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PEAK INTERPOLATION SETS FOR SOME ALGEBRAS OF ANALYTIC FUNCTIONS

A. M. DAVIE AND B. K. ØKSENDAL

For certain algebras of analytic functions on holomorphically convex sets in C^n metric sufficient conditions are given for a set (not necessarily compact) to be an interpolation set. The results extend the Rudin-Carleson theorem for the disc algebra.

Let K be a compact subset of C^n which is holomorphically convex, i.e. K is the intersection of a decreasing sequence of pseudoconvex domains (see [4], Ch. 2). We denote by $H(K)$ the uniform closure on K of the algebra of all functions analytic in a neighborhood of K, and by $A(K)$ the algebra of all continuous functions on K analytic on K^0 (the interior of K). If E is any subset of the boundary ∂K of K then we denote by H_E^∞ the algebra of all bounded continuous functions on $K^0 \cup E$ which are analytic on K^0. We show that if the boundary of K is well behaved at each point of E, and E satisfies a metric condition which says roughly that E has zero 2-dimensional measure in the directions of the complex tangent and zero one dimensional measure in the orthogonal direction, then E is a peak interpolation set (in an appropriate sense) for $H^\infty_{E\cup(\partial K\setminus\bar{E})}$. If E is compact then it is a peak interpolation set in the usual sense ([2], p. 59) for the uniform algebra $H(K)$. We show also that if E has zero one-dimensional measure then the conditions on ∂K can be relaxed.

We say that ∂K is strictly pseudoconvex in a neighborhood of a point $\zeta \in \partial K$ if there is an open neighborhood V of ζ such that $V \cap \partial K$ is a C^2-submanifold of V and the Levi form is positive definite at ζ. Then we can find an open neighborhood V of ζ and a C^2 strictly plurisubharmonic function ρ in V such that $K \cap V = \{z \in V: \rho(z) \leqq 0\}$ and grad $\rho \neq 0$ on $V \cap \partial K$. (See [3] Prop. IX. A4).

LEMMA 1. *Let K be a holomorphically convex compact set in C^n and let ζ be a point of ∂K in a neighborhood of which ∂K is strictly pseudoconvex. We can find positive numbers m_ζ and M_ζ and $G_\zeta \in H(K)$, such that*

(a) $\operatorname{Re} G_\zeta(z) \geqq m_\zeta |\zeta - z|^2$, $z \in K$
(b) $\operatorname{Re} G_\zeta(z) \leqq M_\zeta |\zeta - z|^2$, $z \in \partial K$
(c) $\operatorname{grad}\,(\operatorname{Re} G_\zeta)(\zeta) = -\operatorname{grad}\, \rho(\zeta)$.

Proof. Put

$$F(z) = \sum_{i=1}^{n} \frac{\partial \rho(\zeta)}{\partial \zeta_i} (z_i - \zeta_i) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \rho(\zeta)}{\partial \zeta_i \partial \zeta_j} (z_i - \zeta_i)(z_j - \zeta_j) .$$

Then the Taylor expansion of ρ about ζ is

$$\rho(z) = 2 \operatorname{Re} F(z) + \sum_{i,j=1}^{n} \frac{\partial^2 \rho(\zeta)}{\partial \zeta_i \partial \overline{\zeta_j}} (z_i - \zeta_i)(z_j - \zeta_j) + o(|z - \zeta|^2) .$$

Since ρ is strictly plurisubharmonic at ζ it follows that, shrinking V if necessary, we can find $m > 0$ with $\operatorname{Re} F(z) \leqq -m|\zeta - z|^2$ for $z \in K \cap V$. Since $\rho = 0$ on $\partial K \cap V$ we also deduce that for some constant M

$$\operatorname{Re} F(z) \geqq -M|\zeta - z|^2; \; z \in \partial K \cap V .$$

Choose a pseudoconvex open neighborhood U of K so that $\operatorname{Re} F < 0$ on an open neighborhood W of $\partial V \cap U$ in U. Let $W_1 = W \cup (V \cap U)$ and $W_2 = W \cup (U \backslash \bar{V})$, so that $W_1 \cup W_2 = U$ and $W_1 \cap W_2 = W$. By solving a Cousin problem in U (see [4], Theorem 5.5.1) we can find analytic functions g_1 and g_2 on W_1 and W_2 respectively such that $g_2 - g_1 = F^{-2} \log F$ on W.

