PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 1, 1972

WATTS COHOMOLOGY AND SEPARABILITY

ANDREW T. KITCHEN

A cohomology theory, $H_K^p A$, for commutative K-algebras, A, is discussed for the case where K is a field. This was originally introduced by C. E. Watts in connection with rings of continuous functions. N. Greenleaf computed $H_K^p A$ in the case where A is an extension field of K. In this paper it is shown that, for any K-algebra A, the separable closure of K in A can be identified with $H_K^0 A$. Furthermore Greenleaf's result is extended to a substantial class of local algebras.

1. Let K be a field and A a commutative K-algebra with unit element 1. In [4] Watts described a cochain complex $C_K A$, based on the additive Amitsur complex $F_K A$ [3]. He showed that in the case where $K = \boldsymbol{R}$ and $A = C(X)$, the ring of continuous real valued functions on the compact Hausdorff space X, the cohomology of this complex is naturally isomorphic to the real Čech cohomology of X. At the other extreme Greenleaf in [2] proved the following result. If L is an arbitrary extension field of K then $C_K L$ is naturally isomorphic to $F_{L_s} L$, where L_s is the separable closure of K in L. Thus the homology of $C_K L$ is trivial, except in dimension zero where $H^0(C_K L) \cong L_s$.

In this paper we investigate further the part separability plays in this theory. Letting A_s be the separable closure of K in A (see §2) and writing $H_K^p A$ for $H^p(C_K A)$, we prove the following results.

THEOREM 1. *If A is an arbitrary K-algebra then $H_K^0 A = A_s$.*

THEOREM 2. *Let A be a (not necessarily Noetherian) local K-algebra with unique maximal ideal, m. Suppose the image of A_s, under the canonical map of A onto A/m, is separably closed in A/m; then $C_K A$ is naturally isomorphic to $F_{A_s} A$.*

From Theorem 2 it follows that, for such an algebra, $H_K^p A = 0$ for $p > 0$.

At the end of the paper we mention some interesting classes of local algebras which satisfy the hypothesis of Theorem 2.

2. The complex $F_K A$ is the additive Amitsur complex [3, §4] with a dimension shift of 1: $F_K^p A$ is the $p + 1$ — fold tensor product of A over K, and the coboundary map $d^p: F_K^p A \to F_K^{p+1} A$ is given by $d^p(f_0 \otimes \cdots \otimes f_p) = \sum_{i=0}^{p+1} (-1)^i f_0 \otimes \cdots \otimes f_{i-1} \otimes 1 \otimes f_i \otimes \cdots \otimes f_p$.

PROPOSITION 1. *The complex $F_K A$ has zero homology, except in dimension zero where $H^0(F_K A) \cong K$.*

Proof. See [3, Lemma 4.1].

Let μ_p: $F_K^p A \to A$ by $\mu_p(f_0 \otimes \cdots \otimes f_p) = f_0 \cdots f_p$. The subcomplex $N_K A$ is defined as follows

$$N_K^p A = \{f \in F_K^p A \mid \exists g \in F_K^p A \text{ with } \mu_p g \text{ a unit and } fg = 0\} ,$$

this is easily seen to be equivalent to Watts' definition. The Watts cohomology is then the homology of the complex $C_K A = F_K A / N_K A$.

An element $f \in A$ is said to be *separable over* K if there exists a polynomial $p \in K[X]$, such that $p(f) = 0$ and $p'(f)$ is a unit in A. The *separable closure*, A_s, of K in A is the set of elements of A which are separable over K, it is a subalgebra of A (see §3, Corollary to Theorem 1).

3. From the definition of $C_K A$ it is clear that we can consider $H_K^0 A$ to be embedded in A.

PROPOSITION 2. *If A is an arbitrary K-algebra then $A_s \subset H_K^0 A$.*

Proof. If $f \in A_s$, let $p = a_n X^n + \cdots + a_0 \in K[X]$ be such that $p(f) = 0$ and $p'(f)$ is a unit. Define

$$h_{k-1} = \sum_{i=1}^{k} f^{i-1} \otimes f^{k-i} \in F_K^1 A .$$

Then $\mu_1 h_{k-1} = k f^{k-1}$. If $g = a_n h_{n-1} + \cdots + a_1 h_0$, then $(1 \otimes f - f \otimes 1)g = 0$ and $\mu_1 g = p'(f)$. Thus $d^1 f \in N_K^1 A$, so $f \in H_K^0 A$.

