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FUNCTIONS WHICH OPERATE ON $\mathscr{F}L_p(T)$, $1 < p < 2$

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$\mathscr{F}L_p(T)$ is the algebra of Fourier transforms of functions in L_p of the circle. It is shown that if F is defined on the plane and the composition $F\circ\phi \in \mathscr{F}L_1$ whenever $\phi \in \mathscr{F}L_p$ then for all $\varepsilon > 0$, $F(z) = P(z, \bar{z}) + O(|z|^{q/2-\varepsilon})$ where P is a polynomial in z and $\bar{z}$ and $p^{-1} + q^{-1} = 1$ $(1 < p < 2)$.

1. Introduction.

Throughout, $L_p = L_p(T)$ will denote the usual space of functions on T, the unit circle, normed by

$$||f||_p = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(e^{it})|^p dt\right\}^{1/p} .$$

For $f \in L_1$ the Fourier transform is given by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} f(e^{it})dt \qquad (n = 0, \pm 1, \pm 2, \cdots) .$$

$\mathscr{F}L_p$ is the algebra of Fourier transforms of functions in $L_p(p \geqq 1)$ and $\mathscr{F}C$ is the algebra of transforms of the continuous functions.

Let F be a complex function defined on the plane. F is said to *operate from $\mathscr{F}L_p$ to $\mathscr{F}L_r$* provided the composition $F\circ\phi$ belongs to $\mathscr{F}L_r$ whenever $\phi \in \mathscr{F}L_p$.

We shall write $F(z) = O(G(z))$ to mean $F(z)/G(z)$ is bounded *near the origin*. It is an immediate consequence of Parseval's theorem that F operates from $\mathscr{F}L_2$ to $\mathscr{F}L_2$ if and only if $F(z) = O(z)$. On the other hand it was shown by Helson and Kahane [2] that F operates from $\mathscr{F}L_1$ to $\mathscr{F}L_1$ if and only if F is real analytic in a neighbourhood of the origin and, of course, $F(0) = 0$ (cf. [6, chapter 6]).

For $2 < q \leqq \infty$ it was shown by the author [3] that the functions operating from $\mathscr{F}L_q$ to $\mathscr{F}L_q$ and from $\mathscr{F}C$ to $\mathscr{F}L_q$ are the same and combine the types of behavior of the examples above. We state the result for completeness.

THEOREM 1.1. *Let $2 < q \leqq \infty$ and $p^{-1} + q^{-1} = 1$. The following are equivalent.*

(a) *F operates from $\mathscr{F}L_q$ to $\mathscr{F}L_q$.*

(b) *F operates from $\mathscr{F}C$ to $\mathscr{F}L_q$.*

(c) *$F(z) = c_1 z + c_2\bar{z} + O(|z|^{2/p})$.*

Half of the Hausdorff-Young theorem [8, Thorem 2.3 ii] was used to show that (c) implies (a) in the above. In fact, it is not difficult to see that F operates from $\mathscr{F}L_2$ to $\mathscr{F}L_q$ if and only if $F(z) =$

$O(|z|^{2/p})$.

The other half of the Hausdorff-Young theorem [8, Theorem 2.3 i] shows that if $1 < p < 2$, $p^{-1} + q^{-1} = 1$ and $F(z) = O(|z|^{q/2})$, then F operates from $\mathscr{F}L_p$ to $\mathscr{F}L_2$. It is also easy to see that this is a necessary condition. Since polynomials operate from $\mathscr{F}L_p$ to $\mathscr{F}L_p$, we then have

THEOREM 1.2. *Let $1 < p < 2$ and $p^{-1} + q^{-1} = 1$. If $F(z) = P(z, \bar{z}) + O(|z|^{q/2})$, where P is a polynomial in z and $\bar{z}$ ($P(0) = 0$), then F operates from $\mathscr{F}L_p$ to $\mathscr{F}L_p$ and thus also from $\mathscr{F}L_p$ to $\mathscr{F}L_1$.*

We can assume the polynomial P has order less than $q/2$, for higher order terms can be absorbed into $O(|z|^{q/2})$.

The main result of this paper is the following partial converse to Theorem 1.2.