$$\text{Put } h(z) = \begin{cases} F(z) \exp (F(z)^2 g_1(z)), & z \in W_1 \\ \exp (F(z)^2 g_2(z)), & z \in W_2 . \end{cases}$$

The definitions agree on W so h is an analytic function on U, $h(z) = 0$ only when $z = \zeta$, and in a neighborhood of ζ, $h(z) = F(z) + 0(|z - \zeta|^3)$. Thus $\operatorname{Re} h \leqq 0$ near ζ, so there exists $\varepsilon > 0$ such that if $z \in K$ and $|h(z) - \varepsilon| \leqq \varepsilon$ then $z = \zeta$. Put

$$G(z) = - \frac{h(z)}{\varepsilon - h(z)} , \; z \in K .$$

Then $G \in \bar{H}(K)$, $\operatorname{Re} G(z) > 0$ for $z \in K \backslash \{\zeta\}$. Finally, near ζ, $\operatorname{Re} G(z) = -\varepsilon^{-1} \operatorname{Re} F(z) + \varepsilon^{-2} (\operatorname{Im} F)^2 + 0(|z - \zeta|^3)$ from which it follows that $G_\zeta = 2\varepsilon G$ has the required properties.

If S is a real subspace of C^n and Y is any subset we denote by $d_S(Y)$ the diameter (in the Euclidean metric) of the (real) orthogonal projection of Y on S.

Let K be a compact holomorphically convex subset of C^n and suppose ∂K is strictly pseudoconvex in a neighborhood of a point $\zeta \in \partial D$. Then in a neighborhood of ζ we can write $\partial K = \{z \colon \rho(z) = 0\}$ where $\rho(z)$ is strictly plurisubharmonic in a neighborhood of ζ and grad $\rho \neq 0$. The vector i grad ρ is orthogonal to grad ρ and so lies in the (real) tangent space to ∂K at ζ; let $T(\zeta)$ be the orthogonal complement to i grad ρ in this space. Then $T(\zeta)$ is the unique complex subspace of the real tangent space with complex dimension $n - 1$. Let $L(\zeta)$ be the real line spanned by the vector i grad ρ.

If E is any subset of ∂K we denote by H_E^∞ the set of all bounded continuous functions on $K^0 \cup E$ which are analytic on K^0. We define $A(K) = H_{\partial K}^\infty$.

THEOREM 1. *Let F be a subset of ∂K such that ∂K is strictly pseudoconvex in a neighborhood of F. Suppose that for every $\varepsilon > 0$ the set F can be covered by a sequence $\{V_i\}$ of open sets with diameter $< \varepsilon$ such that if $\zeta_i \in F \cap V_i$ for each i then $\sum_i d_{L(\zeta_i)}(V_i) < \varepsilon$ and $\sum_i \{d_{T(\zeta_i)}(V_i)\}^2 < \varepsilon$. Let V be a neighborhood of F, let $\eta > 0$, and let g be a bounded continuous function on F with $\|g\| \leqq 1$.*

Then we can find $f \in H^\infty_{F \cup (K \setminus \bar{F})}$ with $f|F = g$, $\|f\| \leqq 1$, and $|f| < \eta$ on $K \setminus V$.

The proof will be split up into lemmas.

LEMMA 2. *Let F, V and η be as in the theorem. Then we can find $f \in H^\infty_{F \cup (K \setminus \bar{F})}$ with $f = 1$ on F, $\|f\| \leqq 2$ and $|f| < \eta$ on $K \setminus V$.*

Proof. For each $\zeta \in F$ we choose m_ζ, $M_\zeta > 0$, and a function $G_\zeta \in H(K)$ as in Lemma 1.

If W_ζ is a sufficiently small open neighborhood of ζ, then whenever $\zeta \in U \subseteqq W_\zeta$ and $z \in U \cap \partial K$ we have

$$\begin{aligned} |G_\zeta(z)| &\leqq \operatorname{Re} G_\zeta(z) + |\operatorname{Im} G_\zeta(z)| \\ &\leqq A_\zeta |z - \zeta|^2 + |< \operatorname{grad} (\operatorname{Im} G_\zeta)(\zeta), z - \zeta >| \\ &\leqq 2A_\zeta(d_1^2 + d_2^2) + |\operatorname{grad} \rho(\zeta)| d_1 \\ &\leqq B_\zeta(d_1 + d_2^2) \end{aligned}$$

where $d_1 = d_{L(\zeta)}(U)$, $d_2 = d_{T(\zeta)}(U)$, A_ζ, B_ζ do not depend on z, and $\langle , \rangle$ denotes the real scalar product.

