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RADON-NIKODYM DENSITIES AND JACOBIANS

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A Dirichlet mapping between regions in Euclidean space is a homeomorphism preserving the finiteness of Dirichlet integrals of admissible functions and plays an important role in the potential theory. Two dimensional Dirichlet mappings are known to be characterized geometrically as being quasiconformal mappings. In this paper, higher dimensional Dirichlet mappings will be characterized geometrically as being quasi-isometries. In order to carry out the reasoning it is necessary to study the relation between the Radon-Nikodym density R and the Jacobian J of an arbitrary homeomorphism for which only existences of R and J almost everywhere are assured. It will be proven that $R \leqq |J|$, almost everywhere, which is the main result of this paper.

The change of variables is one of the important subjects in the theory of integrals. Suppose $y = y(x)$ maps an open set D_1 homeomorphically noto an open set D_2 in the m-dimensional Euclidean space $E^m (m \geqq 1)$. Define the outer measure $\nu(X)$ of a subset X of D_1 by

$$\nu(X) = \inf \int \cdots \int_{y(U)} dy^1 \cdots dy^m \tag{1}$$

where the infimum is taken with respect to open sets U in D_1 containing X, and denote by $\boldsymbol{B}_\nu(D_1)$ the field of ν-measurable sets in $\boldsymbol{L}(D_1)$, the field of Lebesgue measurable subsets of D_1. The problem of the change of variables is to study the structure of the measure space $(\nu, \boldsymbol{B}_\nu(D_1), D_1)$. Among contributions in this direction the following theorem of Rademacher [6] is frequently made use of (see also a comprehensive alternative proof of Tsuji [10]):

Suppose $y = y(z)$ is *almost Lipshitzian* in the sense that

$$\limsup_{h \to 0} |h|^{-1} |y(x+h) - y(x)| < \infty \tag{2}$$

for almost every x in D_1. Then

$$\boldsymbol{B}_\nu(D_1) = \boldsymbol{L}(D_1) = \{X \mid X \in \boldsymbol{L}(D_1),\ y(X) \in \boldsymbol{L}(D_2)\} \tag{3}$$

and the *Jacobian*

$$J_y(x) = \det\left(\frac{\partial y^i}{\partial x^j}(x)\right) \tag{4}$$

of $y = y(x)$ exists for almost every x in D_1 and

$$\nu(x) = \int \cdots \int_X | J_y(x) | \, dx^1 \cdots dx^m \tag{5}$$

for every X in $\boldsymbol{B}_\nu(D_1) = \boldsymbol{L}(D_1)$.

The mapping $y = y(x)$ with the property (3) is called *inverse-measurable*. This property is equivalent to $y(X)$ having Lebesgue measure zero along with X. For such a mapping $y = y(x)$ there exists a *Radon-Nikodym density* $R_y(x)$ associated with $y = y(x)$ characterized as follows: $R_y(x) \geqq 0$, $R_y(x)$ is Lebesgue measurable on D_1, and

$$\nu(X) = \int \cdots \int_X R_y(x) dx^1 \cdots dx^m \tag{6}$$

for every X in $\boldsymbol{B}_\nu(D_1) = \boldsymbol{L}(D_1)$. The Rademacher theorem can be restated as follows: if a homeomorphism $y = y(x)$ is almost Lipshitzian, then its Radon-Nikodym density $R_y(x)$ and Jacobian $J_y(x)$ exist almost everywhere and are identical in absolute value. However it frequently occurs that only the existence of $R_y(x)$ and $J_y(x)$ almost everywhere is assured. In such a case what can be said about the relation between $R_y(x)$ and $J_y(x)$? The purpose of our paper is to study this question.

Our main result is that if $R_y(x)$ and $J_y(x)$ exist almost everywhere, then inequality $R_y(x) \leqq | J_y(x) |$ is valid almost everywhere (Theorem 5). As a consequence, if, in addition to the existence of $R_y(x)$ and $J_y(x)$ almost everywhere, the inverse mapping $x = x(y)$ of $y = y(x)$ also has $R_x(y)$ and $J_x(y)$ almost everywhere, and if $J_y(x) \cdot J_x(y(x)) = 1$ almost everywhere, then $R_y(x) = | J_y(x) |$ (Theorem 10).

As an application we shall show that a homeomorphism between open sets in $E^m (m \geqq 3)$ is a Dirichlet mapping, a mapping preserving the finiteness of Dirichlet integrals of admissible functions, if and only if it is a quasi-isometry (Theorem 14).

Asymptotical Differentiability

1. We denote by E^m the m-dimensional real Euclidean space whose points x are n-tuples $x = (x^1, \cdots, x^m)$ of real numbers $(m \geqq 1)$. It is an additive vector space over the real number field $\boldsymbol{R}$. The distance between two points $x = (x^1, \cdots, x^m)$ and $y = (y^1, \cdots, y^m)$ is denoted by $| x - y | = (\sum_{i=1}^m | x^i - y^i |^2)^{1/2}$.

For an open set D in E^m we denote by $\boldsymbol{L}(D)$ the family of all Lebesgue measurable sets in D and by μ the Lebesgue measure on $\boldsymbol{L}(E^m)$. Let X be a bounded set in $\boldsymbol{L}(E^m)$. The number

$$r(X) = \sup \frac{\mu(X)}{\mu(I)} \tag{7}$$

is called the *parameter of regularity* for X where the supremum is taken with respect to m-dimensional cubes $I \supset X$. A sequence $\{X_n\}$ of bounded *closed* sets in E^m is said to be a *regular sequence* converging to a point $x_0 \in E^m$ if $x_0 \in X_n$ $(n = 1, 2, \cdots)$ and there exists a positive number $\alpha > 0$ such that

$$r(X_n) \geqq \alpha \qquad (n = 1, 2, \cdots) \tag{8}$$

and the diameters of X_n tend to zero as n tends to infinity.

Fix a set $X \in \boldsymbol{L}(E^m)$ and consider a regular sequence $\{X_n\}$ converging to $x \in E^m$ such that

$$a\{X_n\} = \lim_{n\to\infty} \frac{\mu(X \cap X_n)}{\mu(X_n)}$$

exists. Denote by $\bar{\delta}_X(x)$ (resp. $\underline{\delta}_X(x)$) the supremum (resp. infimum) of the set of all possible $a\{X_n\}$. If $\bar{\delta}_X(x) = \underline{\delta}_X(x)$, then the common value is denoted by $\delta_X(x)$ and called the *density* (or more precisely the *Lebesgue density*) of X at x. If $\delta_X(x)$ exists, then

$$\delta_X(x) = \lim_{n\to\infty} \frac{\mu(X \cap X_n)}{\mu(X_n)} \tag{9}$$

for *every* regular sequence $\{X_n\}$ converging to x.

The well-known Lebesgue density theorem claims that $\delta_X(x) = 1$ almost everywhere on X and $\delta_X(x) = 0$ almost everywhere on $E^m - X$. This is the special case of the Lebesgue differentiability theorem: if $f(x)$ is integrable on E^m, then, for almost every $x_0 \in E^m$,

$$\lim_{n\to\infty} \mu(X_n)^{-1} \int_{X_n} f(x) d\mu(x) = f(x_0) \tag{10}$$

for every regular sequence $\{X_n\}$ converging to x_0.