THEOREM 1.3. *Let $1 < p < 2$ and $p^{-1} + q^{-1} = 1$. If F operates from $\mathscr{F}L_p$ to $\mathscr{F}L_1$, then, for all $\varepsilon > 0$,*

$$F(z) = P(z, \bar{z}) + O(|z|^{q/2-\varepsilon}) \tag{1.4}$$

where P is a polynomial in z and $\bar{z}$.

I have not been able to remove the ε in (1.4). In fact, I have not been able to show whether or not $z^{q/2} \log |z|$ operates from $\mathscr{F}L_p$ to $\mathscr{F}L_1$. However, as a corollary to Theorems 1.2 and 1.3 we can state the following complete result.

COROLLARY 1.5. *Let $1 < p < 2$ and $p^{-1} + q^{-1} = 1$. The following are equivalent.*

(a) *F operates from $\bigcup_{r>p} \mathscr{F}L_r$ to $\mathscr{F}L_1$.*
(b) *F operates from $\bigcup_{r>p} \mathscr{F}L_r$ to $\bigcup_{r>p} \mathscr{F}L_r$.*
(c) *$F(z) = P(z, \bar{z}) + O(|z|^{q/2-\varepsilon})$ for all $\varepsilon > 0$.*

The proof of Theorem 1.3 uses a factorization of the Rudin-Shapiro polynomials. The idea is to construct polynomials, P, with few coefficients so that small changes in $\hat{P}$ cause large changes in the norms of P. This is done in § 2.

In § 3 these polynomials are used to show that if F operates then, for all complex w, all integers k and certain β,

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} F((w + j)z) = O(|z|^{\beta}) . \tag{1.6}$$

Now any polynomial in z and $\bar{z}$ of degree less than k satisfies (1.6). In § 4 it is shown that, except for a $O(|z|^{\beta})$ term, these are the only functions which satisfy 1.6, at least if β is not an integer and $F(z) =$

$O(1)$. This is then used to obtain a proof of Theorem 1.3.

2. The Rudin-Shapiro polynomials. The Rudin-Shapiro polynomials are defined as follows: let $P_0(x) = Q_0(x) = 1$ and define inductively

$$P_{k+1}(x) = P_k(x) + x^{2^k} Q_k(x)$$
$$Q_{k+1}(x) = P_k(x) - x^{2^k} Q_k(x) .$$

Then

$$P_k(x) = \sum_0^{2^k - 1} \varepsilon(n) x^n \tag{2.1}$$

where $\varepsilon(n) = \pm 1$ is independent of k. As shown in [5] and [7],

$$\left| \sum_0^N \varepsilon(n) e^{int} \right| < 5(N+1)^{1/2} \qquad (0 \leqq t < 2\pi;\ N = 1, 2, \cdots) . \tag{2.2}$$

This definition differs slightly from that given in [5] and [7]. It has also been given by Brillhart and Carlitz [1].

We have the following explicit representation for $\varepsilon(n)$ (cf. [1] and [4, Lemma 2]).

LEMMA 2.3. *If n has a binary expansion*

$$n = \delta_0 + 2\delta_1 + 2^2\delta_2 + \cdots + 2^k \delta_k \qquad (\delta_i = 1 \text{ or } 0)$$

then

$$\varepsilon(n) = \prod_1^k (1 - 2\delta_i \delta_{i-1}) .$$

In the following we will factor $\varepsilon(n)$ in various ways as was done in [4]. Fix positive integers N and k and let $0 \leqq n < 2^{Nk+1}$ so that n has a binary expansion

$$n = \delta_0 + 2\delta_1 + \cdots + 2^{Nk} \delta_{Nk} .$$

Define

$$\rho_j(n) = \prod_{(j-1)N+1}^{jN} (1 - 2\delta_i \delta_{i-1}) \qquad (j = 1, 2, \cdots k) . \tag{2.4}$$

Note also that n can be written in a unique way as

$$n = n_1 + n_2 2^{Nj+1} + n_3 2^{N(j-1)} \tag{2.5}$$

where

$$\begin{aligned} 0 &\leqq n_1 < 2^{N(j-1)} \\ 0 &\leqq n_2 < 2^{N(k-j)} \\ 0 &\leqq n_3 < 2^{N+1} \end{aligned}$$

and, by Lemma 2.3, $\rho_j(n) = \varepsilon(n_3)$. It also follows from Lemma 2.3 that

$$\varepsilon(n) = \prod_1^k \rho_j(n) \; . \tag{2.6}$$

Define

$$R_j(t) = \sum \rho_j(n) e^{int} \qquad (j = 1, 2, \cdots, k) \; ,$$

the sum being from 0 to $2^{Nk+1} - 1$.