LEMMA. *If R is the Jacobson radical of A then $R \cap H_K^0 A = 0$.*

Proof. If $f \in H_K^0 A$, then there exists $g \in F_K^1 A$ such that $\mu_1 g$ is a unit and $(1 \otimes f - f \otimes 1)\, g = 0$. Suppose f is also in R. It then follows that, for each maximal ideal m, the image of f under the natural map φ: $A \to A/m (= L)$ is zero. Thus $(1 \otimes f)g' = 0$ in $L \otimes_K A$, where g' is the image of g under the map $\varphi \otimes 1$: $A \otimes_K A \to L \otimes_K A$. Now g' can be written $g' = \sum \lambda_i \otimes g_i$, where the elements $\lambda_i \in L$ are linearly independent over K. It then follows that $f g_i = 0$ for all i. As $\mu_1 g$ is a unit a simple argument shows that, for some i, $\varphi g_i \neq 0$. So, for each maximal ideal m, there exists g_m such that $g_m \notin m$ and $f g_m = 0$. Therefore $\operatorname{Ann}(f) = A$ and $f = 0$.

Proof of Theorem 1. If $f \in H_K^0 A$ then there exists $g = \sum_{i=1}^n g_i \otimes h_i$

$\in F_K^1 A$ such that $\sum g_i \otimes h_i f = \sum g_i f \otimes h_i$ and $\mu_1 g$ is a unit. In fact

$$\sum g_i \otimes h_i f^k = \sum g_i f^k \otimes h_i$$

for $k = 0, 1, 2, \cdots$. We can assume that $g_1, \cdots, g_n$ are linearly independent over K, in which case $h_i f^k$ is in the K-module spanned by $h_1, \cdots, h_n$. It follows that there exists a polynomial $q_i \in K[X]$ such that $h_i q_i(f) = 0$. Hence, because $\mu_1 g$ is a unit, $q(f) = q_1(f) \cdots q_n(f) = 0$. Thus f is algebraic over K.

For each maximal ideal m, the image of f under $\varphi\colon A \to A/m (= L)$ is in $H_K^0 L$. Hence, by Greenleaf's result [2], there exists an irreducible polynomial $p_m \in K[X]$ such that $p_m(\varphi f) = 0$ and $p_m'(\varphi f) \neq 0$. Now φf satisfies q, so p_m divides q and there are, therefore, only a finite number of distinct p_m. Let $p_1, \cdots, p_r$ be those distinct polynomials and let $p = p_1 \cdots p_r$. Clearly $p(f) \in R \cap H_K^0 A$ so $p(f) = 0$. A simple argument shows that $p'(f)$ is a unit. Thus $H_K^0 A \subset A_s$.

REMARK. The proof shows that $H_K^0 A$, and thus A_s, can be described as follows: $f \in H_K^0 A$ if and only if there exist distinct irreducible separable polynomials $p_1, \cdots, p_r \in K[X]$ such that $p_1(f) \cdots p_r(f) = 0$.

COROLLARY. *The separable closure, A_s, of K in A is a K-algebra. Furthermore if A is a local algebra then A_s is a field extension of K.*

Proof. By Theorem 1 we can identify A_s with $H_K^0 A$. The first part of the result can then by proved easily once we observe the identity

$$1 \otimes fg - fg \otimes 1 = (1 \otimes f - f \otimes 1)(1 \otimes g) + (f \otimes 1)(1 \otimes g - g \otimes 1) .$$

If A is local and f is a nonzero element of $H_K^0 A$, then the minimal polynomial of f, constructed in the proof of Theorem 1, is clearly irreducible over K. Thus the subalgebra $K[f]$ of $H_K^0 A$ is a field, and so $f^{-1} \in H_K^0 A$. Therefore $H_K^0 A$ is a field.