2. Let $f(x)$ be a real-valued function on a measurable set X. The function f is said to be *asymptotically differentiable* at $x_0 \in X$ if there exist real numbers $a_i(x_0)$ $(i = 1, \cdots, m)$ and a measurable subset $S(x_0)$ of X containing x_0 such that

$$f(x) = f(x_0) + \sum_{i=1}^{m} a_i(x_0)(x^i - x_0^i) + \lambda(x; x_0) \ , \tag{11}$$

with

$$\lim_{x \in S(x_0), x\to x_0} \frac{\lambda(x; x_0)}{|x - x_0|} = 0 \ , \tag{12}$$

and

(13) $$\delta_{S(x_0)}(x_0) = 1\ .$$

The set $S(x_0)$ is referred to as an *asymptotic set* of f at x_0 for its asymptotical differentiability.

The following theorem of Stepanoff [9] plays an important role in our reasoning:

Suppose $f(x)$ is measurable on an open set $D \subset E^m$ and $(\partial f/\partial x^i)(x)$ $(i = 1, \cdots, m)$ exist for almost every $x \in D$. Then $f(x)$ is asymptotically differentiable at almost every point $x_0 \in D$ and $a_i(x_0)$ in (11) *is given by*

(14) $$a_i(x_0) = \frac{\partial f}{\partial x^i}(x_0) \qquad (i = 1 \cdots m)\ .$$

For a proof we refer Stepanoff [9; 523–524] or the monograph of Saks [7; 300–303]. In the latter reference, however, the asymptotical difierentiability is replaced by the approximate differentiability which is defined by (11)—(13), with the Lebesgue density replaced by the Saks strong density which is simply called the density in the book. The proof there can be easily modified to fit the present situation. It should also be remarked that proofs given in both references are for the case $m = 2$, but their generalizations to higher dimensions are, as they claim, straightforward.

3. Let $y = y(x)$ be a mapping of a measurable set $X \subset E^m$ into E^m. We call $y(x)$ *asymptotically differentiable* at $x_0 \in X$ if there exists a matrix $A(x_0)$ of (m, m)-type and a measurable set $S(x_0) \subset X$ containing x_0 such that

(15) $$y(x) = y(x_0) + (x - x_0)A(x_0) + \Lambda(x; x_0)$$

where $\Lambda(x; x_0) = (\lambda^1(x; x_0), \cdots, \lambda^m(x; x_0)) \in E^m$,

(16) $$\lim_{x \in S(x_0), x \to x_0} \frac{|\Lambda(x; x_0)|}{|x - x_0|} = 0\ ,$$

and

(17) $$\delta_{S(x_0)}(x_0) = 1\ .$$

The set $S(x_0)$ is again referred to as an *asymptotic set* of $y(x)$ at x_0 for its asymptotical differentiability.

The Stepanoff theorem cited in 2 may be restated as follows:

Suppose $y(x)$ is a mapping of an open set $D \subset E^m$ into E^m, the components $y^i(x)$ $(i = 1, \cdots, m)$ of $y(x)$ are measurable in D, and the $(\partial y^i/\partial x^j)(x)$ $(i, j = 1, \cdots, m)$ exist for almost every x in D. Then $y(x)$

is asymptotically differentiable at almost every point x_0 in D, and $A(x_0)$ in (15) *is given by*

$$A(x_0) = \begin{pmatrix} \frac{\partial y^1}{\partial x^1}(x_0) & \cdots & \frac{\partial y^m}{\partial x^1}(x_0) \\ \vdots & & \vdots \\ \frac{\partial y^1}{\partial x^m}(x_0) & \cdots & \frac{\partial y^m}{\partial x^m}(x_0) \end{pmatrix} . \tag{18}$$

To see this take $E_i \subset D$ for each $i = 1, \cdots, m$ such that $y^i(x)$ is asymptotically differentiable at each point $x \in E_i$ and $\mu(D - E_i) = 0$. Set $E = \bigcap_{i=1}^m E_i$. Then $\mu(D - E) = 0$ and every $y^i(x)$ $(i = 1, \cdots, m)$ is asymptotically differentiable at each point x_0 in E. Let $S_i(x_0)$ be an asymptotic set of $y^i(x)$ at x_0 for its asymptotical differentiability $(i = 1, \cdots, m)$ and put $S(x_0) = \bigcap_{i=1}^m S_i(x_0)$. Then (15) and (16) with (18) are valid. We have only to prove (17).

Let $\{X_n\}$ be an arbitrary regular sequence converging to x_0. Since $\delta_{S_i(x_0)}(x_0) = 1$, we infer that

$$\lim_{n\to\infty} \frac{\mu((E^m - S_i(x_0)) \cap X_n)}{\mu(X_n)} = 0 \tag{19}$$

for every $i = 1, \cdots m$. Observe that

$$\frac{\mu((E^m - S(x_0)) \cap X_n)}{\mu(X_n)} \leqq \sum_{i=1}^{m} \frac{\mu((E^m - S_i(x_0)) \cap X_n)}{\mu(X_n)} ,$$

and therefore (19) implies

$$\lim_{n\to\infty} \frac{\mu((E^m - S(x_0)) \cap X_n)}{\mu(X_n)} = 0 . \tag{20}$$

In view of $\mu(S(x_0) \cap X_n) + \mu((E^m - S(x_0)) \cap X_n) = \mu(X_n)$, we conclude that

$$\lim_{n\to\infty} \frac{\mu(S(X_n) \cap X_n)}{\mu(X_n)} = 1 . \tag{21}$$

Since (21) is true for every regular sequence $\{X_n\}$ converging to x_0, we obtain (17).

The Main Inequality

4. Hereafter we assume that $y = y(x)$ is a *homeomorphism* of an open set D_1 onto an open set D_2 in $E^m (m \geqq 1)$. In terms of components, $y = y(x)$ is expressed by the system of functions on D_1:

$$\begin{cases} y^1 = y^1(x) = y^1(x^1, \cdots, x^m) \\ \cdots \\ y^m = y^m(x) = y^m(x^1, \cdots, x^m) \end{cases} . \tag{22}$$

The mapping $y = y(x)$ is said to be *inverse-measurable* if $X \in \boldsymbol{L}(D_1)$ implies $y(X) \in \boldsymbol{L}(D_2)$, or what amounts to the same, if $\mu(X) = 0$ $(X \subset D_1)$ implies $\mu(y(X)) = 0$. In this case the nonnegative set function $\nu(X)$ on $\boldsymbol{L}(D_1)$ defined by

$$\nu(X) = \mu(y(X)) \tag{23}$$

is μ-absolutely continuous and we have the existence of the *Radon-Nikodym density* $R_y(x)$ of $y = y(x)$, characterized by the properties $R_y(x) \geqq 0$ on D_1, $R_y(x)$ is Lebesgue measurable on D_1, and

$$\nu(X) = \int_X R_y(x) d\mu(x) \qquad (X \in \boldsymbol{L}(D_1)) . \tag{24}$$

We call the mapping $y = y(x)$ *partially differentiable* at $x \in D_1$ if the partial derivatives $(\partial y^i/\partial x^j)(x)$ $(i, j = 1, \cdots, m)$ exist at x. In this case the *Jacobian* $J_y(x)$ of $y = y(x)$ at x exists:

$$J_y(x) = \begin{vmatrix} \dfrac{\partial y^1}{\partial x^1}(x) \cdots \dfrac{\partial y^1}{\partial x^m}(x) \\ \vdots \qquad\qquad \vdots \\ \dfrac{\partial y^m}{\partial x^1}(x) \cdots \dfrac{\partial y^m}{\partial x^m}(x) \end{vmatrix} . \tag{25}$$

Here we append a remark: if $\partial y^i/\partial x^j$ exists almost everywhere on D_1, then it is Lebesgue measurable (cf. Saks [7; p. 299]). Thus if $J_y(x)$ exists almost everywhere on D_1, then $J_x(x)$ is Lebesgue measurable on D_1.