The usefulness of the R_j comes about because if S is the convolution product $S = R_1 * R_2 * \cdots * R_k$, then by (2.6)

$$S = \sum_0^{2^{Nk+1}-1} \varepsilon(n) e^{int} \; .$$

Now, by (2.2), $\| S \|_\infty \leqq 5 \cdot 2^{Nk+1}$ and since $\| S \|_2 = 2^{(Nk+1)/2}$ it follows that

$$\frac{1}{5} 2^{(Nk+1)/2} \leqq \| S \|_1 \leqq \prod_1^k \| R_j \|_1 \; . \tag{2.7}$$

Thus, very roughly, $\| R_j \|_1$ must be as large as $2^{N/2}$. The following shows that $\| R_j \|_1$ is not much larger than this.

PROPOSITION 2.8.

$$\| R_j \|_1 \leqq 2^{N/2} N^2 k^2 C$$

where C is an absolute constant.

Proof. R_j can be written

$$R_j = F_1 F_2 F_3 \tag{2.9}$$

where

$$\begin{aligned} F_1(t) &= \sum_0^{2^{N(j-1)}-1} e^{int} \\ F_2(t) &= \sum_0^{2^{N(k-j)}-1} \exp(in\, 2^{Nj+1} t) \\ F_3(t) &= \sum_0^{2^{N+1}-1} \varepsilon(n) \exp(in\, 2^{N(j-1)} t) \; . \end{aligned}$$

To see that (2.9) holds, note that the product $F_1 F_2 F_3$ consists of 2^{Nk+1} distinct exponentials between 0 and $2^{Nk+1} - 1$. Also the coefficient of e^{int} where n is given as in (2.5) is $\varepsilon(n_3) = \rho_j(n)$ so that $F_1F_2F_3 = R_j$.

It is not difficult to see that $\| F_1F_2 \|_1 \leqq Ck^2N^2$ and since, by (2.2), $\| F_3 \|_\infty \leqq 5\, 2^{(N+1)/2}$ the proposition follows.

PROPOSITION 2.10. *For $1 < p \leqq 2$ and $p^{-1} + q^{-1} = 1$*

$$\| R_j \|_p \leqq C2^{N(1/2+(k-1)/q)}N^2k^2 .$$

Proof. Since $\| R_j \|_2 = 2^{(Nk+1)/2}$ this follows from Hölder's inequality and (2.8).

LEMMA 2.11. *For N and k positive integers there is a decomposition of $\{0, 1, 2, \cdots, 2^{Nk+1} - 1\}$ into $k + 1$ sets $A_0, A_1, \cdots, A_k$ such that if*

$$T_{N,k}(t) = \sum_0^k j \sum_{A_j} e^{int} \tag{2.12}$$

$$R_{N,k}(t) = \sum_0^k j^k \sum_{A_j} e^{int}$$

and

$$S_{N,k}(t) = \sum_0^k (-1)^j \sum_{A_j} e^{int}$$

then

(a) $$\| T_{N,k} \|_1 \leqq C(k)N^2 2^{N/2}$$

(b) $$\| T_{N,k} \|_p \leqq C(k)N^2 2^{N(1/2+(k-1)/q)} \qquad (1 < p \leqq 2)$$

(c) $$\| S_{N,k} \|_1 \geqq C(k)2^{Nk/2}$$

(d) $$\| R_{N,k} \|_1 \geqq C(k)2^{Nk/2}$$

(e) $$\left\| \sum_{A_j} e^{int} \right\|_1 \geqq C(k)2^{Nk/2} \qquad (j = 0, 1, \cdots, k)$$

where the $C(k)$ are (different) positive constants depending only on k.

For $k = 2$ this has been done in [4].

