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APPROXIMATION OF CURVES

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Generalizing recent results of J. M. Sloss we show: A curve in n-space that admits a continuously differentiable first order frame can be C^1 approximated to any desired accuracy by a continuous, piecewise C^{r+2} curve for which the curvature functions are prescribed $C^r(r = 0, 1, \cdots, \infty, \omega)$ functions of the arc length. The result can be extended to riemannian geometry.

The theorem published by James M. Sloss [4] in this journal to the effect that a regular C^3 curve can be approximated by a piecewise helix that either is circular or whose curvature or torsion are those of the given curve, can be generalized. The proof can be simplified by an explicit use of Gronwall's inequality [2] which is one of the strongest tools available in the treatment of ordinary differential equations. Gronwall's inequality says that

$$u(t) \leqq C + \int_a^t u(\tau)v(\tau)d\tau,\ t \geqq a,\ u(t) \geqq 0,\ v(t) \geqq 0\ ,$$

implies

$$u(t) \leqq C \exp \int_a^t v(\tau)d\tau\ .$$

We fix a coordinate system in E^n. Let a_i be the unit vector on the ith coordinate axis and

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

the frame of these vectors. We are given a curve $X(s), 0 \leqq s \leqq L$, as a function of its arc length. We say that X admits a continuously differentiable frame of first order [1] if there exists a continuously differentiable orthogonal matrix function $\Theta(s)$ so that the unit vector $e_1(s) = X'(s)$ is the first vector of the frame $\boldsymbol{e} = \Theta\boldsymbol{a}$. Then X must be a C^2 curve. Conversely, if X is C^2, we may define $\boldsymbol{e}$ as the frame on S^{n-1} for which e_1 is the unit tangent $e_1(s) \in S^{n-1}$ $e_k(1 \leqq k \leqq n-1)$ is tangent to the equatorial S^{k-1} in the S^k defined by $a_1, \cdots, a_{k+1}$ and e_n is tangent to the meridian. Naturally, if X is C^{n+1}, the Frenet frame may be taken as $\boldsymbol{e}$. We put $A(s) = \Theta'(s)\Theta(s)^{-1}$.

For matrices we use the sup-norm of the corresponding linear transformation. The norm of the frame $\boldsymbol{e} = \Theta \boldsymbol{a}$ shall be the norm of Θ.

THEOREM. *A C^2 curve of finite length can be approximated in the C^1 topology by a continuous, piecewise differentiable curve for which the curvature functions are prescribed C^r functions of the arc length, $r \geqq 0$.*

The structure equations of the curve X and the frame $\boldsymbol{e}$ are

$$\begin{aligned} X'(s) &= e_1(s) , \\ \boldsymbol{e}'(s) &= A(s)\boldsymbol{e}(s) , \end{aligned}$$

with a continuous matrix $A(s)$, $0 \leqq s \leqq L$. We want to approximate $X(s)$ by a continuous curve $Y(s)$ so that

$$\begin{aligned} Y'(s) &= e_1^*(s) , \\ \boldsymbol{e}^{*\prime}(s) &= A^*(s)\boldsymbol{e}^*(s) , \end{aligned}$$

except at finitely many points where the equations hold only in the sense of forward one-sided derivatives and where $A^*(s)$ is a given (continuous) matrix function of s. We shall not use the fact that $A^*(s)$ is skew-symmetric and has nonzero entries a_{ij}^* only for $j = i \pm 1$ [3].

We put

$$M = \max \|A(s)\|, N = \max \|A^*(s)\| ,$$

and choose Δs subject to

$$\Delta s \leqq \min (N^{-1}, \varepsilon(e(1 + L)(M + N))^{-1}), \ \frac{L}{\Delta s} \text{ an integer} .$$

For $k\Delta s \leqq s \leqq (k + 1)\Delta s$, $0 \leqq k < L/\Delta s$, define $Y(s)$ by

$$\begin{aligned} \boldsymbol{e}^*(s) &= \boldsymbol{e}(k\Delta s) + \int_{k\Delta s}^{s} A^*(\sigma)\boldsymbol{e}^*(\sigma)d\sigma , \\ Y(s) &= Y(k\Delta s) + \int_{k\Delta s}^{s} e_1^*(\sigma)d\sigma , \\ Y(0) &= X(0) . \end{aligned}$$

Then

$$\begin{aligned} \|\boldsymbol{e}^*(s) - \boldsymbol{e}(s)\| &= \left\| \int_{k\Delta s}^{s} (A^*\boldsymbol{e}^* - A\boldsymbol{e})d\sigma \right\| \\ &\leqq \int_{k\Delta s}^{s} \|A^* - A\| \, \|\boldsymbol{e}\| d\sigma + \int_{k\Delta s}^{s} \|A^*\| \, \|\boldsymbol{e}^* - \boldsymbol{e}\| d\sigma \end{aligned}$$

$$\leqq (M + N)\Delta s + N \int_{k\Delta s}^{s} \| e^* - e \| d\sigma \; .$$

By Gronwall's inequality,

$$\begin{aligned} \| e^*(s) - e(s) \| &\leqq (M + N)\Delta s \exp (N(s - k\Delta s)) \\ &\leqq (M + N)e\Delta s \leqq \varepsilon/(1 + L). \end{aligned}$$

Hence,

$$\begin{aligned} | Y(s) - X(s) | &\leqq | Y(k\Delta s) - X(k\Delta s) | + \int_{k\Delta s}^{s} | e_1^*(\sigma) - e_1(\sigma) | d\sigma \\ &\leqq | Y(k\Delta s) - X(k\Delta s) | + \int_{k\Delta s}^{s} \| e^*(\sigma) - e(\sigma) \| d\sigma \\ &\leqq | Y(k\Delta s) - X(k\Delta s) | + \varepsilon\Delta s(L + 1)^{-1} \; . \end{aligned}$$

By induction, $| Y(k\Delta s) - X(k\Delta s) | \leqq \varepsilon k\Delta s(1 + L)^{-1}$ and

$$| Y(s) - X(s) | \leqq \varepsilon L(1 + L)^{-1} \; ,$$

$$| Y(s) - X(s) | + | Y'(s) - X'(s) | \leqq \varepsilon \frac{L}{1 + L} + \varepsilon \frac{1}{1 + L} = \varepsilon \; .$$

The same argument would hold for a matrix $A^*(s)$ with bounded, integrable entries if the differential equations are supposed to hold a.e., in the sense of Caratheodory. For C^r entries, Y is a C^{r+2} curve. Since the structure of the matrices was not used, the same argument holds for riemannian or Minkowski geometry.

References

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