2 e^{-\phi} *(1) < \infty ,$$

$|u|^2 = \langle u, u\rangle$. The global product will be denoted by $(,)_\phi = \int_M \langle , \rangle e^{-\phi} *(-1)$. The operator $D: P_{p,q}^i \to P_{p,q}^{i+1}$ defines a closed densely defined operator $L^2(P_{p,q}^i, \phi) \to L^2(P_{p,q}^{i+1}, \phi)$, which we denote by $\boldsymbol{D}$. Let us denote by $\boldsymbol{D}^*$ its adjoint; and by $\dot{P}_{p,q}^i$ the space of those sections of $P_{p,q}^i$ over M which can be smoothly extended across the boundary $\partial\bar{M}$ into M'. Because the space $\dot{\mathscr{D}}_{D^*}^i = \dot{P}_{p,q}^i \cap \mathscr{D}_{D^*}$ ($\mathscr{D}_{D^*}$ stands for the domain of $\boldsymbol{D}^*$) is dense in $\mathscr{D}_D \cap \mathscr{D}_{D^*}$ with respect to the graph norm $u \to ||u||_\phi + ||\boldsymbol{D}^* u||_\phi + ||\boldsymbol{D}u||_\phi$, let us look at $\dot{\mathscr{D}}_{D^*}^i$ more closely. It can be shown in the same way as in [6] that $u \in \dot{P}_{p,q}^i$ belongs to $\dot{\mathscr{D}}_{D^*}^i$ if and only if

$$(\boldsymbol{D}v, u)_\phi = (v, \boldsymbol{D}^* u)_\phi \text{ for all } v \in \dot{P}_{p,q}^{i-1} . \tag{3.23}$$

Using this relation let us describe the space $\dot{\mathscr{D}}_{D^*}^i$ explicitely. As we are mainly interested in $\dot{\mathscr{D}}_{D^*}^1$ let us take an element $u \in \dot{P}_{p,q}^1$, $u = (\rho, \eta)$, and $v \in \dot{P}_{p,q}^0$, $v = (\sigma, \gamma)$ (see Lemma 3.1). Then

$$\langle \boldsymbol{D}v, u\rangle = \langle D_0\sigma - \gamma, \rho\rangle + \langle D_0^2\sigma - D_0\gamma, \eta\rangle ,$$

where

$$\langle D_0\sigma - \gamma, \rho\rangle = d\sigma \wedge *\bar{\rho} - (\delta\lambda\sigma + \gamma) \wedge *\bar{\rho} ,$$
$$\langle D_0\gamma, \eta\rangle = d\gamma \wedge *\bar{\eta} - \delta\lambda\gamma \wedge *\bar{\eta} ,$$

and $D_0^2 f\sigma = f D_0^2\sigma$ for any function f because D_0^2 is the curvature form of the connection D_0 defined on the vector bundle $R_1^{p,q} \to M'$. Furthermore we get the formulas

$$\begin{aligned} d\sigma \wedge *\bar{\rho}e^{-\phi} &= d(\sigma \wedge *\bar{\rho}e^{-\phi}) - \sigma \wedge d(*\bar{\rho}e^{-\phi}) \\ &= d(\sigma \wedge *\bar{\rho}e^{-\phi}) - *[e^{\phi} * d * (\bar{\rho}e^{-\phi})]e^{-\phi} \\ d\gamma \wedge *\bar{\eta}e^{-\phi} &= d(\gamma \wedge *\bar{\eta}e^{-\phi}) + \gamma \wedge d(*\bar{\eta}e^{-\phi}) \\ &= d(\gamma \wedge *\bar{\eta}e^{-\phi}) - [e^{\phi} * d * (\bar{\eta}e^{-\phi})]e^{-\phi} \end{aligned}$$

therefore for any v with compact support in U, $U \cap \partial\bar{M} \neq \emptyset$,

$$\begin{aligned} \int_{U\cap M} \langle \boldsymbol{D}v, u\rangle e^{-\phi} *(1) &= \int_{U\cap M} (\langle D_0\sigma - \gamma, \rho\rangle + \langle D_0^2\sigma - D_0\gamma, \eta) e^{-\phi} *(1) \\ &= \int_{U\cap M} \{d(\sigma \wedge \bar{\rho}e^{-\phi}) - d(\gamma \wedge *\bar{\eta}e^{-\phi})\} *(1) \\ &\quad + \int_{U\cap M} \{-(\delta\lambda\sigma + \gamma) \wedge *\bar{\rho} - \sigma \wedge [e^{\phi} * d * (\bar{\rho}e^{-\phi})] \\ &\quad + D_0^2\sigma \wedge *\bar{\eta} + \delta\lambda\gamma \wedge *\bar{\eta} - \gamma \wedge *[e^{\phi} * d * (\bar{\eta}e^{-\phi})]e^{-\phi} *(1) . \end{aligned}$$