Proof. Define

$$2T_{N,k}(t) = \sum_1^k R_j(t) + k \sum_0^{2^{Nk+1}-1} e^{int} .$$

Now

$$T_{N,k}(t) = \sum_0^{2^{Nk+1}-1} \phi(n)e^{int}$$

where

$$\phi(n) = \sum_1^k \frac{\rho_j(n) + 1}{2} .$$

Since $\rho_j(n) = \pm 1$, $\phi(n)$ assumes only the values $0, 1, \cdots, k$ so that if A_j consists of the n with $\phi(n) = j$ then $T_{N,k}$ is as in (2.12). (a) then follows from (2.8) and (b) from (2.10).

Now if $\phi(n) = j$, then precisely $k - j$ of the $\rho_i(n) = -1$, so that, by (2.6), $\varepsilon(n) = (-1)^{k-j}$. Hence

$$S_{N,k}(t) = (-1)^k \sum_0^{2^{Nk+1}-1} \varepsilon(n) e^{int}$$

so that (c) follows from (2.7).

Define $T^0_{N,k} = \sum_1^{2^{Nk+1}-1} e^{int}$, and inductively

$$T^{s+1}_{N,k} = T^s_{N,k} * T_{N,k} .$$

Then $\{T^s_{N,k}\}$ $(s = 0, 1, \cdots, k)$ are $k + 1$ linearly independent polynomials which span the space of polynomials of the form $\sum_0^k c_j \sum_{A_j} e^{int}$. In particular,

$$S_{N,k} = \sum_{s=0}^{k} b_s T^s_{N,k} \tag{2.13}$$

where the b_s depend on k but not on N.

Now it follows from (a) that

$$\| T^s_{N,k} \|_1 \leqq C(k) N^{2s} 2^{Ns/2} \qquad (s = 1, 2, \cdots) . \tag{2.14}$$

Also

$$\| T^0_{N,k} \|_1 \leqq C(k) N$$

so that

$$\begin{aligned} \| S_{N,k} \|_1 &\leqq \sum_0^k | b_s | \, \| T^s_{N,k} \|_1 \\ &\leqq C(k) N^{2(k-1)} 2^{N(k-1)/2} + | b_k | \, \| T^k_{N,k} \|_1 . \end{aligned} \tag{2.15}$$

(d) then follows from (2.15) and (c) since $T^k_{N,k} = R_{N,k}$. (e) holds for the same reasons since, for each j, $\sum_{A_j} e^{int}$ and $\{T^s_{N,k}\}$ $(s = 0, \cdots, k-1)$ are linearly independent.

REMARK. Because $T^k_{N,k} = R_{N,k}$ we must have $\| T_{N,k} \|_1 \geqq C(k) 2^{N/2}$. It would be useful to know if the N^2 in (a) can be removed. Also, by the Hausdorff-Young theorem, $\| T_{N,k} \|_p \geqq C(k) 2^{Nk/q}$. If the right side of (b) could be replaced by $C(k) 2^{Nk/q}$, then the ε in Theorem 1.3 could be removed.

3. The main lemma.

The purpose of this section is to use the polynomials of Lemma 2.11 to prove the following.

LEMMA 3.1. *Let F operate from $\mathscr{F} L_p$ to $\mathscr{F} L_1$ $(1 < p \leqq 2; p^{-1} + q^{-1} = 1)$. Assume that $F(z) = O(| z |^\beta)$ for some $\beta > 0$. Then for each positive integer k and each complex w*

$$\sum_{0}^{k}(-1)^j\binom{k}{j}F((w+j)z) = O(|z|^{\beta'}) \tag{3.2}$$

where

$$\beta' = \min\left(\beta + \frac{q}{4(k+q)}, \frac{qk}{2(k+q)}\right).$$

Before proving this we need the following lemma. If F operates from $\mathscr{F}L_p$ to $\mathscr{F}L_1$ then, for $f \in L_p$, $F \circ f$ will denote the function in L_1 such that $(F \circ f)\hat{}(n) = F(\hat{f}(n))$.

LEMMA 3.3. *Let F operate from $\mathscr{F}L_p$ to $\mathscr{F}L_1$.*
(a) *There are constants M and δ such that $\|f\|_p < \delta$ implies $\|F \circ f\|_1 < M$.*
(b) $F(z) = O(z)$.
(c) $F(0) = 0$.

Proof. The proof of (a) is the same as that of Lemma 1 of [3]. By considering Sidon sets, is is easily seen that F must operate from $\mathscr{F}L_2$ to $\mathscr{F}L_2$ and this gives (b). (c) is obvious.