5. The general conclusion we can make on the relation between $R_y(x)$ and $J_y(x)$ is the following which is the main result of the paper:

THEOREM. *Let $y = y(x)$ be a homeomorphism of an open set D_1 onto an open set D_2 in $E^m (m \geqq 1)$ which is inverse-measurable on D_1 and partially differentiable at almost every point in D_1. Then the Radon-Nikodym density $R_y(x)$ and the Jacobian $J_y(x)$ of $y = y(x)$ satisfy the following inequality almost everywhere on D_1:*

$$R_y(x) \leqq | J_y(x) | . \tag{26}$$

The proof will be given in 6–9.

6. By the Lebesgue differentiability theorem we can find a set $E \subset D_1$ such that $\mu(D_1 - E) = 0$ and

$$\lim_{n\to\infty} \mu(X_n)^{-1} \int_{X_n} R_y(x) d\mu(x) = R_y(x_0) \tag{27}$$

for every regular sequence $\{X_n\}$ converging to any point $x_0 \in E$.

By the Stepanoff theorem in 3 we can also find a set $F \subset D_1$ with $\mu(D_1 - F) = 0$ such that $y = y(x)$ is asymptotically differentiable at every point $x_0 \in F$. Set $G = E \cap F$. We have $\mu(D_1 - G) = 0$.

Fix an arbitrary point $x_0 \in G$ and represent $y = y(x)$ by

$$y^j = y_0^i + \sum_{k=1}^{m} a_{jk}(x^k - x_0^k) + \lambda^j(x; x_0) \qquad (j = 1, \cdots, m) \tag{28}$$

with $y_0 = y(x_0)$ and $a_{jk} = (\partial y^j / \partial x^k)(x_0)$ $(j, k = 1, \cdots, m)$. We also set

$$\tau(x) = |x - x_0|^{-1} \left(\sum_{j=1}^{m} (\lambda^j(x; x_0))^2 \right)^{1/2} . \tag{29}$$

Let $S = S(x_0)$ be an asymptotic set of $y = y(x)$ at x_0 for its asymptotical differentiability. Recall that

$$\delta_S(x_0) = 1 . \tag{30}$$

From this and (16) we obtain

$$\lim_{x \in S, x \to x_0} \tau(x) = 0 . \tag{31}$$

As an approximation to (28) we consider a linear transformation $z = z(x)$ given by

$$z^j = y_0^j + \sum_{k=1}^{m} a_{jk}(x^k - x_0^k) \qquad (j = 1, \cdots, m) . \tag{32}$$

Let $K(r) = \{x \mid |x - x_0| \leqq r\}$ and set $\varepsilon(x) = \sup_{x \in S \cap K(r)} \tau(x)$. Then (31) implies that

$$\lim_{r \to 0} \varepsilon(r) = 0 . \tag{33}$$

Observe that if $x \in S \cap K(a)$, then, by (28), (29), and (32),

$$\begin{aligned} |y(x) - z(x)|^2 &= \sum_{j=1}^{m} |y^j(x) - z^j(x)|^2 \\ &= \sum_{j=1}^{m} (\lambda^i(x; x_0))^2 = (|x - x_0| \tau(x))^2 , \end{aligned}$$

and therefore

$$|y(x) - z(x)| \leqq \varepsilon(r) \cdot r \qquad (x \in S \cap K(r)) . \tag{34}$$

7. For short we will set $S(r) = S \cap K(r)$. By (30) and (9) we obtain

$$\lim_{r \to 0} \frac{\mu(S(r))}{\mu(K(r))} = 1 . \tag{35}$$

Take a *closed* subset $\tilde{S}(r)$ of $S(r)$ such that $x_0 \in \tilde{S}(r)$ and

$$\mu(\tilde{S}(r)) \geqq (1-r)\mu(S(r)) ,$$

$$\int_{\tilde{S}(r)} R_y(x)d\mu(x) \geqq (1-r)\int_{S(r)} R_y(x)d\mu(x) . \tag{36}$$

Then by (35), $\lim_{r\to 0}\mu(\tilde{S}(r))/\mu(K(r)) = 1$, which shows that for any strictly decreasing sequence $\{r_n\}$ converging to zero, $\{\tilde{S}(r_n)\}$ is a regular sequence converging to x_0. Therefore by (27)

$$\lim_{r\to 0}\mu(\tilde{S}(r))^{-1}\int_{\tilde{S}(r)} R_y(x)d\mu(x) = R_y(x_0) .$$

This with (35) and (36) gives

$$\lim_{r\to 0}\frac{\mu(y(S(r)))}{\mu(K(r))} = R_y(x_0) . \tag{37}$$

In view of this, it will suffice to prove that

$$\lim_{r\to 0}\frac{\mu(y(S(r)))}{\mu(K(r))} \leqq |J_y(x_0)| . \tag{38}$$

8. To establish (38) we will distinguish the cases $J_y(x_0) = 0$ and $J_y(x_0) \neq 0$. First suppose $J_y(x_0) = 0$. Then the mapping $z = z(x)$ in (32) is degenerate and therefore $z(S(r))$ lies on a hyperplane P which may be identified with E^{m-1}, the $(m-1)$-demensional Euclidean space. By (32)

$$\sum_{j=1}^{m}|z^j - y_0^j|^2 \leqq a^2\sum_{j=1}^{m}|x^j - x_0^j|^2,\ a = \left(\sum_{j,k=1}^{m}a_{jk}^2\right)^{1/2} ,$$

and a fortiori

$$\mu_{m-1}(z(S(r)) \leqq \Gamma\left(1+\frac{m-1}{2}\right)^{-1}\pi^{(m-1)/2}(ar)^{m-1}$$

where μ_{m-1} is the $(m-1)$-dimensional Lebesgue measure on $P = E^{m-1}$. By virtue of (34), $y(S(r))$ is contained in a cylindrical region with the base congruent to $z(S(r(1+\varepsilon(r))))$, and of heigh $2\cdot\varepsilon(r)\cdot r$. Hence

$$\mu(y(S(r))) \leqq 2\left(\Gamma\left(1+\frac{m-1}{2}\right)^{-1}\pi^{(m-1)/2}\right)\cdot a^{m-1}r^m\varepsilon(r)(1+\varepsilon(r))^{m-1} .$$

Since $\mu(K(r)) = \Gamma(1+m/2)^{-1}\pi^{m/2}r^m$, (33) implies that

$$\lim_{r\to 0}\frac{\mu(y(S(r)))}{\mu(K(r))} = 0 ,$$

and (38) is trivially true with the equality.