By Stoke's formula we get

$$(\boldsymbol{D}v, u)_\phi - (v, \boldsymbol{D}^*u)_\phi = \int_{U \cap \partial \bar{M}} (\sigma \wedge *\bar{\rho} - \gamma \wedge *\bar{\eta}) e^{-\phi} *(dr) ,$$

where

$$\boldsymbol{D}^*u = (-e^\phi * d * (\rho e^{-\phi}) + \cdots, -e^\phi * d * (\eta e^{-\phi}) + \cdots) ,$$

and $\cdots$ stands for the terms which do not involve differentiation of u or ϕ. From the above remarks it follows that

$$\int_{U \cap \partial \bar{M}} (\sigma \wedge *\bar{\rho} - \gamma \wedge *\bar{\eta}) e^{-\phi} *(dr) = 0$$

for any $v \in \dot{P}^0_{p,q}$ with compact support in U if and only if $u \in \mathscr{D}^1_{D^*}$. Because σ and γ can vanish independently we get instead

$$\int_{U \cap \partial \bar{M}} \sigma \wedge *\bar{\rho} e^{-\phi} = 0 , \qquad \int_{U \cup \partial \bar{M}} \gamma \wedge *\bar{\eta} e^{-\phi} = 0 .$$

If we use the usual notation for the decomposition of forms into the tangent and normal parts $\Phi = t\Phi + n\Phi$ on $\partial \bar{M}$, we conclude from above that $\sigma \wedge *\bar{\rho} = n(\sigma \wedge *\bar{\rho})$, $\gamma \wedge *\bar{\eta} = n(\gamma \wedge *\bar{\rho})$ because

$$\int_{U \cap \partial \bar{M}} \sigma \wedge *\bar{\rho} e^{-\phi} = \int_{U \cap \partial M} t(\sigma \wedge *\bar{\rho}) e^{-\phi} = 0$$

and analogously for the second integral. But, if for any form Φ, we have $\Phi = n\Phi$ on $\partial \bar{M}$, then $dr \wedge \Phi = 0$. Therefore

$$dr \wedge \sigma \wedge *\bar{\rho} = 0 , \qquad dr \wedge \gamma \wedge *\bar{\eta} = 0 . \tag{3.24}$$

From the first identity we can conclude that $dr \wedge *\bar{\rho} = 0$, because σ is a 0-form with values in $R_1^{p,q}$. Therefore $t\bar{\rho} = 0$ and from the formulas

$$*n = t* , \qquad *t = n*$$

we conclude that $t*\bar{\rho} = *n\bar{\rho} = \overline{*n\rho} = 0$, so that $n\rho = 0$. Recall that γ is a $(T^* \otimes A_R^{p,q})$-valued 1-form (as $\delta(T^* \otimes g_2^{p,q}) \subset T^* \otimes T \otimes A_R^{p,q}$). Such a form γ splits into (1, 0) and (0, 1)-parts, $\gamma = \gamma_1 + \gamma_2$. The secon condition (3.24) should hold for any γ with compact support in U. Therefore

$$dr \wedge \gamma_1 \wedge *\bar{\eta} = 0 , \qquad dr \wedge \gamma_2 \wedge *\bar{\eta} = 0 \tag{3.25}$$

should hold for any η_1 and η_2. From Proposition 3.1 follows that we can write

$$\eta = \Sigma \sum_{k=1}^{m} \eta_{I\bar{J};k}\omega^k \otimes \omega^I \wedge \bar{\omega}^J + \Sigma \sum_{k=1}^{m} \eta_{I\bar{J};\bar{k}}\bar{\omega}^k \otimes \omega^I \wedge \bar{\omega}^J$$
$$+ \Sigma \sum_{k=1}^{m} \eta_{\bar{I}J;k}\omega^k \otimes \bar{\omega}^I \wedge \omega^J + \Sigma \sum_{k=1}^{m} \eta_{\bar{I}J;\bar{k}}\bar{\omega}^k \otimes \bar{\omega}^I \wedge \omega^J ,$$

where $\eta_{I\bar{J};k}$, etc $\cdots$ are 2-forms (3.11). We shall use the obvious notation

$$\eta = \sum_{\substack{|U|=p \\ |V|=q}} \sum_{w=1}^{m} \eta_{UV;w}\omega^w \otimes \omega^U \wedge \omega^V$$

where U, V and w stand for barred as well as for unbarred indices. Then we can write, instead of (3.25), for any (1, 0)-form ϕ and any (0, 1)-form ψ

$$dr \wedge \phi \wedge *\bar{\eta}_{UV;w} = 0 , \qquad dr \wedge \psi \wedge *\bar{\eta}_{UV;w} = 0 .$$

And these identities have to be satisfied for all components of η.