Proof of 3.1. k and w are fixed throughout this proof. If $0 < |z| < 1$, then a positive integer N can be chosen so that

$$2^{-N((k+q)/q)} \leqq |z| < 2^{-(N-1)((k+q)/q)}. \tag{3.4}$$

Let $T_{N,k}$ be as in Lemma 2.11 and define

$$f(t) = z\{T_{N,k}(t) + wT^0_{N,k}(t)\}.$$

Then by (3.4) and (2.11 (b))

$$\|f\|_p \leqq C(k, w)N^2 2^{-N(1/2+1/q)}.$$

Thus if M and δ are as in Lemma 3.2 and $|z|$ is small enough then $\|f\|_p < \delta$ so that

$$\|F \circ f\|_1 < M. \tag{3.5}$$

Now

$$\begin{aligned} F \circ f &= \sum_{0}^{k} F((w+j)z) \sum_{A_j} e^{int} \\ &= \sum_{0}^{k} b_s T^s_{N,k} \end{aligned} \tag{3.6}$$

where the b_s satisfy

$$F((w+j)z) = \sum_0^k b_s j^s \qquad (j = 0, 1, \cdots, k) .$$

Solving for the b_s and using the assumption that $F(z) = O(|z|^\beta)$ gives that, for $|z|$ small enough,

$$(3.7) \qquad |b_s| \leqq C(k)|z|^\beta \qquad (s = 0, 1, \cdots)$$

and

$$(3.8) \qquad b_k = \frac{\det\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & F(wz) \\ 1 & 1 & 1 & \cdots & 1 & F((w+1)z) \\ 1 & 2 & 2^2 & \cdots & 2^{k-1} & F((w+2)z) \\ 1 & 3 & 3^2 & \cdots & 3^{k-1} & F((w+3)z) \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & k & k^2 & \cdots & k^{k-1} & F((w+k)z) \end{pmatrix}}{\det\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2^2 & \cdots & 2^{k-1} & 2^k \\ 1 & 3 & 3^2 & \cdots & 3^{k-1} & 3^k \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & k & k^2 & \cdots & k^{k-1} & k^k \end{pmatrix}}$$

$$= A(k) \sum_0^k (-1)^j \binom{k}{j} F((w+j)z)$$

where $A(k) \neq 0$ is independent of z. Now by (3.5) and (3.6)

$$(3.9) \qquad |b_k| \, \| T_{N,k}^k \|_1 \leqq M + \sum_0^{k-1} |b_s| \, \| T_{N,k}^s \|_1 .$$

Lemma 2.11d, (2.14), (3.7), (3.8) and (3.9) then give, if $|z|$ is small enough,

$$(3.10) \qquad \begin{aligned} \Big| \sum_0^k (-1)^j \binom{k}{j} F((w+j)z) \Big| &\leqq C(k)\left\{\frac{M}{2^{Nk/2}} + \frac{|z|^\beta N^{2(k-1)}}{2^{N/2}}\right\} \\ &\leqq C(k)\left\{\frac{M}{2^{Nk/2}} + \frac{|z|^\beta}{2^{N/4}}\right\}. \end{aligned}$$

By (3.4) the right side of (3.10) is bounded by

$$C(k)\{M|z|^{kq/(2(k+q))} + |z|^{\beta+q/4(k+q)}\}$$

and this gives (3.2).

4. Proof of Theorem 1.3. We can now prove Theorem 1.3 provided we have the following theorem.

THEOREM 4.1. *Suppose F is bounded near the origin and for some positive integer k and each complex w, F satisfies*

$$\sum_{0}^{k} (-1)^j \binom{k}{j} F((w + j)z) = O(|z|^\beta) \tag{4.2}$$

where $\beta > 0$ and is not an integer. Then

$$F(z) = P(z, \bar{z}) + H(z)$$

where P is a polynomial in z and $\bar{z}$ of degree less than k,

$$H(z) = O(|z|^\beta) \text{ and } H(0) = 0 \text{ .}$$

REMARKS. Since $\beta > 0$ and $H(0) = 0$ it follows that H and thus also F is continuous at 0. F need not be continuous anywhere else.

