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AN ALGEBRAIC PROPERTY OF THE TOTALLY SYMMETRIC LOOPS ASSOCIATED WITH KIRKMAN-STEINER TRIPLE SYSTEMS

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The concept of an x-root of degree r in a loop of order n is introduced. It is shown that the totally symmetric loop of order $n+1$ derived from any Kirkman-Steiner triple system of order n admits a maximal identity-root. A statistical-combinatorial application of this algebraic property is then indicated. Finally, two open problems are also given.

A mathematical system consisting of an n-set Ω and a binary operation $*$ is said to form a loop of order n if the following axioms are satisfied:

(1) Ω contains an identity element e such that $x*e = e*x = x$ for every x in Ω.

(2) Any two of the elements in the equation $x*y = z$ uniquely determine the third.

Since the notation $x*y$ is too bulky we shall use, hereafter, the notation xy instead. A loop is said to be a totally symmetric loop if it also satisfies

(3) $xy = yx$ and $x(xy) = y$ for all x and y in Ω.

In this paper, we shall introduce and study an algebraic property of totally symmetric loops of order $n \equiv 3 \pmod 6$. In the final part of this paper we shall indicate, briefly, a statistical-combinatorial application of this study. A few open questions are also stated.

We begin by introducing and reviewing certain concepts and results that will be relevant to our forthcoming results.

DEFINITION 1. We say a loop $\mathscr{L}$ of order n accepts a

$$(k_1, k_2, \cdots, k_r)$$

orthogonal partition if the n^2 cells in the Cayley table of $\mathscr{L}$ can be divided into r mutually disjoint exhaustive sets $S_1, S_2, \cdots, S_r$; in such a way that (1) S_i has k_i cells from each row and each column, (2) each element of $\mathscr{L}$ appears k_i times in the cells of S_i,

$$(3) \qquad k_1 + k_2 + \cdots + k_r = n\ .$$

In particular a set S_t is called a transversal of $\mathscr{L}$ if $k_t = 1$. If two transversals have no cells in common, they are said to be parallel; if they have exactly one cell in common, they are called orthogonal.

A set $\{t_1, t_2 \cdots, t_r\}$ mutually orthogonal transversals of $\mathscr{L}$ is said to be an x-root of degree r if these transversals are all sharing a unique cell containing the element x. Clearly any x-root of degree r occupies $r(n-1)+1$ cells of the Cayley table of a loop of order n. An x-root of degree r in the Cayley table of a loop of order n is said to be a maximal x-root if $r = n-2$. The following lemma justifies this terminology.

LEMMA 1. *For any x-root of degree r in a loop of order n, $r \leqq n-2$.*

Proof. Let the cell in the given x-root that contains the element x occur in row i and column j. Then the remaining $2n-2$ cells of row i and column j, together with the $n-1$ other cells containing the element x, cannot be in the x-root. Thus there remains only $n^2 - 3n + 3$ cells to accommodate the given x-root. However, as pointed out before, this x-root must occupy $r(n-1)+1$ cells. Hence $r \leqq n-2$.

DEFINITION 2. Let Σ be an n-set, $n \equiv 1, 3 (\text{mod } 6)$. Then a Steiner triple system of order n on Σ is a collection of $n(n-1)/6$ unordered triples (x, y, z) with x, y, z in Σ, such that every pair of distinct elements of Σ belongs to exactly one triple. A triple system of order $n \equiv 3 (\text{mod } 6)$ is said to be a Kirkman-Steiner triple system of order n if it is a Steiner triple system with the following additional stipulation: the set of triples can be partitioned into $r = (n-1)/2$ disjoint classes such that the totality of elements in each class exhaust the set on which the system is defined.

While Reiss [9] has shown the sufficiency of $n \equiv 1, 3 (\text{mod } 6)$ for the existence of a Steiner triple system of order n, Ray-Chaudhuri and Wilson [8] have proved the sufficiency of $n \equiv 3 (\text{mod } 6)$ for the existence of a Kirkman-Steiner triple system of order n.

The coextensiveness of totally symmetric loops of order $n+1$ with Steiner triple systems of order n has been shown by Bruck [2] who proved the following theorem:

THEOREM 1. *A totally symmetric loop of order $n+1$ exists if and only if there exists a Steiner triple system of order n.*

For the sake of clarity of later arguments, we shall sketch a proof of this theorem here.