PROPOSITION 3.5. *An element $u \in \dot{P}^1_{p,q}$, $u = (\rho, \eta)$, belongs to $\dot{\mathscr{D}}^1_{D^*}$ if and only if for any $\phi \in A^{1,0}$, $\psi \in A^{0,1}$ with compact support in U, $U \cap \partial\bar{M} \neq \varnothing$,*

$$(3.26)\qquad \begin{gathered} n\rho = 0 , \\ dr \wedge \phi \wedge *\bar{\eta}_{UV;w} = 0 , \qquad dr \wedge \psi \wedge *\bar{\eta}_{UV;w} = 0 , \end{gathered}$$

holds (on $U \cap \partial\bar{M}$).

Using the explicit coordinate description (3.11) of an element $u \in \dot{P}^1_{p,q}$, we get, by direct computation,

COROLLARY. *An element $u \in \dot{P}^1_{p,q}$* belongs to $\dot{\mathscr{D}}^1_{D^*}$ *if and only if for its components* (via (3.11)) *the following identities hold:*

$$(3.27)\qquad \begin{gathered} n\rho_{I\bar{J}} = 0 , \qquad n\rho_{I\bar{J};l} = 0 , \qquad n\rho_{I\bar{J};\bar{l}} = 0 \\ \sum_{l=1}^{m} \frac{\partial r}{\partial\bar{\omega}^l}\eta_{I\bar{J};k,lj} = 0 , \qquad \sum_{l=1}^{m} \frac{\partial r}{\partial\bar{\omega}^l}\eta_{\bar{I}J;k,lj} = 0 , \\ \sum_{l=1}^{m} \frac{\partial r}{\partial\omega^j}\eta_{I\bar{J};k,l\bar{j}} = 0 , \qquad \sum_{l=1}^{m} \frac{\partial r}{\partial\omega_j}\eta_{\bar{I}J;k,lj} = 0 , \\ \sum_{l=1}^{m} \frac{\partial r}{\partial\omega^l}\eta_{I\bar{J};\bar{k},\bar{l}j} = 0 , \qquad \sum_{l=1}^{m} \frac{\partial r}{\partial\omega^l}\eta_{\bar{I}J;\bar{k},\bar{l}j} = 0 , \end{gathered}$$

together with their complex conjugates.

REMARK. If we write down only formally the conjugate equations to (3.27), than using the remark following Proposition 3.1 we get the boundary conditions for an element u in complex situation.

For an element $u \in \dot{\mathscr{D}}^1_{D^*}$ write $\boldsymbol{D}u = \boldsymbol{A}u + \cdots$ and $\boldsymbol{D}^*u = \boldsymbol{B}u + \cdots$, where $\cdots$ stands for those terms where u and ϕ do not get differentiated. Then for such an $u = (\rho, \eta)$ we have

$$\boldsymbol{A}u = \boldsymbol{A}(\rho, \eta) = (d\rho, d\eta) \ . \tag{3.28}$$

Let us introduce symbols

$$d^\phi w = e^\phi d(w \cdot e^{-\phi})$$

and

$$d^\phi = \sum_{l=1}^m \omega^l \wedge d_l^\phi + \sum_{l=1}^m \bar{\omega}^l \wedge d_{\bar{l}}^\phi \ .$$

The differentiating part of the adjoint $\boldsymbol{D}^*$ gives then

$$\begin{aligned} \boldsymbol{B}u = \boldsymbol{B}(\rho, \eta) = (&-\Sigma d_{\bar{l}}^\phi \phi_l - \Sigma d_l^\phi \rho_{\bar{l}} \ , \\ &\Sigma d_{\bar{j}}^\phi \eta_{lj} \omega^l - \Sigma d_{\bar{l}}^\phi \eta_{l\bar{j}} \bar{\omega}^j - \Sigma d_l^\phi \eta_{l\bar{j}} \omega^l \\ &- \Sigma d_{\bar{j}}^\phi \eta_{\bar{l}j} \bar{\omega}^l - \Sigma d_l^\phi \eta_{\bar{l}j} \omega^j + \Sigma d_j^\phi \eta_{\bar{l}\bar{j}} \bar{\omega}^l) \ . \end{aligned} \tag{3.29}$$