The theorem is false if β is an integer as can be seen by letting $\beta = 1$, $k = 2$ and $F(z) = z \log |z|$ $(F(0) = 0)$.

It is also false if $F(z) \neq O(1)$. For there are functions defined on the plane which are unbounded near the origin and satisfy $F(z + w) = F(z) + F(w)$ for all z and w. The left side of (4.2) is then 0 for all $k > 1$. Being unbounded F cannot satisfy the conclusion of the theorem.

Proof of 1.3. F operates from $\mathscr{F}L_p$ to $\mathscr{F}L_1$ where $1 < p < 2$. There is a positive integer r such that $r < q/2 \leqq r + 1$. We will prove the theorem by induction on r.

First, we can assume that

$$F(z) = O(|z|^{r-\delta}) \qquad \text{for all } \delta > 0. \tag{4.3}$$

For if $r = 1$ then, by Lemma 3.3b, (4.3) holds even with $\delta = 0$. On the other hand, suppose $r > 1$ and the theorem holds when $r - 1 < q'/2 \leqq r$. Since F operates from $\mathscr{F}L_p$ to $\mathscr{F}L_1$, it operates from $\mathscr{F}L_s$ to $\mathscr{F}L_1$ where $s^{-1} + (2r)^{-1} = 1$. Thus $F(z) = P(z, \bar{z}) + O(|z|^{r-\varepsilon})$ for all $\varepsilon > 0$. Since polynomials operate we can assume $p = 0$, that is (4.3).

Next choose k so large and then δ so small that $\beta' = \min(r - \delta + q/4(k + q),\ q/2(k + q)) > r$ and also so that β' is not an integer. Then by (4.3), Lemma 3.1 and Theorem 4.1

$$F(z) = P(z, \bar{z}) + O(|z|^{\beta'}) \text{ .}$$

Thus, by subtracting another polynomial from F, we can assume

$$F(z) = O(|z|^{\beta'}) \text{ for some } \beta' > r \text{ .} \tag{4.4}$$

Finally, let $\gamma = \sup \beta'$ such that (4.4) holds. If $\gamma < q/2$ then we can choose k so large and then $r < \beta' < \gamma$ so that

$$\beta'' = \min\left(\beta' + q/4(k+q), \frac{qk}{2(k+q)}\right) > \gamma \tag{4.5}$$

and β'' is not an integer.

Then by Lemma 3.1 and Theorem 4.1 again

$$F(z) = P(z, \bar{z}) + O(|z|^{\beta''}) .$$

Since $F(z) = O(|z|^{\beta'})$ and $r < \beta' < \beta'' < r + 1$ we must have $P(z, \bar{z}) = O(|z|^{r+1})$ so that $F(z) = O(|z|^{\beta''})$. Since $\beta'' > \gamma$ this is a contradiction. Thus (4.4) holds for all $\beta' < q/2$ and this completes the proof of the theorem.

It now remains to give a proof of Theorem 4.1.

LEMMA 4.6. *Suppose F, defined on the plane—$\{0\}$, satisfies*

$$F(qz) - q^s F(z) = O(|z|^{\beta})$$

where $q > 1$.

(a) *If $F = O(1)$ and $s > \beta > 0$, then $F(z) = O(|z|^{\beta})$.*

(b) *If $\beta > s > 0$ then $F(z) = K(z) + O(|z|^{\beta})$ where $K(qz) = q^s K(z)$. If also $F(z) = O(1)$ then $K(s) = O(|z|^{s})$.*

The proof of (a) is simple and that of (b) is the same as the proof of Lemma 3 of [3].

LEMMA 4.7. *Suppose F is bounded near the origin and, for some positive integer k and each nonnegative integer p, F satisfies*

$$\sum_{0}^{k} (-1)^j \binom{k}{j} F((p+j)z) = O(|z|^{\beta}) \tag{4.8}$$

where $\beta > 0$ and β is not an integer. Then

$$F(z) = F(0) + \sum_{1}^{k-1} F_j(z) + O(|z|^{\beta}) \tag{4.9}$$

where

$$F_j(qz) = q^j F_j(z) \tag{4.10}$$

for all positive integers q and $F_j(z) = O(|z^j|)$.

Note that it follows from the conclusion that F is continuous at 0.

