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TOPOLOGIES ON STRUCTURE SPACES OF LATTICE GROUPS

S. J. BERNAU

A structure space of a lattice group G is, coventionally, a set of prime subgroups of G with the hull-kernel topology. The set of all prime subgroups of G, together with G when G has no strong unit, carries a natural topology, stronger than the hull-kernel topology, which is compact and Hausdorff. There is a natural closed subspace which is a quotient of the Stone space of the complete Boolean algebra of polar subgroups. Under the hull-kernel topology this subspace is a retract of the space of prime subgroups, but no longer closed. These topologies are compared, with particular reference to coincidences.

Consideration of structure spaces for lattice groups is not new. Nakano [15], for complete vector lattices and Amemiya [1] for arbitrary vector lattices were the first to treat the question systematically. Indeed the germ of our compactness proof is already in [1] (Theorem 2.1). More recently Isbell [11] and Isbell and Morse [12] have introduced other structure spaces to solve a specific problem in the theory of f-rings.

1. Prime subgroups.

We recall some definitions. Let G be a lattice group written additively, but not necessarily commutative. A subgroup K of G is *solid* if $x \in K$ and $|y| \leqq x$ imply $y \in K$; K is *prime* if K is solid, proper and $x \wedge y = 0$ implies $x \in K$ or $y \in K$ (normality of K is *not* required). Properties of prime subgroups are listed in many places [6, 7, 9, 13]. We record some which will be used later. The solid subgroups containing prime K from a chain under set inclusion and are prime. If K is prime K contains a minimal prime subgroup.

If S is a nonempty subset of G, $S^{\perp} = \{y \in G: |x| \wedge |y| = 0$ for all $x \in S\}$. We have $S \subset S^{\perp\perp}$, $S \subset T$ gives $S^{\perp} \supset T^{\perp}$, $S^{\perp} = S^{\perp\perp\perp}$ and $S^{\perp}$ is a solid subgroup of G. Subgroups M of G such that $M = M^{\perp\perp}$ are called *polar subgroups*. Under set inclusion and $^{\perp}$ for complementation the set of polar subgroups is a complete Boolean algebra [16, 2]. A prime subgroup K is minimal if and only if for each $x \in G$ exactly one of $x^{\perp\perp}$ and $x^{\perp}$ is a subset of K, [9].

A subgroup K of G is a *z-subgroup* if $x^{\perp\perp} \subset K$ for each $x \in K$. This definition makes K solid and is equivalent to the definition given by Bigard [4]. A z-subgroup of G which is prime will be called a *z-prime subgroup* of G.

Let $P(G)$, $Z(G)$ denote the set of all prime subgroups and all z-prime subgroups of G respectively. Recall that a *weak unit* of G is an element e such that $e > 0$ and $e^{\perp\perp} = G$; e is a *strong unit* if $e > 0$ and for each $x \in G$ there is an integer n such that $x \leqq ne$. There are three possibilities for G, (i) G has no weak unit, (ii) G has a weak unit but no strong unit (iii) G has a strong unit. We define $P^*(G) = P(G)$ in case (iii) and $P^*(G) = P(G) \cup \{G\}$ otherwise; we define $Z^*(G) = Z(G) \cup \{G\}$ in case (i) and $Z^*(G) = Z(G)$ otherwise; finally we define $Z^{**}(G) = Z^*(G)$ in case (iii) and $Z^{**}(G) = Z(G) \cup \{G\}$ otherwise. We have equality of $Z^*(G)$ and $Z^{**}(G)$ except in case (ii) and always $Z^*(G) \subset Z^{**}(G) \subset P^*(G)$. For each $K \in P^*(G)$ there is a unique z-subgroup $\zeta(K)$ generated by K. The map $\zeta \colon P^*(G) \to Z^{**}(G)$ is (algebraically) a retraction with the inclusion map as coretraction. We state the following without proof.