It is an easy computation to show that for any $u \in \dot{\mathscr{D}}^1_{D^*}$ there is an inequality

$$\begin{aligned} &|\, \|\boldsymbol{A}u\|_\phi^2 + \|\boldsymbol{B}u\|_\phi^2 - \|\boldsymbol{D}u\|_\phi^2 - \|\boldsymbol{D}^*u\|_\phi^2 \,| \\ &\leqq C\|u\|_\phi(\|\boldsymbol{D}u\|_\phi + \|\boldsymbol{D}^*u\|_\phi + \|u\|_\phi) \end{aligned} \tag{3.30}$$

where C is a constant independent of u and ϕ.

We have chosen the local coframe $(\omega^1, \cdots, \omega^m)$, where ω^i is $(1, 0)$-form. Therefore there are smooth (C^∞) functions c^i_{jk}, and a^i_{jk} such that

$$\begin{aligned} \bar{\partial}\omega^i &= \sum_{j,k=1}^m c^i_{jk} \bar{\omega}^j \wedge \omega^k, \ \partial\bar{\omega}^i = \sum_{j,k=1}^m \bar{c}^i_{jk} \omega^j \wedge \bar{\omega}^k \ , \\ \partial\omega^i &= \sum_{j,k=1}^m a_{jk} \omega^j \wedge \omega^k, \ \bar{\partial}\bar{\omega}^i = \sum_{j,k=1}^m \bar{a}^i_{jk} \bar{\omega}^j \wedge \bar{\omega}^k \ . \end{aligned}$$

If w is any function, then

$$\begin{aligned} &\partial\bar{\partial}w = -\bar{\partial}\partial w = -\sum_{k,j=1}^m w_{k\bar{j}} \bar{\omega}^j \wedge \omega^k \ , \\ &w_{\bar{j}k} = \frac{\partial^2 w}{\partial\bar{\omega}^j \partial\omega^k} + \sum_{i=1}^m c^i_{jk} \frac{\partial w}{\partial\omega^i}, \ w_{k\bar{j}} = \frac{\partial^2 w}{\partial\omega^k \partial\bar{\omega}^j} + \sum_{i=1}^m \bar{c}_{k\,j}^{\,i} \frac{\partial w}{\partial\bar{\omega}^i} \ , \\ &w_{\bar{j}k} = w_{k\bar{j}} \ . \end{aligned} \tag{3.31}$$

And we introduce other symbols, namely

$$w_{jk} = \frac{\partial^2 w}{\partial\omega^j\partial\omega^k} + \sum_{i=1}^m a^i_{jk}\frac{\partial w}{\partial\omega^i},\ w_{\bar{j}k} = \frac{\partial^2 w}{\partial\bar{\omega}^j\partial\bar{\omega}^k} + \sum_{i=1}^m \bar{a}^i_{jk}\frac{\partial w}{\partial\bar{\omega}^i}, \tag{3.32}$$
$$w_{\bar{j}\bar{k}} = w_{\bar{k}\bar{j}},\ w_{jk} = w_{kj} .$$

Because $*(dr)/|dr|$ is the volume element on $\partial\bar{M}$ we have $dr \wedge *dr = |dr|^2*(1)$, where $*(1) = (\sqrt{-1})^m\omega^1 \wedge \cdots \wedge \omega^m \wedge \bar{\omega}^1 \wedge \cdots \bar{\omega}^m$ is the volume element on M'. Let f, g be any two functions with support in a coordinate neighborhood $U \subset M'$. Then

$$d(f\bar{g}e^{-\phi}(\sqrt{-1})^m\omega^1 \wedge \cdots \wedge \omega^{k-1} \wedge \omega^{k+1} \wedge \cdots \wedge \omega^m \wedge \bar{\omega}^1 \wedge \cdots \wedge \bar{\omega}^m)$$
$$= (-1)^{k-1}\frac{\partial f}{\partial\omega^k}\bar{g}e^{-\phi}*(1) + (-1)^{k-1}f\cdot d^\phi_k\bar{g}e^{-\phi}*(1) + (-1)^{k-1}f\bar{g}e^{-\phi}\sigma_k*(1)$$

where σ_k is defined by this relation. By Stoke's fromula we get

PROPOSITION 3.5. *Let f, g be complex-valued functions (C^∞) with supports in U, then we have the formula*

$$\int_{U\cap M}\frac{\partial f}{\partial\omega^k}\bar{g}e^{-\phi}*(1) = -\int_{U\cap M} f\cdot d^\phi_k\bar{g}e^{-\phi}*(1) - \int_{U\cap M} f\bar{g}e^{-\phi}\sigma_k*(1) \tag{3.33}$$
$$+ \int_{U\cap\partial\bar{M}} f\bar{g}\frac{\partial r}{\partial\omega^k}e^{-\phi}*(dr) .$$

There is an analogous formula for $\int_{U\cup M}\partial f/\partial\bar{\omega}^k\bar{g}e^{-\phi}*(1)$ which is obvious.

