PACIFIC JOURNAL OF MATHEMATICS
Vol. 40, No. 3, 1972

NORM CONVERGENCE OF MARTINGALES OF RADON-NIKODYM DERIVATIVES GIVEN A σ-LATTICE

R. B. DARST AND G. A. DEBOTH

Suppose that $\{\mathscr{M}_k\}$ is an increasing sequence of sub σ-lattices of a σ-algebra $\mathscr{A}$ of subsets of a non-empty set Ω. Let $\mathscr{M}$ be the sub σ-lattice generated by $\bigcup_k \mathscr{M}_k$. Suppose that L^Φ is an associated Orlicz space of $\mathscr{A}$-measurable functions, where Φ satisfies the Δ_2-condition, and let $h \in L^\Phi$. It is verified that the Radon-Nikodym derivative, f_k, of h given $\mathscr{M}_k$ is in L^Φ and shown that the sequence $\{f_k\}$ converges to f in L^Φ, where f is the Radon-Nikodym derivative of h given $\mathscr{M}$.

1. Introduction. H. D. Brunk defined conditional expectation given a σ-lattice and established several of its properties in [1]. Subsequently S. Johansen [5] described a Radon-Nikodym derivative given a σ-lattice and showed that the Radon-Nikodym derivative was the conditional expectation in the appropriate case. Then H. D. Brunk and S. Johansen [2] proved an almost everywhere martingale convergence theorem for the Radon-Nikodym derivatives given an increasing sequence of σ-lattices. We shall establish norm convergence of these derivatives in L_1 and in the Orlicz spaces L^Φ, where Φ satisfies the Δ_2-condition. The theory of these Orlicz spaces can be found in [6], so we shall assume and build upon the results therein. Thereby, we can place fewer restrictions on Φ and obtain L_1-convergence as a byproduct.

2. Notation. Let $\mathscr{A}$ be a σ-algebra of subsets of a (non-empty) set Ω, and let μ be a non-negative (bounded) σ-additive function defined on $\mathscr{A}$.

For our purposes the following information about Φ will suffice: Φ is an even, convex function defined on the real numbers, R, with $\Phi(0) = 0$ and $\Phi(x) \neq 0$ for some x. Moreover, there exists $K > 0$ with $\Phi(2x) \leqq K\Phi(x)$ for all $x \in R$. This latter property is called the Δ_2-condition; it implies

$$(1) \quad \Phi(x + y) = \Phi\left(2\left(\frac{x+y}{2}\right)\right) \leqq K\Phi\left(\frac{x+y}{2}\right) \leqq \left(\frac{K}{2}\right)[\Phi(x) + \Phi(y)] .$$

Then L^Φ denotes the collection of (real valued) $\mathscr{A}$-measurable functions h defined on Ω with $\int_\Omega \Phi(h)d\mu < \infty$. Since Φ is convex and not identically zero, $L^{\Phi} \subset L_1$; L^{Φ} is usually a proper subset of L_1 if $\lim_{x\to\infty} \Phi(x)/x = \infty$. This latter property and $\lim_{x\to 0} \Phi(x)/x = 0$ are required of an Orlicz space; but, these two properties are not necessary for our estimates to be valid. Examples are $\Phi(x) = |x|^p$, $1 \leqq p < \infty$.

Let $h \in L^{\Phi}$ and $\lambda(E) = \int_E h d\mu$, where $E \in \mathscr{A}$. Let $\mathscr{M}$ be a sub σ-lattice of $\mathscr{A}$ and let f be the Radon-Nikodym derivative of λ with respect to μ. Thus, f is the $\mathscr{M}$-measurable function defined on Ω (ϕ: the empty set, Ω, and $[f > a]$ belong to $\mathscr{M}$, for all $a \in R$) satisfying

$$\lambda(A \cap [f \leqq b]) \leqq b\mu(A \cap [f \leqq b]) , \quad \text{where } A \in M \text{ and } b \in R , \tag{2}$$

and

$$\lambda([f > a] \cap B^C) \geqq a\mu([f > a] \cap B^C) , \quad \text{where } B^C = \Omega - B,\ B \in \mathscr{M}, \text{ and } a \in R . \tag{3}$$

Our first result is a preliminary step to an L^{Φ} martingale convergence theorem.