For each positive integer n let

$$F_n = \{\zeta \in F \colon B_\zeta < n, \Delta(\zeta, 1/n) \subseteqq W_\zeta, m_\zeta d(\zeta, K \setminus V)^2 > 1/n, m_\zeta > 1/n\} .$$

Then $F = \bigcup_n F_n$. For each n we choose a sequence $\{V_i^{(n)}\}$ of open sets with diameter less than $1/n$ such that each point of F_n is contained in infinitely many $V_i^{(n)}$, and $\sum_i \{d_{L(\zeta_i^n)}(V_i^n) + (d_{T(\zeta_i^n)}(V_i^n))^2\} < \eta n^{-2} 2^{-n-2}$ for some choice of $\zeta_i^{(n)} \in V_i^{(n)} \cap F_n$. Renumber the collection of all $V_i^{(n)}$ as $V_1, V_2, \cdots$. For each j choose n_j so that $V_j = V_i^{(n_j)}$ for some i, and let $\zeta_j = \zeta_i^{(n_j)}$. Let $G_j = G_{\zeta_j}$. Writing $c_j = d_{L(\zeta_j)}(V_j) + \{d_{T(\zeta_j)}(V_j)\}^2$ we define

$$B_r(z) = \prod_{j=1}^r \frac{G_j(z)}{2n_j c_j + G_j(z)} , \; z \in K, \; r = 1, 2, \cdots .$$

Then $B_r \in H(K)$ and $|B_r| \leqq 1$ on K. We claim that $\{B_r\}$ converges pointwise on $F \cup (K \backslash \bar{F})$ to a limit B which is continuous on $F \cup (K \backslash \bar{F})$, analytic on K^0, zero at each point of F, with $\|B\| \leqq 1$ and $|1 - B| < \eta$ on $K \backslash V$.

If $z \in K \backslash V$ then $\operatorname{Re} G_j(z) \geqq m_{\zeta_j} |z - \zeta_j|^2 > 1/n_j$, so

$$\sum_{j=1}^{\infty} \left|1 - \frac{G_j(z)}{2n_j c_j + G_j(z)}\right| = \sum_{j=1}^{\infty} \frac{2n_j c_j}{|2n_j c_j + G_j(z)|} \leqq \sum_{j=1}^{\infty} 2n_j^2 c_j < \eta/2 ,$$

which proves that $B_r(z)$ converges to a limit $B(z)$ with $|1 - B(z)| < \eta$.

If $z \in K \backslash \bar{F}$ then

$$\sum_{j=j_0}^{\infty} \frac{2n_j c_j}{|2n_j c_j + G_j(z)|} \leqq \sum_{j=j_0}^{\infty} \frac{2n_j^2 c_j}{|z - \zeta_j|^2} \leqq d(z, F)^{-2} \sum_{j=j_0}^{\infty} 2n_j^2 c_j .$$

The series on the right converges, so B_r converges uniformly to a limit B on sets at positive distance from F, so B is continuous on $K \backslash \bar{F}$ and analytic on K^0.

Finally let $z \in F$. Then $z \in V_j$ for infinitely many j. For each such j we have $V_j \subseteqq W_{\zeta_j}$ and for all $w \in W_{\zeta_j}$,

$$\left|\frac{G_j(w)}{2n_j c_j + G_j(w)}\right| \leqq \frac{n_j c_j}{2n_j c_j} = \frac{1}{2} .$$

It follows that $B_r(z) \to 0$ and $\lim |B_r|$ is continuous at z. Thus B has the asserted properties, and $f = 1 - B$ satisfies the requirements of the theorem.

LEMMA 3. *Let X be a compact subset of K, W a neighborhood of X, and h a continuous function on K with support in X such that $\|h\| \leqq 1$. Let $\eta > 0$.*

Then there exists $f \in H^\infty_{F \cup (K \backslash \bar{F})}$ such that $|f - h| < \eta$ on F, $\|f\| \leqq 3$, and $|f| < \eta$ on $K \backslash W$.