One more technical device is needed for obtaining the basic estimate—the commutation relations. Using the definition of d^ϕ_i and replacing w in (3.31) by ϕ we get

$$d^\phi_k\frac{\partial w}{\partial\bar{\omega}^j} - \frac{\partial}{\partial\bar{\omega}^j}d^\phi_k w = w\cdot\phi_{k\bar{j}} + \sum_{i=1}^m c^i_{jk}d^\phi_i w - \sum_{i=1}^m c^i_{kj}\frac{\partial w}{\partial\bar{\omega}^i} , \tag{3.34}$$
$$d^\phi_k\frac{\partial w}{\partial\omega^j} - \frac{\partial}{\partial\omega^j}d^\phi_k w = w\cdot\phi_{kj} + \sum_{i=1}^m a^i_{jk}d^\phi_i w - \sum_{i=1}^m a^i_{kj}\frac{\partial w}{\partial\omega^i} .$$

DEFINITION 3.1. Let η be a tangent vector at $\partial\bar{M}$. The quadratic form

$$\langle\bar{\partial}\partial r, \bar{\eta} \wedge \eta\rangle \tag{3.35}$$

is called the Levi form.

If we use the orthonormal coframe $(\omega^1, \cdots, \omega^m)$ then the Levi form can be written in the form

$$\sum_{i,j=1}^m r_{i\bar{j}}\eta_i\bar{\eta}_j, \qquad \eta = \sum_{j=1}^m \eta_j\omega^j .$$

Now, let us compute explicitely $||Au||^2_\phi + ||Bu||^2_\phi$ for $u \in \dot{\mathscr{D}}^1_{D^*}$ and use the estimate (3.30) to make the results of [4] immediately applicable. The computation is rather long and routine. Using (3.28) and (3.29) together with (3.33) and (3.34) we get, for $u \in \dot{\mathscr{D}}^1_{D^*}$ with support in U, the terms involving ρ only:

$$(||Au||^2_\phi + ||Bu||^2_\phi)_\rho = \sum_{l,k} \int_{U \cap M} \left\{\left|\frac{\partial \rho_l}{\partial \omega^k}\right|^2 + \left|\frac{\partial \rho_{\bar{l}}}{\partial \omega^k}\right|^2 + \left|\frac{\partial \rho_l}{\partial \bar{\omega}^k}\right|^2 + \left|\frac{\partial \rho_{\bar{l}}}{\partial \bar{\omega}^k}\right|^2\right\} e^{-\phi} *(1) + \cdots \tag{3.36}$$

and the terms involving η only:

$$\begin{aligned}(||Au||^2_\phi + ||Bu||^2_\phi) &= \Sigma \int_{U \cap M} \left\{\left|\frac{\partial \eta_{lj}}{\partial \omega^k}\right|^2 + \left|\frac{\partial \eta_{\bar{l}\bar{j}}}{\partial \omega^k}\right|^2 + \left|\frac{\partial \eta_{lj}}{\partial \bar{\omega}^k}\right|^2 + \left|\frac{\partial \eta_{l\bar{j}}}{\partial \bar{\omega}^k}\right|^2 + \left|\frac{\partial \eta_{\bar{l}\bar{j}}}{\partial \bar{\omega}^k}\right|^2 \right. \\ &\quad \left. + \left|\frac{\partial \eta_{\bar{l}j}}{\partial \bar{\omega}^k}\right|^2 + \left|\frac{\partial \eta_{l\bar{j}}}{\partial \omega^k}\right|^2 + \left|\frac{\partial \eta_{\bar{l}j}}{\partial \omega^k}\right|^2\right\} e^{-\phi} *(1) + \cdots .\end{aligned} \tag{3.37}$$