4. The following proposition is proved in [2].

PROPOSITION 3. *If L is a separable (algebraic) extension field of K then $N_K^p L = \ker \mu_p$.*

Using an inductive argument based on Proposition 2, we can in fact remove the restriction that L be a field.

PROPOSITION 4. *If the field L is separable over K and A is an L-algebra, then the natural map $\theta\colon F_K A \to F_L A$ induces an isomorphism, $C_K A \cong C_L A$.*

Proof. The induced map is certainly a surjection. On the other hand, by Proposition 3, the sequence

$$0 \longrightarrow N_K^p L \longrightarrow F_K^p L \longrightarrow L \longrightarrow 0$$

is exact. Applying the exact functor $F_K^p A \otimes_B (\ \ )$, where $B = F_K^p L$, we obtain the exact sequence

$$0 \longrightarrow F_K^p A \otimes_B N_K^p L \longrightarrow F_K^p A \longrightarrow F_K^p A \otimes_B L \longrightarrow 0 .$$

However in $F_K^p A \otimes_B L$

$$a_0 \otimes \cdots \otimes \lambda a_i \otimes \cdots \otimes a_p \otimes 1 = a_0 \otimes \cdots \otimes a_i \otimes \cdots \otimes a_p \otimes \lambda .$$

So the map of $F_K^p A \otimes_B L$ onto $F_L^p A$, induced by taking $a_0 \otimes \cdots \otimes a_p \otimes \lambda$ to $a_0 \otimes \cdots \otimes \lambda a_p$, is an isomorphism. The composition of this map with $1 \otimes \mu_p$ is θ_p, and the kernel of θ_p is thus the image of $F_K^p A \otimes_B N_K^p L$ in $F_K^p A$. It follows therefore that $\ker \theta_p \subset N_K^p A$. Suppose $f \in F_K^p A$ with $\theta_p f \in N_L^p A$; then there exists $g \in F_K^p A$ such that $\mu_p g$ is a unit and $fg \in \ker \theta_p$. So there exists $h \in F_K^p A$, such that $\mu_p h$ is a unit and $fgh = 0$. Since $\mu_p hg = (\mu_p h)(\mu_p g)$ is a unit, $f \in N_K^p A$. This completes the proof.

A ring in which every zero divisor is nilpotent we will call a *ZDN* ring.

PROPOSITION 5. *Let A and A' be K-algebras which are ZDN rings, and let N be the ideal of nilpotents of A. Suppose K is separably closed in the field of quotients of A/N, then $A \otimes_K A'$ is a ZDN ring.*

Proof. If B is a subring of A then it is a *ZDN* ring, with ideal of nilpotents $N \cap B$. The domain $B/(N \cap B)$ embeds in A/N, so K is separably closed in the quotient field of $B/(N \cap B)$. We can therefore restrict ourselves to a finitely generated subalgebra of A, and so assume that A is Noetherian. Let L be the quotient field of A/N, then $(A/N) \otimes_K A' \subset L \otimes_K A'$. So by [2, Proposition 3] $(A/N) \otimes_K A'$ is a *ZDN* ring and hence $N \otimes_K A'$ is primary. However (0) is a primary ideal of A with associated prime N. Thus it follows, putting $E = A$ and $F = B = A \otimes_K A'$ in [1, Chapter IV, §2.6, Theorem 2], that the associated primes of (0) in $A \otimes_K A'$ are also the associated primes of $N \otimes_K A'$. Hence (0) is a primary in $A \otimes_K A'$ also, and so $A \otimes_K A'$ is a *ZDN* ring.

Note that if A is a local ring (A has a unique maximal ideal m)

and n is a positive integer, then A/m^n is a ZDN ring.