LEMMA 1.1. (i) *If $K \in P^*(G)$ and M is a polar subgroup of G then $M \subset K$ or $M^\perp \subset K$.* (ii) *If K is a minimal prime subgroup of G then $K \in Z(G)$.*

By a *filter* $\mathscr{F}$ in G we mean a nonempty subset of

$$G^+ = \{x \in G \colon x \geqq 0\}$$

such that $x \in \mathscr{F}$ and $y \geqq x$ imply $y \in \mathscr{F}$ and $x, y \in \mathscr{F}$ imply $x \wedge y \in \mathscr{F}$. An *ultrafilter* in G is a maximal filter. Note that K is minimal prime if and only if $G^+ \sim K$ is an ultrafilter [9].

Our next result is well known for prime subgroups.

LEMMA 1.2. *Let $\mathscr{F}$ be a filter in G and K a z-subgroup of G such that $\mathscr{F} \cap K = \varnothing$. If L is a z-subgroup of G maximal with respect to $K \subset L$ and $\mathscr{F} \cap L = \varnothing$, then L is z-prime.*

Proof. Suppose $x \wedge y = 0$, $x \notin L$, $y \notin L$. By maximality there exist $u, v \in L^+$ such that $x^{\perp\perp} \vee u^{\perp\perp}$ and $y^{\perp\perp} \vee v^{\perp\perp}$ meet $\mathscr{F}$ (commutativity not used here. Thus $\mathscr{F}$ meets $(x^{\perp\perp} \vee (u+v)^{\perp\perp}) \wedge (y^{\perp\perp} \vee (u+v)^{\perp\perp}) = (x^{\perp\perp} \wedge y^{\perp\perp}) \vee (u+v)^{\perp\perp} = (u+v)^{\perp\perp} \subset L$. This is impossible so L is z-prime.

DEFINITION. For $a \in G$,

$$U_a = \{\theta \in P^*(G) \colon a \notin \theta\}$$
$$V_a = \{\theta \in P^*(G) \colon a \in \theta\} = P^*(G) \sim U_a .$$

The next Lemma summarizes some well-known and easy properties of these sets.

LEMMA 1.3. (i) *If* $a, b \in G^+$, $U_a \cap U_b = U_{a\wedge b}$ *and* $U_a \cup U_b = U_{a+b} = U_{a\vee b}$. (ii) $U_a \subset U_b$ *if and only if there is an integer* n *such that* $|a| \leqq n|b|$. (iii) $U_a \subset V_b$ *if and only if* $b \in a^\perp$.

Proof. For (i) see [1, 13]. For (ii) and (iii) use the fact that a maximal solid subgroup which omits $b \in G$ is prime [7] (and called a value of b).

If we write $\bar{U}_a = U_a \cap Z^*(G)$ and $\bar{V}_a = V_a \cap Z^*(G)$, we have the following results.

LEMMA 1.4. (i) $\bar{U}_a \subset \bar{U}_b$ *if and only if* $a^{\perp\perp} \subset b^{\perp\perp}$. (ii) $\bar{U}_a \subset \bar{V}_b$ *iff* $b^{\perp\perp} \subset a^\perp$. (iii) $\bar{U}_x = \bar{U}_a \cap \bar{V}_b$ *if and only if* $x^{\perp\perp} = a^{\perp\perp} \cap b^\perp$.

Proof. (i) and (ii) are immediate, Lemma 1.2 is needed.

For (iii) assume $x, a, b \in G^+$ and $x^{\perp\perp} = a^{\perp\perp} \cap b^\perp$, we have [2] $a^{\perp\perp} = (a^{\perp\perp} \cap b^{\perp\perp}) \vee (a^{\perp\perp} \cap b^\perp) = (a \wedge b)^{\perp\perp} \vee x^{\perp\perp} = (a \wedge b + x)^{\perp\perp}$. Hence, using Lemma 1.3 we have $\bar{U}_a = \bar{U}_{a\wedge b} \cup \bar{U}_x = (\bar{U}_a \cap \bar{U}_b) \cup U_x$. It follows that $\bar{U}_a \cap \bar{V}_b = \bar{U}_x$.