Let us put for the moment:
k_1 = the boundary integral in (3.36),
k_2 = the integral following k_1 in (3.36),
k_3 = the terms involving the $d^\phi \rho$'s,
k_4 = the remaining terms.
Therefore (3.36) can be written in the form

$$\begin{aligned}&\Sigma \int_{U \cap M} \{| \quad |^2 + | \quad |^2 + | \quad |^2 + | \quad |^2\} e^{-\phi} *(1) \\ &+ \Sigma \int_{U \cap M} \{\phi_{k\bar{j}}(\rho_k \cdot \overline{\rho_j} + \rho_{\bar{j}} \cdot \overline{\rho_{\bar{k}}}) + \phi_{kj}\rho_j \cdot \overline{\rho_{\bar{k}}} + \phi_{\bar{k}\bar{j}}\rho_{\bar{j}} \cdot \overline{\rho_k}\} e^{-\phi} *(1) \\ &+ k_1 + k_2 + k_3 + k_4 .\end{aligned}$$

The integral k_3 splits into k_3' and k_3''; $k_3 = k_3' + k_3''$, where

$$\begin{aligned}k_3' = \Sigma \int_{U \cap \partial M} &\left\{\bar{c}^k_{ij}\rho_j \cdot \overline{\rho_i}\frac{\partial r}{\partial \bar{\omega}^k} + c^k_{ji}\rho_{\bar{j}} \cdot \overline{\rho_{\bar{i}}}\frac{\partial r}{\partial \omega^k} + \bar{a}^k_{ji}\rho_j \cdot \overline{\rho_{\bar{i}}}\frac{\partial r}{\partial \bar{\omega}^k} \right. \\ &\left. + a^k_{ji}\rho_{\bar{j}} \cdot \overline{\rho_i}\frac{\partial r}{\partial \omega^k}\right\} e^{-\phi} *(dr) .\end{aligned}$$

Using the boundary conditions (3.27) all terms involving the $d^\phi \rho$'s are zero, because $n\rho = 0$ implies

$$\sum_{j=1}^m \rho_j \frac{\partial r}{\partial \bar{\omega}^j} = 0 , \qquad \sum_{j=1}^m \rho_{\bar{j}} \frac{\partial r}{\partial \omega^j} = 0$$

and simple substitution does it. As for the remaining part of k_1 we get by integration by parts

$$k_1 = \Sigma \int_{U \cap \partial \bar{M}} \left\{ \frac{\partial^2 r}{\partial \omega^j \partial \bar{\omega}^i} \rho_j \cdot \overline{\rho_i} + \frac{\partial^2 r}{\partial \bar{\omega}^j \partial \omega^i} \rho_{\bar{j}} \cdot \overline{\rho_{\bar{i}}} + \frac{\partial^2 r}{\partial \omega^j \partial \omega^i} \rho_j \cdot \overline{\rho_{\bar{i}}} \right.$$
$$\left. + \frac{\partial^2 r}{\partial \bar{\omega}^j \partial \bar{\omega}^i} \rho_{\bar{j}} \cdot \overline{\rho_i} \right\} e^{-\phi} * (dr) \ .$$

Then

$$k_1 + k_3' = \Sigma \int_{U \cap \partial \bar{M}} \{ r_{j\bar{i}} \rho_j \cdot \overline{\rho_i} + r_{i\bar{j}} \rho_{\bar{j}} \cdot \overline{\rho_{\bar{i}}} + r_{ji} \cdot \rho_j \cdot \overline{\rho_{\bar{i}}}$$
$$+ r_{\bar{j}\bar{i}} \cdot \rho_{\bar{j}} \cdot \overline{\rho_i} \} e^{-\phi} * (dr) \ .$$

But the special choice of the local frame in U with the property (3.21) shows that the last two terms are zero on $\partial \bar{M}$ so that

$$k_1 + k_3' = \Sigma \int_{U \cap \partial \bar{M}} r_{j\bar{i}} (\rho_j \cdot \overline{\rho_i} + \rho_{\bar{i}} \cdot \overline{\rho_{\bar{j}}}) e^{-\phi} * (dr) \ . \tag{3.38}$$

Let us denote by

$$|||\rho|||_\phi^2 = \Sigma \int_{U \cap M} \left\{ \left| \frac{\partial \rho_l}{\partial \omega^k} \right|^2 + \left| \frac{\partial \rho_{\bar{l}}}{\partial \omega^k} \right|^2 + \left| \frac{\partial \rho_l}{\partial \bar{\omega}^k} \right|^2 + \left| \frac{\partial \rho_{\bar{l}}}{\partial \bar{\omega}^k} \right|^2 \right\} e^{-\phi} * (1)$$
$$+ ||\rho||_\phi^2 \ . \tag{3.39}$$