9. Next we treat the case $|J_y(x_0)| > 0$. Let (b_{ij}) be the inverse matrix of (a_{ij}), the existence of which is assured by $\det(a_{ij}) = J_y(x_0) \neq 0$. The inverse transformation of (32) is then given by

$$x^j = x_0^j + \sum_{k=1}^m b_{jk}(z^k - y_0^k) \qquad (j = 1, \cdots, m) . \tag{39}$$

Denote by ${}^t(b_{ij})$ the transposed matrix of (b_{ij}); let $(B_{ij}) = {}^t(b_{ij}) \cdot (b_{ij})$; then consider the strictly positive definite bilinear form

$$B[\xi, \eta] = \sum_{j,k=1}^m B_{jk}\xi^j\eta^k$$

for $\xi = (\xi^1, \cdots, \xi^m)$ and $\eta = (\eta^1, \cdots, \eta^m)$ in E^m; set

$$\begin{aligned} B[\xi] = B[\xi, \xi] &= \sum_{j,k=1}^m B_{jk}\xi^j\xi^k \\ &= \sum_{j=1}^m \left(\sum_{k=1}^m b_{jk}\xi^k\right)^2 . \end{aligned}$$

Let

$$H(r) = \{\xi \in E^m \mid B[\xi - y_0] \leqq r^2\} .$$

This set is a closed ellipsoid with center y_0. Since (39) implies that

$$B[z - y_0] = \sum_{j=1}^m \left(\sum_{k=1}^m b_{jk}(z^k - y_0^k)\right)^2 = \sum_{j=1}^m (x^j - x_0^j)^2 ,$$

the image of $H(r)$ under (39) is $K(r)$, and therefore $z(K(r)) = H(r)$. Hence

$$\mu(H(r)) = \mu(z(K(r))) = \int_{K(r)} |J_z(x)| \, d\mu(x) .$$

The Jacobian $J_z(x)$ of the mapping $z = z(x)$ in (32) is given by

$$J_z(x) \equiv \det(a_{ij}) = J_y(x_0)$$

and we obtain

$$\mu(H(r)) = |J_y(x_0)| \, \mu(K(r)) . \tag{40}$$

Let $x \in S(r) = S \cap K(r)$. Since $B[\cdot]^{1/2}$ is a norm on E^m,

$$B[y(x) - y_0]^{1/2} \leqq B[y(x) - z(x)]^{1/2} + B[z(x) - y_0]^{1/2} . \tag{41}$$

Observe that $x \in S(r) \subset K(r)$ implies $z(x) \in z(K(r)) = H(r)$, which in turn gives

$$B[z(x) - y_0] \leqq r^2 . \tag{42}$$

In view of (34) we also conclude that

$$B[y(x) - z(x)] \leqq b^2 \,|\, y(x) - z(x) \,|^2 \leqq b^2(\varepsilon(r)\cdot r)^2 \tag{43}$$

with $b = (\sum_{j,k=1}^{m} b_{jk}^2)^{1/2}$. From (41)—(43), it follows that

$$B[y(x) - y_0] \leqq (r(1 + b\varepsilon(r)))^2$$

for x in $S(r)$ and therefore

$$y(S(r)) \subset H(r(1 + b\varepsilon(r))) \ .$$

Thus (40) can be used to conclude that

$$\begin{aligned} \mu(y(S(r))) &\leqq \mu(H(r(1 + b\varepsilon(r)))) \\ &= |\, J_y(x_0) \,|\, \mu(K(r)))(1 + b\varepsilon(r))^m \ . \end{aligned}$$

Again by (33), we now have

$$\lim_{r\to 0} \frac{\mu(y(S(r)))}{\mu(K(r))} \leqq |\, J_y(x_0) \,| \ ,$$

i.e., (38) is proved.

The proof of Theorem 5 is herewith complete.

Integration by Change of Variables

10. As a direct consequence of the main inequality (26) we obtain the following result on integration by change of variables:

THEOREM. *Let $y = y(x)$ be a homeomorphism of an open set D_1 onto an open set D_2 in E^m and let $x = x(y)$ be the inverse mapping of $y = y(x)$. Suppose both $y = y(x)$ and $x = x(y)$ are inverse-measurable and also partially differentiable almost everywhere. Moreover suppose the Jacobians $J_y(x)$ and $J_x(y)$ of $y = y(x)$ and $x = x(y)$ satisfy*

$$J_y(x)\cdot J_x(y(x)) = 1 \tag{44}$$

almost everywhere on D_1. Then $f(y)$ is Lebesgue measurable on D_2 if and only if $f(y(x))$ is Lebesgue measurable on D_1, and if $f(y)$ is Lebesgue integrable on D_2, then $f(y(x))J_y(x)$ is Lebesgue integrable on D_1 and

$$\int\cdots\int_{D_2} f(y)dy^1\cdots dy^m = \int\cdots\int_{D_1} f(y(x))\,|\,J_y(x)\,|\,dx^1\cdots dx^m \ . \tag{45}$$

11. For the proof take the Radon-Nikodym densities $R_y(x)$ and $R_x(y)$ of $y = y(x)$ and $x = x(y)$, and observe that

$$\int_X d\mu(x) = \int_{x(y(X))} d\mu(x)$$
$$= \int_{y(X)} R_x(y)d\mu(y) = \int_X R_x(y(x))R_y(x)d\mu(x)$$

for every $X \in \boldsymbol{L}(D_1)$. Then

$$R_y(x) \cdot R_x(y(x)) = 1 \tag{46}$$

almost everywhere on D_1.

Then main inequality (26) applied to $x = x(y)$ yields $R_x(y) \leqq |J_x(y)|$ almost everywhere on D_2, and since $x(y)$ is inverse-measurable, we obtain

$$R_x(y(x)) \leqq |J_x(y(x))| \tag{47}$$

almost everywhere on D_1. By (44), (46), and (47), we infer that

$$R_y(x) \geqq |J_y(x)| \tag{48}$$

almost everywhere on D_1. Again by the main inequality (26), $R_y(x) \leqq |J_y(x)|$ almost everywhere on D_1, and this with (48) implies that $R_y(x) = |J_y(x)|$ almost everywhere on D_1, i.e.,

$$\int_Y d\mu(y) = \int_{x(Y)} |J_y(x)|\, d\mu(x) \tag{49}$$

for every $Y \in \boldsymbol{L}(D_2)$.

Since $X \in \boldsymbol{L}(D_1)$ if and only if $y(X) \in \boldsymbol{L}(D_2)$, $f(y)$ is Lebesgue measurable on D_2 if and only if $f(y(x))$ is on D_1, and in this case $f(y(x))J_y(x)$ is Lebesgue measurable on D_1. Thus by the definition of integrals and by (49), (45) is shown to be valid first for bounded nonnegative measurable f, and then, by the Lebesgue convergence theorem, for nonnegative integrable f, and finally, by decomposing f into positive and negative parts, for general integrable f.

Dirichlet Mappings

12. In the remainder of this paper we deduce an application of the theorems given thus far. Especially the main inequality (26) in Theorem 5 will play a crucial role.

Let D be an open set in $E^m (m \geqq 2)$. We denote simply by $W(D)$ the *Sobolev space* $W^{1,2}(D)$, the space of real-valued functions φ on D such that φ and their distribution derivatives $[\partial\varphi/\partial x^i]_{\text{dis}}$ $(i = 1, \cdots, m)$ are square integrable functions on D. Since the $(\partial\varphi/\partial x^i)$ $(i=1, \cdots, m)$ exist almost everywhere on D and are square integrable, we can define the *Dirichlet integral* $\boldsymbol{D}(\varphi)$ of φ by

$$D(\varphi) = \int \cdots \int_D \sum_{i=1}^{m} \left(\frac{\partial \varphi}{\partial x^i}(x)\right)^2 dx^1 \cdots dx^m . \tag{50}$$

We also write $D_D(\varphi)$ if it is necessary to indicate the dependence on the integrating domain D. For Sobolev spaces, see e.g., Yosida [11].