3. The derivative of an L^{Φ}-function is an L^{Φ}-function. We shall verify this assertion by establishing a sequence of estimates, the first of which is

$$\int_{[f>a]} \Phi(f)d\mu \leqq \int_{[f>a]} \Phi(h)d\mu , \qquad \text{for all } a \geqq 0 . \tag{4}$$

To verify (4), choose $\delta > 0$ and $a = a_0 < a_1 < a_2 < \cdots$ with $\Phi(a_k) = \Phi(a_{k-1}) + \delta$. Let $A_k = [a_k \geqq f > a_{k-1}]$ and notice that (3) implies

$$|\lambda|(\Omega) \geqq \lambda([f > a_k]) \geqq a_k\mu([f > a_k]) .$$

Thus, $\mu([f > a_k]) \to 0$ and

$$\int_{[f>a]} \Phi(\cdot)\, d\mu = \sum_{k=1}^{n} \int_{A_k} \Phi(\cdot)\, d\mu + \int_{[f>a_n]} \Phi(\cdot)\, d\mu = \sum_{k=1}^{\infty} \int_{A_k} \Phi(\cdot)\, d\mu .$$

Applying (3) again, $\int_{A_k} h d\mu = \lambda(A_k) \geqq a_{k-1}\mu(A_k)$, so

$$a_{k-1} \leqq \frac{1}{\alpha_k}\int_{A_k} h d\mu , \quad \text{where} \quad \alpha_k = \mu(A_k) > 0 .$$

Then, applying Jensen's inequality,

$$\Phi(a_{k-1}) \leqq \Phi\left(\frac{1}{\alpha_k}\int_{A_k} h d\mu\right) \leqq \frac{1}{\alpha_k}\int_{A_k} \Phi(h) d\mu .$$

Next, notice that

$$\int_{A_k} \Phi(f)d\mu \leqq \Phi(a_k)\mu(A_k) = (\Phi(a_{k-1}) + \delta)\,\mu(A_k) \leqq \int_{A_k} \Phi(h)d\mu + \delta\mu(A_k) .$$

Thus $\int_{[f>a]} \Phi(f)d\mu \leqq \int_{[f>a]} \Phi(h)d\mu + \delta\mu(\Omega)$, for all $\delta > 0$, which implies (4).

By a similar argument, one obtains

$$\int_{[f\leqq a]} \Phi(f)d\mu \leqq \int_{[f\leqq a]} \Phi(h)d\mu , \qquad \text{for all } a \leqq 0 . \tag{5}$$

Hence, splitting Ω into two pieces, $[f > 0]$ and $[f \leqq 0]$, and applying (4) and (5), yields

$$\int_{\Omega} \Phi(f)d\mu \leqq \int_{\Omega} \Phi(h)d\mu; \tag{6}$$

thus verifying Theorem 1.

THEOREM 1. *The Radon-Nikodym derivative of an L^{Φ}-function is an L^{Φ}-function.*

4. A Martingale convergence theorem. Suppose that $\{\mathscr{M}_k\}_{k=1}^{\infty}$ is an increasing sequence of σ-lattices of subsets of Ω, and $\mathscr{M}$ is the σ-lattice generated by the lattice $\mathscr{M}_{\infty} = \bigcup_k \mathscr{M}_k$. Denote by $\mathscr{A}_k$ the σ-algebra that is generated by $\mathscr{M}_k$ and by λ_k and μ_k the restrictions of λ and μ to $\mathscr{A}_k$. Let h_k be an $\mathscr{A}_k$-measurable function satisfying $\lambda(E) = \int_E h_k d\mu$, where $E \in \mathscr{A}_k$, and denote by f_k the Radon-Nikodym derivative of λ_k with respect to μ_k on $\mathscr{M}_k$.

THEOREM 2. *The sequence $\{f_k\}$ converges to f in L^{Φ}-norm:*

$$\lim_{k\to\infty} \int_{\Omega} \Phi(f - f_k)d\mu = 0 . \tag{7}$$

Proof. To begin, notice that applying (4) and (5) to f_k yields

$$\int_{[f_k>a]} \Phi(h_k)d\mu \geqq \int_{[f_k>a]} \Phi(f_k)d\mu , \qquad \text{for all } a \geqq 0 , \tag{8}$$

and

$$\int_{[f_k\leqq a]} \Phi(h_k)d\mu \geqq \int_{[f_k\leqq a]} \Phi(f_k)d\mu , \qquad \text{for all } a \leqq 0 . \tag{9}$$

Since λ_k is the restriction of λ to $\mathscr{A}_k$, a variation on the theme which established (4) verifies

$$\int_E \Phi(h)d\mu \geqq \int_E \Phi(h_k)d\mu , \qquad \text{for all } E \in \mathscr{A}_k: \tag{10}$$

To substantiate this latter assertion, suppose $a \geqq 0$, $\delta > 0$, $b > a$, $\Phi(b) = \Phi(a) + \delta$, $E \in \mathscr{A}_k$, $F = E \cap [b \geqq h_k > a]$, and $\mu(F) > 0$. Then $\int_F h_k d\mu = \int_F h d\mu$, since $F \in \mathscr{A}_k$. Moreover,

$$\int_F \Phi(h_k) d\mu \leqq \Phi(b)\mu(F) = [\Phi(a) + \delta]\, \mu(F) ,$$

and

$$\Phi(a) \leqq \Phi\left(\frac{1}{\mu(F)} \int_F h_k d\mu\right) = \Phi\left(\frac{1}{\mu(F)} \int_F h d\mu\right) \leqq \frac{1}{\mu(F)} \int_F \Phi(h) d\mu .$$