Proof. Let A be a totally symmetric loop of order $n+1$ and let $H = A - \{e$, the identity element in $A\}$. Then the collection of all unordered triples (x, y, z) with x, y, z in H, such that $xy = z$, forms a Steiner triple system on H. Conversely, given a Steiner triple system of order n on an n-set W, we can then form a totally symmetric loop of order $n+1$ from these triples as follows: Define an operation $\circ$ on the set $\mathscr{L}^* = WU\{e\}$ by: (1) $a \circ b = c$ if and only if (a, b, c) is in $\mathscr{L}^*$, (2) $e \circ a = a \circ e = a$, and (3) $a^2 = e^2 = e$ for all a in $\mathscr{L}^*$. Then $\mathscr{L}^*$ together with the binary operation $\circ$ forms a totally symmetric loop of order $n+1$.

Let Σ be an n-set, $n \equiv 3 (\text{mod } 6)$ and let $\mathscr{K}$ be a Kirkman-Steiner triple system on Σ. Let also $\mathscr{L}^*$ be the totally symmetric loop of order $n+1$ derived from $\mathscr{K}$. Denote the identity element in $\mathscr{L}^*$ by e. Partition $\mathscr{L}^*$ into $r = (n-1)/2$ disjoint classes C_i, $i = 1, 2, \cdots, r$ as described in Definition 2. Then we have the following lemma.

LEMMA 2. *C_i determines an e-root of degree 2 in the Cayley table of $\mathscr{L}^*$.*

Proof. Denote an arbitrary triple in C_i by

$$(a_{ij}, b_{ij}, c_{ij}),\ j = 1, 2, \cdots, n/3 .$$

Identify three cells in the Cayley table of $\mathscr{L}^*$ by the 2-tuples (a_{ij}, b_{ij}), (b_{ij}, c_{ij}) and (c_{ij}, a_{ij}), the components of each 2-tuple being the row and column indicies respectively. Now let j run through all the $n/3$ triples in C_i. Then the corresponding $3 \times n/3 = n$ cells determined by the preceding rule, together with the cell corresponding to row and column indices (e, e), form a transversal for $\mathscr{L}^*$. Denote this transversal by t_{i_1}. Another transversal t_{i2} is obtained by considering the cell (e, e) and the three cells in the Cayley table described by the 2-tuples (b_{ij}, a_{ij}), (c_{ij}, b_{ij}) and (a_{ij}, c_{ij}), where we let j run through the values $1, 2, \cdots, n/3$. These exhibition rules clearly guarantee that t_{i_1} is orthogonal to t_{i2} and that the point of intersection is the cell (e, e).

We shall now prove the following:

THEOREM 2. *The totally symmetric loop $\mathscr{L}^*$ derived from any Kirkman-Steiner triple system contains a maximal identity-root.*

Proof. By Lemma 2 every class in the given Kirkman-Steiner triple system determines an e-root of degree 2 in the Cayley table of $\mathscr{L}^*$, where e is the identity in $\mathscr{L}^*$. The method of exhibition in the lemma together with the fact that every pair or distinct elements in the triple system appears exactly once reveals that the transversal $t_{ik}(\mathrm{k} = 1, 2)$ is orthogonal to $t'_{ik}(k = 1, 2)$ if $i \neq i'$ with cell (e, e) as the intersection point. Since there are $(n - 1)/2$ classes, we have $2(n - 1)/2 = n - 1$ pairwise orthogonal transversals sharing the cell (e, e), *i.e.*, an identity-root of degree $n - 1$. Since the order of $\mathscr{L}^*$ is $n + 1$, the proof is complete.

As an immediate application we have

COROLLARY. *Every totally symmetric loop of order $n + 1$ derived from a Kirkman-Steiner triple system of order n implies the existence of a set consisting of at least a pair of mutually orthogonal Latin squares of order n.*

A proof of this corollary, together with some additional results, will be given in another paper. However, we should remark that, in particular, for $n = 15$, the corresponding pair of orthogonal Latin squares can be embedded in a set of three mutually orthogonal Latin squares of order 15, thus disproving MacNeish's [5] conjecture for order 15.

Before finishing, let us mention a few open problems.

(1) Prove or disprove that the totally symmetric loop of order $n + 1$ derived from any arbitrary Steiner triple system of order n admits a maximal x-root.

(2) Characterize those loops whose Cayley tables admit a $(1, 1, \cdots, 1)$ orthogonal partition.

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