Conversely suppose $\bar{U}_x = \bar{U}_a \cap \bar{V}_b (x, a, b \in G^+)$, then $\bar{U}_a = \bar{U}_{a\wedge b} \cup \bar{U}_x$ and $a^{\perp\perp} = (a \wedge b)^{\perp\perp} \vee x^{\perp\perp}$. It follows that $x^{\perp\perp} \supset a^{\perp\perp} \cap b^\perp$ and the reverse inclusion is a consequence of (i) and (ii).

2. Topologies. Until now the only topology that has been studied on $P(G)$ is the hull-kernel topology. Since this is well known [1, 13, 17] and readily extended to $P^*(G)$ we will take most properties of the hull-kernel topology of $P^*(G)$ for granted.

Write $P^*(G)_h$ for the topological space $P^*(G)$ with the hull-kernel topology. A base for $P^*(G)_h$ is $\{U_a : a \in G\} \cup \{P^*(G)\}$. Note that this topology is not T_0 if $G \in P^*(G)$. If $G \notin P^*(G)$, $P^*(G)_h$ is a T_0 space but not usually Hausdorff (see Theorem 3.3 to follow). In either case $P^*(G)_h$ is a compact space and each U_a is compact [1, 17].

At this point the lattice structure of G plays a vital role. As has been pointed out, if $K \in P^*(G)$, $G^+ \sim K$ is a filter (possibly empty) which is a prime lattice ideal in the dual ordering of $G^+ (x, y \in G^+$ and $x \vee y \in G^+ \sim K$ imply $x \in G^+ \sim K$ or $y \in G^+ \sim K)$. Thus there is a natural, dual hull-kernel topology on $P^*(G)$ giving a second topological space $P^*(G)^h$. It is easy to check that $\{V_a : a \in G\}$ is a base for this topology and that $P^*(G)^h$ is always a T_0-space. It will follow (Corollary 2.2) that $P^*(G)^h$ is again compact.

DEFINITION. The *strong topology* on $P^*(G)$ is the supremum of the hull-kernel and dual hull-kernel topologies. $P^*(G)_s$ denotes $P^*(G)$

with its strong topology.

A subbase for $P^*(G)_S$ is the set $\{U_a: a \in G\} \cup \{V_a: a \in G\}$ and $P^*(G)_S$ is a Hausdorff space. Our main result is the compactness of $P^*(G)$. This is close to the surface in Theorem 2.1 of [1].

THEOREM 2.1. *The space $P^*(G)_S$ is compact.*

Proof. By Alexander's sub-base theorem [14, p. 139] it is sufficient to show that any covering by sets from the sub-base has a finite sub-cover. If $G \in P^*(G)$ then $G \notin U_a$ for any $a \in G$. Thus any cover by sub-basic open sets must contain a set V_a for some $a \in G$. If $G \notin P^*(G)$ then $P^*(G) = U_a$ for some strong unit a. Hence it is sufficient to prove that each U_a is compact.

Choose $a \in G^+$ and let $\mathscr{C} = \{U_x: x \in X\} \cup \{V_y: y \in Y\}$ be a cover of U_a which has no finite subcover. Assume, as we may that $X \cup Y \subset G^+$. If $x_1, \cdots, x_n \in X$ then $U_a \not\subset U_{x_1} \cup \cdots \cup U_{x_n} = U_{x_1+\cdots+x_n}$; thus a is not in the solid subgroup K of G which is generated by X. If $y_1, \cdots, y_m \in Y$ then $U_a \not\subset V_{y_1} \cup \cdots \cup V_{y_m} = V_{y_1\wedge\cdots\wedge y_m}$ (Lemma 1.3 (i) and take complements). Hence, Lemma 1.3, $y_1 \wedge \cdots \wedge y_m \notin a^\perp$ and $a \wedge y_1 \wedge \cdots \wedge y_m \neq 0$ for any $y_1, \cdots, y_m \in Y$. Thus $\{a\} \cup Y$ generates a filter $\mathscr{F}$ in G^+.