13. Let $y = y(x)$ be a homeomorphism of an open set D_1 onto an open set D_2 in E^m, of dimension $m \geqq 2$.

DEFINITION. The homeomorphism $y = y(x)$ is said to be a Dirichlet mapping of D_1 onto D_2 if $\varphi(y) \in W(D_2)$ is equivalent to $\varphi(y(x)) \in W(D_1)$.

The notion of Dirichlet mapping was introduced in Nakai-Sario [5], where the following additional requirement was imposed; there exists a finite constant $K \geqq 1$ such that

$$K^{-1} D_{D_2}(\varphi) \leqq D_{D_1}(\varphi \circ y) \leqq K D_{D_2}(\varphi) \tag{51}$$

for every φ in $W(D_2)$. However it is known that (51) is a consequence of the very definition of Dirichlet mapping (see Nakai [4]).

As is well-known the solution of the variational problem min $D_D(\varphi)$ among functions φ with a fixed boundary condition is harmonic. Therefore $W(D)$ is, in a sense, characteristic of the potential-theoretic structure of D, and thus Dirichlet mappings may be viewed as mappings which preserve, again in a sense, potential-theoretic structures of open sets. In view of this it is important to characterize Dirichlet mappings geometrically. For $m = 2$ it is known that a mapping is a Dirichlet mapping if and only if it is quasiconformal (see Nakai [3] and Sario-Nakai [8]). For quasiconformal mappings, see e.g., the monograph of Künzi [2] and also Gehring [1], among others.

Our object is, in contrast, to characterize Dirichlet mappings geometrically for $m \geqq 3$. Hereafter we always assume that $m \geqq 3$.

14. Let D be a region, i.e., connected open set in E^m. For two points x_1 and x_2, we denote by $[x_1, x_2]$ the line segment connecting x_1 and x_2:

$$[x_1, x_2] = \{x \mid x = \alpha x_1 + (1 - \alpha)x_2,\ 0 \leqq \alpha \leqq 1\} .$$

For two points x_0 and x in D, the Riemannian flat metric $\rho_D(x_0, x)$ is defined by

$$\rho_D(x_0, x) = \inf \sum_{j=1}^{m} | x_j - x_{j-1} | , \tag{52}$$

where the infimum is taken with respect to every finite sequence

$\{x_j\}_{j=0}^n$ such that $x_n = x$ and $[x_{j-1}, x_j] \subset D$ $(j = 1, \cdots, n)$. Clearly ρ_D is a metric on D, and, provided $[x_0, x] \subset D$, we have

$$\rho_D(x_0, x) = |x_0 - x| . \tag{53}$$

DEFINITION. A homeomorphism $y = y(x)$ of a region D_1 onto a region D_2 in E^m is a quasi-isometry if there exists a finite constant $K \geqq 1$ such that for any two points x_1 and x_2 in D_1

$$K^{-1}\rho_{D_1}(x_1, x_2) \leqq \rho_{D_2}(y(x_1), y(x_2)) \leqq K\rho_{D_1}(x_1, x_2) . \tag{54}$$

In view of (53) and (52), the condition (54) is equivalent to the following:

$$K^{-1}|x_1 - x_2| \leqq |y(x_1) - y(x_2)| \leqq K|x_1 - x_2| \tag{55}$$

for any two points x_1 and x_2 such that

$$\begin{aligned} &\{x \mid |x - x_1| \leqq |x_2 - x_1|\} \subset D_1 , \\ &\{y \mid |y - y(x_1)| \leqq |y(x_2) - y(x_1)|\} \subset D_2 . \end{aligned} \tag{56}$$

15. A complete geometric characterization of Dirichlet mappings for $m \geqq 3$ is given as follows:

THEOREM. *A homeomorphism $y = y(x)$ of a region D_1 onto a region D_2 in $E^m (m \geqq 3)$ is a Dirichlet mapping if and only if it is a quasi-isometry of D_1 onto D_2.*

It is not difficult to generalize the concepts of Dirichlet mapping and quasi-isometry and also the above theorem to the case where D_1 and D_2 are Riemannian manifolds.

To prove the theorem we have a rather long way to go: 16–23. An application of Theorem 5 will appear in 21, which is one of the crucial steps in our proof.

16. First we suppose $y = y(x)$ is a quasi-isometry of D_1 onto D_2, and shall prove it is a Dirichlet mapping. A somewhat indirect proof for this was already given in Nakai-Sario [5], but we furnish here a more direct proof for the sake of completeness. Observe that

$$\varlimsup_{h \to 0} \frac{|y(x + h) - y(x)|}{|h|} \leqq K , \tag{57}$$

and by the Rademacher theorem mentioned in the introduction, (3) and (5) are valid for the present $y(x)$. In particular, (57) assures the uniform boundedness of partial derivatives of the components of $y(x)$ in the essential supremum norm, and the same is true of the inverse $x(y)$ of $y(x)$.

Therefore there exists a finite constant $K_1 \geqq 1$ such that

$$K_1^{-1}\boldsymbol{D}_{D_2}(\varphi) \leqq \boldsymbol{D}_{D_1}(\varphi\circ y) \leqq K_1\boldsymbol{D}_{D_2}(\varphi) \tag{58}$$

for every $\varphi \in W(D_2) \cap C^\infty(D_2)$. Here $\varphi\circ y \in W(D_1)$ is obviously true.

17. We pause here to insert the following remark. Consider the norms

$$\|f\|_{2,D}^2 = \int\cdots\int_D |f(x)|^2\, dx^1 \cdots dx^m\,,$$
$$\|f\|_{\infty,D} = \operatorname*{ess\cdot sup}_{x\in D} |f(x)|$$

for $f \in W(D)$. By the standard mollifier method (or the regularization method (cf. e.g., Yosida [11; p. 29, 58]), see also 23 below), we can see easily that $W(D) \cap C^\infty(D)$ is dense in $W(D)$ with respect to the combined norm

$$|||f|||_D = \|f\|_{2,D} + (\boldsymbol{D}_D(f))^{1/2}\,,$$

and also that $W(D) \cap C^\infty(D)$ is dense in $W(D) \cap C(D)$ with respect to the combined norm

$$\|f\|_D = \|f\|_{\infty,D} + (\boldsymbol{D}_D(f))^{1/2}\,.$$

Here, as usual, C^∞ stands for infinitely continuously differentiable, and C for continuous.

18. Fix an arbitrary $\varphi \in W(D_2)$ and choose $\{\varphi_n\} \subset W(D_2) \cap C^\infty(D_2)$ such that $|||\varphi - \varphi_n|||_D$ tends to zero as n tends to infinity. Since $J_y(x)$ is essentially bounded by K_2^2, say, we see that

$$\|\varphi_n\circ y - \varphi_{n+p}\circ y\|_{2,D_1} \leqq K_2\|\varphi_n - \varphi_{n+p}\|_{2,D_2}$$

and by (58) we can find a constant K_3 such that

$$|||\varphi_n\circ y - \varphi_{n+p}\circ y|||_{D_1} \leqq K_3|||\varphi_n - \varphi_{n+p}|||$$

for every $n, p = 1, 2, \cdots$. Since $W(D_1)$ is complete with respect to $|||\cdot|||$ (cf. Yosida [11; p. 55]) and a suitable subsequence of $\{\varphi_n\}$ converges to φ almost everywhere on D_2 and $y(x)$ is inverse-measurable, we conclude that $\varphi\circ y \in W(D_1)$. We remark in passing that (58) is valid for the present φ. Similarly $\psi \in W(D_1)$ implies that $\psi\circ x \in X(D_2)$. Thus we have seen that $y(x)$ is a Dirichlet mapping.