Then it is easy to show that there are constants C_2, C_3, C_4 such that

$$|k_3''| \leqq C_3 |||\rho|||_\phi \cdot ||\rho||_\phi, \ |k_2| \leqq C_2 |||\rho|||_\phi \cdot ||\rho||_\phi$$

and

$$|k_4| \leqq C_4 ||\rho||_\phi \cdot |\rho|_\phi \ .$$

Similarly let us define t_1, t_2, t_3, t_4 in (3.37). And as we did for k_3 we can also split t_3 into $t_3' + t_3''$, get an estimate for t_2, t_3'', t_4 and write the boundary integral

$$t_1 + t_3' = \Sigma \int_{U \cap \partial \bar{M}} r_{i\bar{j}} (\eta_{li} \cdot \overline{\eta_{lj}} + \eta_{l\bar{j}} \cdot \overline{\eta_{l\bar{i}}} + \eta_{li} \cdot \overline{\eta_{\bar{l}j}}$$
$$+ \eta_{\bar{l}\bar{j}} \cdot \overline{\eta_{\bar{l}\bar{i}}} + \eta_{i\bar{l}} \cdot \overline{\eta_{j\bar{l}}} + \eta_{\bar{j}l} \cdot \overline{\eta_{\bar{i}l}}) e^{-\phi} * (dr) \ . \tag{3.39}$$

By direct computation we get

PROPOSITION 3.6. *For an element $u \in \dot{\mathscr{D}}_{D^*}^1$ vanishing outside a fixed compact subset of a coordinate neighborhood U in M' and for any $\phi \in C^2(\bar{M})$, $\partial \bar{M} \in C^3$ the following estimate holds*

$$
\begin{aligned}
&| \, ||\boldsymbol{D}^*u|_\phi^2 + ||\boldsymbol{D}u||_\phi^2 - Q_1(u, u) - Q_2(u, u) - Q_3(u, u) \, | \\
&\leqq C||u||_\phi(||\boldsymbol{D}u||_\phi + ||\boldsymbol{D}^*u||_\phi + |||u|||_\phi)
\end{aligned} \tag{3.40}
$$

where

$$
\begin{aligned}
Q_1(u, u) = \Sigma \int_{U \cap M} \Big\{ & \left|\frac{\partial \rho_l}{\partial \omega^k}\right|^2 + \left|\frac{\partial \rho_{\bar l}}{\partial \omega^k}\right|^2 + \left|\frac{\partial \rho_l}{\partial \bar\omega^k}\right|^2 + \left|\frac{\partial \rho_{\bar l}}{\partial \bar\omega^k}\right|^2 + \left|\frac{\partial \eta_{lj}}{\partial \omega^k}\right|^2 \\
& + \left|\frac{\partial \eta_{l\bar j}}{\partial \omega^k}\right|^2 + \left|\frac{\partial \eta_{\bar l\bar j}}{\partial \omega^k}\right|^2 + \left|\frac{\partial \eta_{\bar l j}}{\partial \omega^k}\right|^2 + \left|\frac{\partial \eta_{lj}}{\partial \bar\omega^k}\right|^2 + \left|\frac{\partial \eta_{l\bar j}}{\partial \bar\omega^k}\right|^2 \\
& + \left|\frac{\partial \eta_{\bar l\bar j}}{\partial \bar\omega^k}\right|^2 + \left|\frac{\partial \eta_{\bar l j}}{\partial \bar\omega^k}\right|^2 \Big\} e^{-\phi} * (1) \\
Q_2(u, u) = \Sigma \int_{U \cap M} \{ & \phi_{k\bar j}(\rho_j \cdot \overline{\rho_k} + \rho_{\bar j} \cdot \overline{\rho_{\bar k}}) + \phi_{kj}\rho_j \cdot \overline{\rho_{\bar k}} + \phi_{\bar k\bar j}\rho_{\bar j} \cdot \overline{\rho_k} \\
& + \phi_{i\bar j}(\eta_{ki} \cdot \overline{\eta_{kj}} + \eta_{l\bar j} \cdot \overline{\eta_{l\bar i}} + \eta_{\bar l i} \cdot \overline{\eta_{\bar l j}} + \eta_{\bar l\bar j} \cdot \overline{\eta_{\bar l\bar i}} + \eta_{i\bar l} \cdot \overline{\eta_{j\bar l}} \\
& + \eta_{\bar j l} \cdot \overline{\eta_{\bar i l}}) + \phi_{ij}\eta_{\bar l j} \cdot \overline{\eta_{\bar i\bar l}} + \phi_{\bar i\bar j}(\eta_{l\bar i} \cdot \overline{\eta_{lj}} + \eta_{\bar j\bar l} \cdot \overline{\eta_{\bar l\bar i}} \\
& + \eta_{lj} \cdot \overline{\eta_{l\bar i}})\} e^{-\phi} * (1) \\
Q_3(u, u) = \Sigma \int_{U \cap \partial \bar M} & r_{i\bar j}(\rho_i \cdot \overline{\rho_j} + \rho_{\bar j} \cdot \overline{\rho_{\bar i}} + \eta_{li} \cdot \overline{\eta_{lj}} + \eta_{l\bar j} \cdot \overline{\eta_{l\bar i}} + \eta_{\bar l i} \cdot \overline{\eta_{\bar l j}} \\
& + \eta_{\bar l\bar j} \cdot \overline{\eta_{\bar l\bar i}} + \eta_{i\bar l} \cdot \overline{\eta_{j\bar l}} + \eta_{\bar j l} \cdot \overline{\eta_{\bar i l}}) e^{-\phi} * (dr) \, .
\end{aligned}
$$