Thus,

$$\int_F \Phi(h_k) d\mu \leqq \int_F \Phi(h) d\mu + \delta\mu(F) .$$

Hence, appealing to the proof of (4) and to the sentence containing (5), we claim (10). Consequently,

$$\int_{[f_k > a]} \Phi(h) d\mu \geqq \int_{[f_k > a]} \Phi(f_k) d\mu , \quad \text{where } a \geqq 0 \text{ and } k = 1, 2, \cdots , \tag{11}$$

and

$$\int_{[f_k \leqq a]} \Phi(h) d\mu \geqq \int_{[f_k \leqq a]} \Phi(f_k) d\mu , \quad \text{where } a \leqq 0 \text{ and } k = 1, 2, \cdots . \tag{12}$$

Moreover, $a\mu([|f_k| > a]) \leqq |\lambda|([|f_k| > a]) \leqq |\lambda|(\Omega)$, where $a \geqq 0$; thus,

$$\lim_{n\to\infty} \sup_k \int_{[|f_k| > n]} \Phi(f_k) d\mu = 0 . \tag{13}$$

So we can truncate the functions and still approximate them uniformly as follows. Whenever n is a positive integer and u is a (real valued) function defined on Ω, let $u^n(x) = u(x)$, where $|u(x)| \leqq n$, and $u^n(x) = nu(x)/|u(x)|$ otherwise. Then, using (1) and setting $M = \max\{(K/2), (K^2/4)\}$,

$$\int_\Omega \Phi(f - f_k) d\mu = \int_\Omega \Phi(\{f - f^n\} + \{f^n - (f_k)^n\} + \{(f_k)^n - f_k\})\, d\mu \leqq M(A_n + B_n + C_n) ,$$

where

$$A_n = \int_{[|f|>n]} \Phi(f) d\mu ,$$

$$B_n = \int_{\Omega} \Phi(f_n - (f_k)^n)\, d\mu ,$$

and

$$C_n = \int_{[|f_k|>n]} \Phi(f_k) d\mu .$$

From (4), (5) and (13), we obtain $A_n \to 0$ and $C_n \to 0$. Moreover, for each $\delta > 0$,

$$\begin{aligned} B_n &\leqq \Phi(2n)\mu([|f^n - (f_k)^n| > \delta]) + \Phi(\delta)\mu(\Omega) \\ &\leqq \Phi(2n)\mu([|f - f_k| > \delta]) + \Phi(\delta)\mu(\Omega) . \end{aligned}$$

But, Brunk and Johansen have shown that $\lim_k \mu([|f - f_k| > \delta]) = 0$, where $\delta > 0$, so Theorem 2 is established.

Because of the approximation properties which are verified in [4], the results of this paper extend immediately to analogous results for the derivatives of additive set functions defined on algebras of subsets of Ω given a sub lattice (cf. [3]). Results for vector valued functions with respect to lattices which are related to the results: [7], [8], [9], of J. J. Uhl, Jr. for vector valued functions with respect to algebras should appear subsequently.

REFERENCES

1. H. D. Brunk, *Conditional expectation given a σ-lattice and applications*, Annals. Math. Statist., **36** (1965), 1339-1350.

2. H. D. Brunk and S. Johansen, *A generalized Radon-Nikodym derivative*, Pacific J. Math., **34** (1970), 585-617.

3. R. B. Darst, *The Lebesgue decomposition, Radon-Nikodym derivative, conditional expectation and martingale convergence for lattices of sets*, Pacific J. Math., **35** (1970), 581-600.

4. R. B. Darst and G. A. DeBoth, *Two approximation properties and a Radon-Nikodym derivative for lattices of sets*, Indiana Univ. Math. J., **21** (1971), 355-362.

5. S. Johansen, *The descriptive approach to the derivative of a set function with res pect to a σ-lattice*, Pacific J. Math., **21** (1967), 49-58.

6. M. A. Krasnosel'skii and Ya. B. Rutickii, *Convex functions and Orlicz spaces* (Translation), Groningen, 1961.

7. J. J. Uhl, Jr., *Orlicz spaces of finitely additive set functions*, Studia Math., T. **XXIX** (1967), 19-58.

8. ———, *Applications of Radon-Nikodym theorems to martingale convergence*, Trans. Amer. Math. Soc., **145** (1969), 271-285.

9. J. J. Uhl. Jr., *Martingales of vector valued set functions*, Pacific J. Math., **30** (1969), 533-548.

Received April 7, 1971, R. B. Darst was supported in part by the National Science Foundation under grant no. GP 9470 and G. A. DeBoth was supported by a National Science Foundation Science Faculty Fellowship.

PURDUE UNIVERSITY
COLORADO STATE UNIVERSITY
AND
ST. NORBERT COLLEGE