19. The main part of the proof is to conclude that $y = y(x)$ is a quasi-isometry of D_1 onto D_2 under the assumption that $y = y(x)$ is

a Dirichlet mapping of D_1 onto D_2. Needless to say the inverse mapping $x = x(y)$ of D_2 onto D_1 of $y = y(x)$ is also a Dirichlet mapping.

Take an arbitrary relatively compact region D_1' in D_1. Let $\varphi \in C^\infty(D_2)$ such that φ has a compact sopport in D_2 and $\varphi \mid y(D_1') = 1$. For each $i = 1, \cdots, m$, $\varphi(y)\cdot y^i \in W(D_2)$ and hence $\varphi(y(x))y^i(x) \in W(D_1)$. Since $\varphi(y(x))y^i(x) = y^i(x)$ for $x \in D_1'$, we conclude that $y^i(x) \in W(D_1')$ $(i = 1, \cdots, m)$ for every relatively compact region D_1' in D_1. The same is true of $x^i(y)$ $(i = 1, \cdots, m)$.

Since the partial derivatives of the $y^i(x)$ exist almost everywhere and are measurable on D_1,

$$\lambda(x) = \sum_{i,j=1}^{m} \left(\frac{\partial y^i}{\partial x^j}(x)\right)^2 \tag{59}$$

is defined almost everywhere on D_1 and measurable. We shall first prove that

$$\lambda(x) > 0 \tag{60}$$

almost everywhere on D_1.

For this purpose we consider the set

$$E = \{x \in D_1 \mid \lambda(x) = 0\} ,$$

which is clearly measurable. Take arbitrary relatively compact open sets U_1 and V_1 such that $\bar{U}_1 \subset V_1 \subset \bar{V}_1 \subset D_1$, and set $U_2 = y(U_1)$ and $V_2 = y(V_1)$. Also choose an open set V_2' with $\bar{V}_2 \subset V_2' \subset \bar{V}_2' \subset D_2$. Since $x^i(y) \in W(V_2') \cap C(V_2')$ $(i = 1, \cdots, m)$, we can find sequences $\{x_n^i(y)\} \subset W(V_2') \cap C^\infty(V_2')$ such that

$$\lim_{n\to\infty} \| x_n^i(\cdot) - x^i(\cdot) \|_{V_2'} = 0 \qquad (i = 1, \cdots, m) \tag{61}$$

(cf. 17). Let $h \in C^\infty(D_2)$ such that $h \mid U_2 = 1$ and $h \mid (D_2 - V_2) = 0$. Set

$$\varphi(y) = h(y) \sum_{i=1}^{m} x^i(y),\ \varphi_n(y) = h(y) \sum_{i=1}^{m} x_n^i(y) .$$

These functions can be considered to be in $W(D_2) \cap C^\infty(D_2)$. In view of (61) we have

$$\lim_{n\to\infty} \boldsymbol{D}_{D_2}(\varphi - \varphi_n) = 0 . \tag{62}$$

We also write $u(x) = \varphi(y(x))$ and $u_n(x) = \varphi_n(y(x))$. They are in $W(D_1)$. Observe that if $x \in U_1$, then $u(x) = \sum_{i=1}^m x^i$ and $u_n(x) = \sum_{i=1}^m x_n^i(y(x))$. Using (51) we infer

$$\begin{aligned} D_{U_1-E}(u - u_n) &\leqq D_{U_1}(u - u_n) \leqq D_{D_1}(\varphi\circ y - \varphi_n\circ y) \\ &\leqq KD_{D_2}(\varphi - \varphi_n) \end{aligned}$$

and by (62) we conclude that

$$\lim_{n\to\infty} D_{U_1-E}(u - u_n) = \lim_{n\to\infty} D_{U_1}(u - u_n) = 0 . \tag{63}$$

In particular

$$D_{U_1-E}(u) = \lim_{n\to\infty} D_{U_1-E}(u_n),\ \ D_{U_1}(u) = \lim_{n\to\infty} D_{U_1}(u_n) . \tag{63'}$$

Since $x_n^i(y) \in C^\infty(U_1)$ $(i = 1, \cdots, m)$, we see that

$$\begin{aligned} \frac{\partial u_n}{\partial x_i}(x) &= \sum_{i=1}^m \frac{\partial}{\partial x^j} x_n^i(y(x)) \\ &= \sum_{i=1}^m \sum_{k=1}^m \frac{\partial x_n^i}{\partial y^k}(y(x)) \cdot \frac{\partial y^k}{\partial x^j}(x) = 0 \quad (j = 1, \cdots, m) \end{aligned}$$

for $x \in U_1 \cap E$, i.e.,

$$\sum_{j=1}^m \left(\frac{\partial u_n}{\partial x^j}(x)\right)^2 = 0$$

for $x \in U_1 \cap E$. Therefore

$$D_{U_1-E}(u_n) = D_{U_1}(u_n)$$

and by (63) we conclude that

$$D_{U_1-E}(u) = D_{U_1}(u) . \tag{64}$$

However for x in U_1

$$\frac{\partial u}{\partial x^j}(x) = \frac{\partial}{\partial x^j} \sum_{i=1}^m x^i = 1$$

and (61) takes on the form

$$\int_{U_1-E} m\, d\mu(x) = \int_U m\, d\mu(x) .$$

Since this is true for every relatively compact open set U_1 in D_1, we must have $\mu(E) = 0$ and (60) is herewith established.

20. For each $\xi = (\xi^1, \cdots, \xi^m) \in E^m$ with $|\xi| = 1$ we consider

$$\lambda_\xi(x) = \sum_{i,j=1}^m \sum_{k=1}^m \frac{\partial y^i}{\partial x^k}(x) \cdot \frac{\partial y^j}{\partial x^k}(x) \cdot \xi^i \xi^j , \tag{65}$$

which is defined almost everywhere on D_1. Fix an arbitrary point

$x_0 \in D_1$ and an arbitrary positive number $\varepsilon > 0$. Let

$$V_\varepsilon = V_\varepsilon(y_0) = \{y \mid |y - y_0| < \varepsilon\} , \qquad y_0 = y(x_0) .$$

Choose the function $h(y) \in C(D_2)$ characterized by $h \mid \bar{V}_\varepsilon = 1$, $h \mid (D_2 - V_{2\varepsilon}) = 0$, and

$$h(y) = 1 - ((2\varepsilon)^{2-m} - \varepsilon^{2-m})^{-1}(|y - y_0|^{2-m} - \varepsilon^{2-m})$$

for $y \in V_{2\varepsilon} - \bar{V}_\varepsilon$, i.e., h is the harmonic function on $V_{2\varepsilon} - \bar{V}_\varepsilon$ with boundary values 1 at the interior boundary and 0 at the exterior boundary. Consider

$$\varphi(y) = h(y) \sum_{i=1}^{m} \xi^i y^i ,$$

which is clearly in $W(D_2)$. By (51), we see that

$$\boldsymbol{D}_{x(V_\varepsilon)}(\varphi \circ y) \leqq \boldsymbol{D}_{D_1}(\varphi \circ y) \leqq K\boldsymbol{D}_{D_2}(\varphi) = K\boldsymbol{D}_{V_{2\varepsilon}}(\varphi)$$

and therefore

$$\int_{x(V_\varepsilon)} \lambda_\xi(x) d\mu(x) \leqq K\left(\int_{\bar{V}_\varepsilon} d\mu(y) + \boldsymbol{D}_{V_{2\varepsilon}-\bar{V}_\varepsilon}(\varphi)\right) . \tag{66}$$