Proof. Choose $0 < \delta < d(X, K \backslash W)$ so small that $|h(x) - h(y)| < \eta/8$ whenever $x, y \in K$, $|x - y| < \delta$. We can easily find an integer $N > 0$, compact sets $X_1 \cdots X_r$ contained in X, and open sets $W_1 \cdots W_r$, with diameters $< \delta$, with $W_i \supseteqq X_i$, $W_i \subseteqq W$, such that

(a) if $x \in X$ and N_x is the number of integers i in $\{1, \cdots, r\}$ for which $x \in X_i$, $|N_x - N| < \eta N/8$

(b) if $x \in \boldsymbol{C}^n$ the number of integers i for which $x \in W_i \backslash X_i$ is less than ηN.

Let $F_i = F \cap X_i$. For $i = 1, 2 \cdots r$ we can find by Lemma 2 functions $f_i \in H^\infty_{F_i \cup (K \backslash \bar{F}_i)}$ with $f_i = 1$ on F_i, $\|f_i\| \leqq 2$ and $|f_i| < \eta/3r$ on $K \backslash W_i$.

Choose $x_i \in X_i$ for each i and put $f(z) = 1/N \sum_{i=1}^r f_i(z)h(x_i)$, $z \in F \cup (K\backslash\bar{F})$. Clearly $f \in H^\infty_{F\cup(K\backslash\bar{F})}$ and $\|f\| \leqq 3$ by (a). If $z \in K\backslash W$ then $|f_i(z)| < \eta/r$ for each i so $|f(z)| < \eta$.

Finally let $z \in F$. Then

$$\begin{aligned} f(z) &= \frac{1}{N}\Big(\sum_{z\in X_i} + \sum_{z\in W_i\backslash X_i} + \sum_{z\notin W_i}\Big) f_i(z)h(x_i) \\ &= f_1(z) + f_2(z) + f_3(z), \text{ say.} \end{aligned}$$

We have

$$\begin{aligned} |f_1(z) - h(z)| &\leqq \left|\frac{1}{N}\sum_{z\in X_i} f_i(z)(h(z) - h(x_i))\right| \\ &\quad + \left|1 - \frac{N_z}{N}\right| < \frac{\eta N_z}{8N} + \left|1 - \frac{N_z}{N}\right| < \eta/3 , \end{aligned}$$

by (a), since $|z - x_i| < \delta$. Moreover, $|f_2(z)| < \sum_{i=1}^r \eta/3$, by (b) and $|f_3(z)| < \sum_{i=1}^r \eta/3r = \eta/3$, so that we have $|f(z) - h(z)| < \eta$ as required.

LEMMA 4. *With F as in the theorem, if W is any open neighborhood of F and h a bounded continuous function on W with $\|h\| \leqq 1$, we can find $G \in H_{F\cup(K\backslash\bar{F})}$ with $|G - h| < \eta$ on F, $\|G\| \leqq 7$, and $|G| < \eta$ outside W.*

Proof. Choose a sequence $\{W_n\}$ of relatively compact open subsets of W with $W = \bigcup_{n=1}^\infty W_n$, such that $\bar{W}_m \cap \bar{W}_n = \emptyset$ if $|m - n| > 1$. We can write $h = \sum_{n=1}^\infty h_n$ on W where $h_n \in C(K)$ has support in W_n and $\|h_n\| \leqq 1$. By Lemma 3 for each n we can find $f_n \in H^\infty_{F\cup(K\backslash\bar{F})}$ with $|f_n - h_n| < 2^{-n}\eta$ on F, $|f_n| < 2^{-n}\eta$ on $K\backslash W$, and $\|f_n\| \leqq 3$. Then $G = \sum_{n=1}^\infty f_n$ has the required properties.