Proof. The lemma is clear if $k = 1$, so assume $k > 1$ and the lemma holds for $k - 1$. Fix $q > 1$, an integer and for a nonnegative integer p consider the polynomial

$$S(\lambda) = \sum_0^{k-1} (-1)^j \binom{k-1}{j} (\lambda^{(p+j)q} - q^{k-1}\lambda^{p+j}) .$$

Now S has a zero of order k at 1 and thus can be written

$$S(\lambda) = (1-\lambda)^k \sum_0^b a_j \lambda_j \qquad (b = (p+k-1)q - k)$$
$$= \sum_0^b a_j \sum_0^k (-1)^s \binom{k}{s} \lambda^{s+j} .$$

By comparing the coefficients of λ^n in the two forms of S it is seen that for any function F

$$\sum_0^{k-1} (-1)^j \binom{k-1}{j} (F((p+j)qz) - q^{k-1}F((p+j)z)) = \sum_0^b a_j \sum_0^k (-1)^s \binom{k}{s} F((s+j)z) .$$

Thus if F satisfies the hypotheses of the lemma for k then the function $T(z) = F(qz) - q^{k-1}F(z)$ satisfies them for $k-1$. Thus

$$T(z) = T(0) + \sum_1^{k-2} T_j(z) + O(|z|^\beta)$$

where the T_j satisfy (4.10). Let

$$H(z) = F(z) - F(0) - \sum_0^{k-1} \frac{T_j(z)}{q^j - q^{k-1}} . \tag{4.11}$$

Then $H(qz) - q^{k-1}H(z) = O(|z|^\beta)$. Since β is not an integer and $H(z) = O(1)$ one of the two cases of Lemma 4.6 holds so that H can be written

$$H(z) = K(z) + O(|z|^\beta)$$

where $K(qz) = q^{k-1}K(z)$ and $K(z) = O(|z^{k-1}|)$. If $\beta < k-1$ then we can assume $K = 0$ and by using any q, (4.11) gives the desired form for F. If $\beta > k-1$, then it is easily seen that $F_j = T_j/(q^j - q^{k-1})$ and $F_{k-1} = K$ are independent of the choice of q. All the F_j then satisfy (4.10), and by (4.11), F is given by (4.9).

Proof of Theorem 4.1. We have that for each complex w

$$\sum_0^k (-1)^j \binom{k}{j} F((w+j)z) = O(|z|^\beta) . \tag{4.12}$$

Because of the previous lemma we need only consider functions of the form

$$F(z) = F(0) + \sum_1^{k-1} F_s(z)$$

where the F_s satisfy (4.10) and $F_s = 0$ if $s > \beta$. Also since constant functions satisfy (4.12) we can assume $F(0) = 0$. If $\beta < 1$ there is nothing left to prove so assume $\beta > 1$.

Now by (4.10) and (4.12), for each positive integer q,

$$\sum_1^{k-1} \frac{q}{q^s} \sum_0^k (-1)^j \binom{k}{j} F_s((w+j)z)$$
$$= q \sum_0^k (-1)^j \binom{k}{j} F((w+j)z/q) = qO\left(\frac{|z|^\beta}{q^\beta}\right).$$

Fixing z and letting $q \to \infty$ then gives

$$\sum_0^k (-1)^j \binom{k}{j} F_1((w+j)z) = 0$$

so that

$$\sum_0^k (-1)^j \binom{k}{j} F_1(w+jz) = 0 \tag{4.13}$$

for all z and w. Similarly (4.13) holds for $F_2, F_3, \cdots, F_{k-1}$. Then, for each complex w, the function $H(z) = F_s(w+z)$ satisfies the hypotheses of Lemma 4.7, but this implies that H is continuous at 0 so that F_s is continuous everywhere and $F_s(xz) = x^s F_s(z)$ for all $x \geqq 0$. Finally, for each integer n,

$$K_n(z) = \int_0^{2\pi} F_s(ze^{it}) e^{-int} dt$$

satisfies (4.13) and for $x \geqq 0$

$$K_n(xe^{it}) = x^s e^{int} K_n(1).$$

It can be easily seen directly that $K_n(1)$ must be zero unless $s+n$ is even and $|n| \leqq s$ which implies $F_s(z) = \sum_0^s c_r z^r \bar{z}^{s-r}$ and this completes the proof of the theorem.

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