PROPOSITION 6. *Let A be a Noetherian local K-algebra; then the natural map of $F_K^p A$ into the projective limit (inverse limit) of the system $\{F_K^p(A/m^n)\}_n$ is an injection.*

Proof. As A is Noetherian, $\bigcap_{n=1}^{\infty} m^n = 0$, and so $A \to \text{proj lim}_n (A/m^n)$ is an injection. The proof can be completed by induction on p, using the following lemma, the demonstration of which is straightforward.

LEMMA. *If $\{M_i, f_{ji}\}$ and $\{N_i, g_{ji}\}$ are projective systems of K-modules (K a field) indexed over the same directed set, and if M and N are the projective limits of these systems, then the natural map of $M \otimes_K N$ into* proj $\lim_i (M_i \otimes_K N_i)$ *is an injection.*

PROPOSITION 7. *Let A be a local K-algebra and let K be separably closed in A/m. If z is a zero divisor in $F_K^p A$ then $\mu_p z \in m$, and hence $N_K^p A = 0$.*

Proof. Suppose z is a zero divisor in $F_K^p A$; then there exists $w \neq 0$ such that $zw = 0$. Choose a finitely generated subalgebra, B, of A such that w and z are in $F_K^p B$. The ideal $B \cap m$ is prime in B. So, localizing B at $B \cap m$, we get a local Noetherian subalgebra B' of A, such that $B' \cap m$ is the maximal ideal of B', and z and w are elements of $F_K^p(B')$. We can therefore assume that A is Noetherian. By Proposition 6, there exists n such that the image of w in $F_K^p(A/m^n)$ is nonzero. Thus z', the image of z, is a zero divisor in $F_K^p(A/m^n)$. However K is separably closed in A/m and so, by induction from Proposition 5, we see that $F_K^p(A/m^n)$ is a ZDN ring. The image z' is thus nilpotent and the same is true of $\mu_p z' \in A/m^n$. As the image of $\mu_p z$ in A/m^n is $\mu^p z'$, it follows that $\mu_p z \in m$.

Proof of Theorem 2. As A_s is a field we can apply Proposition 4 to get $C_K A \cong C_{A_s} A$. However Proposition 7 shows that $C_{A_s} A = F_{A_s} A$. This completes the proof.

The following corollary to Theorem 2 is immediate on applying Proposition 1.

COROLLARY. *If A satisfies the hypotheses of Theorem* 2, *then $H_K^p A = 0$ for $p > 0$.*

Clearly any local algebra over a separably closed field (i.e. separably closed in its algebraic closure) satisfies the hypotheses of Theorem 2.

If A is a complete Noetherian local K-algebra, there exists [5, Chapter VIII, §12, Theorem 27] a subfield L of A which is mapped onto A/m by the natural map. Under these circumstances A_s is mapped isomorphically onto the separable closure of K in A/m. Thus it follows that, for such an algebra also, the hypotheses of Theorem 2 are satisfied.

Our ultimate goal is to prove the conclusion of Theorem 2 for all local K-algebras; then, loosely speaking, to study this cohomology theory for an arbitrary K-algebra by using sheaf theoretic methods to patch the algebra together from its localizations (at prime or maximal ideals). Partial results in this direction have been obtained.

REFERENCES

1. N. Bourbaki, *Algèbra Commutative*, Chapter IV, Hermann, Paris, 1961.
2. N. Greenleaf, *Watts cohomology of field extensions*, Proc. Amer. Math. Soc., **21** (1969), 208-210.
3. A. Rosenberg and D. Zelinski, *On Amitsur's complex*, Trans. Amer. Math. Soc., **97** (1960), 327-356.
4. C. Watts, *Alexander-Spanier cohomology and rings of continuous functions*, Proc. Nat. Acad. Sci. U.S.A., **54** (1965), 1027-1028.
5. O. Zariski and P. Samuel, *Commutative Algebra*, Vol. II, D. Van Nostrand, Princeton, N.J., 1960.

Received May 21, 1971 and in revised form November 23, 1971.

INSTITUTO DE MATEMÁTICA
UNIVERSIDADE FEDERAL DO RIO DE JANEIRO
CAIXA POSTAL 1835 ZC-00
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