At each $y \in V_{2\varepsilon} - \bar{V}_\varepsilon$, we have

$$\sum_{i=1}^{m} \left(\frac{\partial \varphi}{\partial y^i}(y)\right)^2 \leqq 8\varepsilon^2 \sum_{i=1}^{m} \left(\frac{\partial h}{\partial y^i}(y)\right)^2 + 2$$

and an integration of both sides gives

$$\boldsymbol{D}_{V_{2\varepsilon}-\bar{V}_\varepsilon}(h) \leqq 8\varepsilon^2 \cdot \Gamma(1 + m/2)^{-1}\pi^{m/2}(m - 2)(2^{m-2} - 1)\varepsilon^{m-2} + 2\cdot\mu(V_{2\varepsilon} - \bar{V}_\varepsilon) .$$

Therefore, putting $K_1 = K + 8(m - 2)(2^{m-2} - 1) + 2(2^m - 1)$, we conclude that

$$\int_{x(V_\varepsilon)} \lambda_\xi(x) d\mu(x) \leqq K_1 \mu(V_\varepsilon) . \tag{67}$$

If we choose $\xi = (\delta^{i1}, \cdots, \delta^{im})$ $(i = 1, \cdots, m)$ and add (67) with these choices of ξ, we obtain, on setting $mK_1 = K_2$,

$$\int_{x(V_\varepsilon)} \lambda(x) d\mu(x) \leqq K_2 \mu(V_\varepsilon) . \tag{68}$$

This is true for every $V_\varepsilon = V_\varepsilon(x_0) \subset D_2$.

Let Y be an arbitrary set in D_2 with $\mu(Y) = 0$. For each $n = 1, 2, \cdots$ we can find suitable balls $V_{ni} = V_{\varepsilon_{ni}}(y_{ni}) \subset D_2$ such that $Y \subset \bigcup_{i=1}^{\infty} V_{ni}$ and $\sum_{i=1}^{\infty} \mu(V_{ni}) < 1/n$. Let $B_n = \bigcup_{i=1}^{\infty} V_{ni}$ and $B = \bigcap_{n=1}^{\infty} B_n$. Since $x(B) = \bigcap_{n=1}^{\infty} x(B_n) = \bigcap_{n=1}^{\infty} (\bigcup_{n=1}^{\infty} x(V_{ni}))$ is a G_δ-set, $x(B)$ is Lebesgue measurable and

$$\int_{x(B)} \lambda(x)d\mu(x) \leqq \int_{x(B_n)} \lambda(x)d\mu(x)$$
$$\leqq \sum_{i=1}^{\infty} \int_{x(V_{ni})} \lambda(x)d\mu(x)$$

for each $n = 1, 2, \cdots$. In view of (68)

$$\int_{x(B)} \lambda(x)d\mu(x) \leqq K_2 \sum_{i=1}^{\infty} \mu(V_{ni}) \leqq K_2/n$$

for every $n = 1, 2, \cdots$, and we conclude that $\int_{x(B)} \lambda(x)d\mu(x) = 0$. By (60), we must have $\mu(x(B)) = 0$. Since $x(B) \supset x(Y)$, we see that $\mu(x(Y)) = 0$, i.e., $x = x(y)$ is inverse-measurable. The same should be true of $y = y(x)$ and we can consider the Radon-Nikodym density $R_y(x)$ of $y = y(x)$.

21. Take an arbitrary set $X \in \boldsymbol{L}(D_1)$. Since $y(X) \in \boldsymbol{L}(D_2)$, we can find balls $V_{ni} = V_{\varepsilon_{ni}}(y_{ni}) \subset D_2$ such that $y(X) \subset B_n = \bigcup_{i=1}^{\infty} V_{ni}$ and

$$\mu(B_n - y(X)) \leqq \frac{1}{n}, \ \sum^{\infty} \mu(V_{ni}) \leqq \mu(B_n) + \frac{1}{n} .$$

Set $B = \bigcap_{n=1}^{\infty} B_n$. Then $\mu(B - y(X)) = 0$ and thus $\nu(x(B) - X) = 0$. Therefore

$$\int_X \lambda_\xi(x)d\mu(x) = \int_{x(B)} \lambda_\xi(x)d\mu(x) \leqq \int_{x(B_n)} \lambda_\xi(x)d\mu(x)$$
$$\leqq \sum_{i=1}^{\infty} \int_{x(V_{ni})} \lambda_\xi(x)d\mu(x)$$

for every n. An application of (67) to the right-hand terms gives

$$\int_X \lambda_\xi(x)d\mu(x) \leqq K_1 \sum_{i=1}^{\infty} \mu(V_{ni}) \leqq K_1 \left(\mu(B_n) + \frac{1}{n}\right)$$
$$\leqq K_1 \left(\mu(y(X)) + \frac{2}{n}\right) .$$

On letting $n \to \infty$, we obtain

$$\int_X \lambda_\xi(x)d\mu(x) \leqq K_1 \int_X R_y(x)d\mu(x) \tag{69}$$

for every $X \in \boldsymbol{L}(D_1)$ and every $\xi \in E^m$ with $|\xi| = 1$.

Now we are in a position to use (26) in Theorem 5: $R_y(x) \leqq J_y(x)$. From (69) it follows that

$$\int_X \lambda_\xi(x)d\mu(x) \leqq K_1 \int_X |J_y(x)|\, d\mu(x) \tag{70}$$

for every $X \in L(D_1)$ and every $\xi \in E^m$ with $|\xi| = 1$. Let $\{\xi_i\}$ be a countable dense subset of $\{\xi \in E^m \mid |\xi| = 1\}$. Since (70) is true for $\xi = \xi_i$ and every $X \in L(D_1)$, there exists a set $F_i \subset D_1$ such that $\mu(F_i) = 0$ and

$$\lambda_{\xi_i}(x) \leqq K_1 |J_y(x)| \qquad (x \in D_1 - F_i) . \tag{71}$$

Put $F = \bigcup_{i=1}^\infty F_i \subset D_1$. Then $\mu(F) = 0$ and (71) is true for each $x \in D_1 - F$ and every $i = 1, 2, \cdots$. Since $\{\xi_i\}$ is dense in $|\xi| = 1$, we finally conclude that

$$\lambda_\xi(x) \leqq K_1 |J_y(x)| \qquad (x \in D_1 - F,\ \mu(F) = 0) \tag{72}$$

for every $\xi \in E^m$ with $|\xi| = 1$.