This proposition corresponds to Proposition 3.1.1 is [4]. Now applying the technique of [4] to our situation we get

LEMMA 3.1. *If the Levi from* (3.35) *has at least* $(n-2)$ *positive eigenvalues or at least* 3 *negative eigenvalues at every point on* $\partial \bar M$ *then there exists a constant* $C > 0$ *such that*

$$
|\boldsymbol{D}u|_\phi^2 + |\boldsymbol{D}^*u|_\phi^2 + |u|^2 \geqq C \int_{\partial \bar M} |u|^2 e^{-\phi} * (dr) \tag{3.41}
$$

for $u \in \dot{\mathscr{D}}^1_{D^*}$.

We are now in the position that the Kohn-Nirenberg Theorem can be applied (Theorem 5, § 2 [7]). Let us denote by $\boldsymbol{N}^1$ the subspace of $\dot{\boldsymbol{P}}^1_{p,q}$ composed of all sections $u \in \dot{\boldsymbol{P}}^1_{p,q}$ satisfying the boundary conditions (3.23) and

$$
(\boldsymbol{D}v, \boldsymbol{D}u)_\phi = (u, \boldsymbol{D}^*\boldsymbol{D}u)_\phi \text{ for all } v \in \dot{\boldsymbol{P}}^1_{p,q} \, . \tag{3.42}
$$

Let $\boldsymbol{H}^1$ be the subspace of $\boldsymbol{N}^1$ which is annihilated by the laplacian $\boldsymbol{D}\boldsymbol{D}^* + \boldsymbol{D}^*\boldsymbol{D}$, i.e., $\boldsymbol{H}^1 = \{u \in \boldsymbol{N}^1 \mid \boldsymbol{D}u = \boldsymbol{D}^*u = 0\}$.

THEOREM 3.1. *For an open manifold* $M \in M'$, $\partial \bar M \subset C^3$, *satisfying* *the assumptions of the previous Lemma* 3.1, *the Neumann problem is solvable for the operator* $\boldsymbol{D}: P^0_{p,q} \to P^1_{p,q}$ (related to dd_c by (1.1)) *at* $P^1_{p,q}$. *This means that* H^1 *is closed in* $L^2(P^1_{p,q}, \phi)$, *and that there exists a bounded operator* $N: L^2(P^1_{p,q}, \phi) \to L^2(P^1_{p,q}, \phi)$ *such that its range is in* $\boldsymbol{N}^1$, *and*

(i) $NH = HN$, *where* $H: L^2(P^1_{p,q}, \phi) \to \boldsymbol{H}^1$ *is the orthogonal projection,*

(ii) *each element* $u \in L^2(P^1_{p,q}, \phi)$ *can be written in the form* $u = \boldsymbol{D}\boldsymbol{D}^*Nu + \boldsymbol{D}^*\boldsymbol{D}Nu + Hu$, *where the terms are mutually orthogonal,*

(iii) $\boldsymbol{D}N = N\boldsymbol{D}$.

REMARKS. 1. If one drops the "side conditions" (3.12) and considers the operator $\partial\bar{\partial}$ instead of dd_c then exactly the same conditions on the Levi form are sufficient for the solvability of the Neumann problem related to $\partial\bar{\partial}$.

2. All the computations have been done at $P^1_{p,q}$ only. It would be only a technical problem to get an extension in that direction and show that on strongly pseudoconvex manifolds the Neumann problem is solvable (for $\partial\bar{\partial}$ and dd_c).

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