Proof of Theorem 1. By Lemma 4 and using the fact that g can be approximated uniformly by functions continuous in a neighborhood of F, we can construct by induction on n a sequence $\{G_n\}_{n=0}^\infty$ in $H^\infty_{F\cup(K\backslash\bar{F})}$ such that, writing $f_n = G_0 + \cdots + G_n$ we have:

$$(1) \qquad |G_0 - g| < \lambda/7 \text{ on } F ,$$

$$(1)_n \qquad |G_n + f_{n-1} - (1 + \lambda + \cdots + \lambda^n)g| < \frac{\lambda^{n+1}}{7}$$

on F, $n > 1$, where $\lambda = 9/10$

$$(2)_n \qquad \|G_n\| \leqq 7\|f_{n-1} - (1 + \lambda + \cdots + \lambda^n)g\|_F < 8\lambda^n$$

$$(3)_n \qquad \|f_n\| \leqq 1 + \lambda + \cdots + 9\lambda^n .$$

(To get $(3)_n$ observe that by $(1)_{n-1}$ we have $|f_{n-1}| < 1 + \lambda + \cdots + \lambda^{n-1} + \lambda^n/7$ on F, and hence on a neighborhood of F; if we make $|G_n| < \lambda^{n-1}/10$ outside this neighborhood then $(3)_n$ follows from $(2)_n$ and $(3)_{n-1}$).

$$(4)_n \qquad |G_n| < 2^{-n} \text{ on } K \backslash V .$$

Then $(2)_n$ shows that $f_n \to G$ say uniformly on K, so $G \in H^\infty_{F \cup (K \backslash \bar{F})}$; by $(1)_n$ $G = 10g$ on F and by $(3)_n$ $\|G\| \leqq 10$. Finally by $(4)_n |G| < \eta$ on $K \backslash V$. Then $f = (1/10)G$ is the required function.

REMARK. The metric condition on F in Theorem 1 is clearly satisfied if F has zero one-dimensional Hausdorff measure; however it is also satisfied by sets which are thicker in the direction of the complex tangent space, e.g. any smooth arc in ∂K whose tangent at each point lies in the complex tangent space.

If F is compact then of course it is a peak interpolation set, so Theorem 1 extends the Rudin-Carleson theorem. The extension to non-closed sets in the case of the disc has been obtained independently by Detraz [1], and subsequently generalized to other domains in the plane by A. Stray (private communication).

If we assume that F has zero one-dimensional Hausdorff measure then we can make do with a weaker pseudoconvexity hypothesis at the points of F. We say that ∂K is point pseudoconvex at ζ if there exists a neighborhood N of ζ and a real C^2 strictly plurisubharmonic function ρ in N such that $\rho(\zeta) = 0$ and $\rho(z) \leqq 0$ in $N \cap K$.

THEOREM 2. *Let K be holomorphically convex, and let F be a subset of ∂K with zero one-dimensional Hausdorff outer measure such that ∂K is point pseudoconvex at each point of F. Let V be a neighborhood of F in K, let $\eta > 0$, and let g be a bounded continuous function on F with $\|g\| \leqq 1$.*

Then we can find $f \in H^\infty_{F \cup (K \backslash \bar{F})}$ with $f | F = g$, $\|f\| \leqq 1$ and $|f| < \eta$ on $K \backslash V$.

Proof. We show that the conclusion of Lemma 2 holds; the rest of the proof is just as before. As in the proof of Lemma 2 for each $\zeta \in F$ we can find positive constants m_ζ and M_ζ, a neighborhood W_ζ of ζ, and $G_\zeta \in H(K)$ such that

(a) $m_\zeta |\zeta - z|^2 \leqq \operatorname{Re} G_\zeta(z), \qquad z \in K$

(b) $|G_\zeta(z)| \leqq M_\zeta |\zeta - z|, \qquad z \in K .$

Then whenever $\zeta \in U \subset W_\zeta$ and $z \in U$ we have $|G_\zeta(z)| \leqq M_\zeta \operatorname{diam}(U)$. We define F_n as before and cover F_n by balls $\Delta_i^{(n)}$ such that $\sum_i \operatorname{diam}(\Delta_i^n) < \varepsilon n^{-2} 2^{-n-2}$. The rest of the proof goes just as before, with c_j replaced by diam (Δ_j).

COROLLARY. *Let F be a compact subset of ∂K with zero 1-dimensional Hausdorff measure and assume ∂K is point pseudoconvex at each point of F. Then F is a peak interpolation set for $A(K)$.*

Finally we remark that the functions obtained in Theorem 1 and 2 are actually pointwise limits on K^0 of bounded sequences in $H(K)$; this follows from the construction. If F is compact the peak-interpolating functions constructed are in $\bar{H}(K)$; in this case the proof simplifies somewhat since it is only necessary to take finite products in Lemma 2 and the theorem follows from Lemma 2 by general theorems on peak interpolation sets.

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UNIVERSITY OF CALIFORNIA, LOS ANGELES