If we write

$$M_y(x) = \left(\frac{\partial y^i}{\partial x^i}\right) \text{ and } E = (\delta^{ij}) ,$$

i indicating the row and j the column, then (72) takes on the following form in the matrix inequality:

$$\xi \cdot M_y(x) \cdot {}^tM_y(x) \cdot {}^t\xi \leqq \xi \cdot K_1 |J_y(x)| E \cdot {}^t\xi .$$

Since this is true for every $\xi \in E^m$, or rather, for every $(1, m)$-matrix, we conclude that

$$M_y(x) \cdot {}^tM_y(x) \leqq K_1 |J_y(x)| E .$$

The relation is preserved if we take determinants of both sides, and we obtain

$$|J_y(x)|^2 \leqq K_1^m |J_y(x)|^m$$

and a fortiori

$$|J_y(x)| \geqq K_3 \tag{73}$$

almost everywhere on D_1, where $K_3 = K_1^{-m/(m-2)}$. At this point the assumption $m \geqq 3$ is made essential use of. The same argument can be applied to $J_x(y)$ to yield

$$|J_x(y)| \geqq K_3 \tag{74}$$

almost everywhere on D_2.

22. Let D_2' and D_2'' be arbitrary relatively compact regions such

that $\bar{D}_2' \subset D_2'' \subset \bar{D}_2'' \subset D_2$. Set $D_1' = x(D_2')$ and $D_1'' = x(D_2'')$. Let $h \in C^\infty(D_2)$ such that the support of h is contained in D_2'' and $h \mid D_2' = 1$. Choose $\{x_n^i(y)\} \subset W(D_2'') \cap C^\infty(D_2'')$ with

$$\lim_{n\to\infty} \| x_n^i(\cdot) - x^i(\cdot) \|_{D_2'} = 0 \qquad (i = 1, \cdots, m) \tag{75}$$

(cf. 17). Set $u_n^i(y) = x_n^i(y) \cdot h(y)$ and $u^i(y) = x^i(y) \cdot h(y)$; they can be viewed as members of $W(D_2)$. From (75) and (51), we conclude that

$$\begin{aligned} \| x_n^i \circ y - x^i \circ y \|_{D_1'} &\leqq \| u_n^i \circ y - u^i \circ y \|_{D_1} \\ &\leqq \| u_n^i - u^i \|_{D_2} \to 0 \qquad (n \to \infty) \end{aligned} \tag{76}$$

for every $i = 1, \cdots, m$. By choosing a suitable subsequence, we may assume that

$$\begin{aligned} \frac{\partial}{\partial x^j} x^i(y(x)) &= \lim_{n\to\infty} \frac{\partial}{\partial x^j} x_n^i(y(x)) \\ &= \lim_{n\to\infty} \sum_{k=}^{m} \frac{\partial x_n^i}{\partial y^k}(y(x)) \cdot \frac{\partial y^k}{\partial x^i}(x) \end{aligned} \tag{77}$$

almost everywhere on D_1'. We might also assume that $\partial x_n^i / \partial y^k$ converges to $\partial x^i / \partial y^k$ almost everywhere on D_2', and since $y(x)$ is measurable (i.e., $x(y)$ is inverse-measurable), we conclude by (77) that

$$\sum_{k=1}^{m} \frac{\partial x^i}{\partial y^k}(y(x)) \; \frac{\partial y^k}{\partial x^j}(x) = \delta^{ij} \tag{78}$$

almost everywhere on D_1' and hence on D_1. This means that

$$J_y(x) \cdot J_x(y(x)) = 1 \tag{79}$$

almost everywhere on D_1. From (74), (79), and the fact that $x(y)$ is inverse-measurable, it follows that, by setting $K_4 = K_3^{-1}$,

$$| J_y(x) | \leqq K_4 < \infty \tag{80}$$

almost everywhere on D_1. On combining this with (72) we see that

$$\left| \frac{\partial y^i}{\partial x^j}(x) \right| \leqq K_5 \qquad (i, j = 1, \cdots, m) \tag{81}$$

almost everywhere on D_1, with $K_5 = K_1 \cdot K_4$.

23. We are ready to prove (54), or equivalently (55). To this end we fix two arbitrary points x_1 and x_2 satisfying (56). Set

$$\bar{B} = \{x \in E^m \mid | x - x_1 | \leqq | x_2 - x_1 |\} \subset D_1 ,$$

and let $\eta > 0$ be the distance, which may be ∞, between the boundary of D_1 and $\bar{B}$.

Let $\theta_1(x) = \exp((|x|^2 - 1)^{-1})$ for $|x| < 1$ and $\theta_1(x) = 0$ for $|x| \geqq 1$; it is in $C^\infty(E^m)$. Choose $c_n > 0$ so that the function $\theta_n(x) = c_n\theta_1(nx)(\geqq 0)$ which is also in $C^\infty(E^m)$ with its support in $|x| \leqq 1/n$, satisfies

$$\int_{E^m} \theta_n(x) d\mu(x) = 1 \tag{82}$$

for each $n = 1, 2, \cdots$. We will consider only those n which meet the condition $1/n < \eta$. Consider *regularizations* of $y^i(x)$:

$$\begin{aligned} y^i_n(x) &= \int_{|\xi| \leqq 1/n} y^i(x - \xi)\theta_n(\xi) d\mu(\xi) \\ &= \int_{|x-\xi| \leqq 1/n} y^i(\xi)\theta_n(x - \xi) d\mu(\xi) \end{aligned}$$

where x is in the interior B of $\bar{B}$. Then $y^i_n(x) \in C^\infty(B)$ $(i = 1, \cdots, m;$ $m = 1, 2, \cdots)$ and

$$\lim_{n\to\infty} \| y^i_n(\cdot) - y^i(\cdot) \|_B = 0 \qquad (i = 1, \cdots, m) \tag{83}$$

(cf. Yosida [11; p. 29, 58]). Since the $\partial y^i/\partial x^j$ $(i, j = 1, \cdots, m)$ are square integrable, integration by parts implies

$$\frac{\partial y^i_n}{\partial x^j}(x) = \int_{|x-\xi| \leqq 1/n} \frac{\partial y^i}{\partial \xi^j}(\xi)\theta_n(x - \xi) d\mu(\xi) \; .$$

By (81) and (83), we conclude that

$$\left| \frac{\partial y^i_n}{\partial x^j}(x) \right| \leqq K_5 \qquad (x \in B;\, i, j = 1, \cdots, m) \; . \tag{84}$$

The mean value theorem yields

$$y^i_n(x_1) - y^i_n(x_2) = \sum_{k=1}^m (x^k_1 - x^k_2)\frac{\partial y^i_n}{\partial x^k}(x_2 + t_i(x_1 - x_2))$$

for some $t_i \in (0, 1)$ and every $i = 1, \cdots, m$. In view of (84), we have

$$| y^i_n(x_1) - y^i_n(x_2) |^2 \leqq mK_5^2 \, | x_1 - x_2 |^2 \; .$$

On letting $n \to \infty$, we conclude in view of (83), that

$$| y^i(x_1) - y^i(x_2) |^2 \leqq mK_5^2 \, | x_1 - x_2 |^2 \qquad (i = 1, \cdots, m) \; . \tag{85}$$

Adding (85) with respect to $i = 1, \cdots, m$, we finally obtain

$$| y(x_1) - y(x_2) | \leqq K_6 \, | x_1 - x_2 | \; ,$$

with $K_6 = mK_5$, for every x_1 and x_2 in D_1 with (56). If we interchange the roles of $y = y(x)$ and $x = x(y)$, we also have

$$K_6^{-1} |x_1 - x_2| \leqq |y(x_1) - y(x_2)| \leqq K_6 |x_1 - x_2| ,$$

i.e., (55) and thus (54) is valid.

The proof of Theorem 15 is herewith complete.

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