Suppose $K \cap \mathscr{F} \neq \varnothing$, then there exist $x_1, \cdots, x_n \in X$ and $y_1, \cdots, y_m \in Y$ such that

$$a \wedge y_1 \wedge \cdots \wedge y_m \leqq x_1 + \cdots + x_n .$$

Let $\theta \in U_a$. If $a \wedge y_1 \wedge \cdots \wedge y_m \in \theta$ then, since θ is prime and $a \notin \theta$ some $y_i \in \theta$ and $\theta \in V_{y_i}$. If $a \wedge y_1 \wedge \cdots \wedge y_m \notin \theta$, then $\theta \in U_{a\wedge y_1\wedge\cdots\wedge y_m} \subset U_{x_1+\cdots+x_n} = U_{x_1} \cup \cdots \cup U_{x_n}$. Thus $\{V_{y_1},\ldots, V_{y_m}\} \cup \{U_{x_1},\ldots, U_{x_n}\}$ covers U_a contrary to hypothesis.

It follows that $K \cap \mathscr{F} = \varnothing$ so by the 'non-z' version of Lemma 1.2 there is a prime subgroup ψ of G such that $K \subset \psi$ and $\psi \cap \mathscr{F} = \varnothing$. Since $X \subset K \subset \psi$, $\psi \notin U_x (x \in X)$. Since $\{a\} \cup Y \subset \mathscr{F} \subset G \sim \psi$, $\psi \notin V_y (y \in Y)$ and $\psi \in U_a$. Thus $\mathscr{C}$ does not cover U_a, a contradiction. Compactness of $P^*(G)_S$ is established.

COROLLARY 2.2. *The spaces $P^*(G)_h$ and $P^*(G)^h$ are both compact.*

Now we consider the subspaces $Z(G)$, $Z^*(G)$ and $Z^{**}(G)$ of $P^*(G)$ and use subscripts h, S and superscripts h, to denote the subspace topologies inherited from $P^*(G)_h$, $P^*(G)_S$ and $P^*(G)^h$.

PROPOSITION 2.2. *The space $Z^{**}(G)^h$ is a topological retract of $P^*(G)^h$. The space $Z^{**}(G)_S$ is a closed subspace of $P^*(G)_S$ and $Z^*(G)_S$ is an open and closed subspace of $Z^{**}(G)_S$.*

Proof. We have $\zeta^{-1}(V_a \cap Z^{**}(G)) = \cup\{V_x: a \in x^{\perp\perp}\}$ so $\zeta: P^*(G)^h \rightarrow Z^{**}(G)^h$ is continuous. If $\theta \notin Z^*(G)$ and $\theta \in P^*(G)$ there exist $a, b \in G$ such that $a \in \theta$ and $b \in a^{\perp\perp} \sim \theta$. Then $\theta \in V_a \cap U_b$ and $V_a \cap U_b \cap Z^*(G) = \emptyset$. Finally, if $Z^{**}(G) \neq Z^*(G)$ then G has a weak unit e, but no strong unit. Thus $Z^{**}(G) \sim Z^*(G) = \{G\} = V_e \cap Z^{**}(G)$ which is open and closed in $Z^{**}(G)_S$.

We now consider the Stone representation space E of the complete Boolean algebra $\mathfrak{B}$ of polar subgroups of G. We consider E as the set of maximal ideals of $\mathfrak{B}$ with the hull-kernel topology. A base of open sets is the collection $\{\{t \in E: M \notin t\}: M \in \mathfrak{B}\}$ and each member of the base is also compact (and hence closed). Define a map $\xi: E \rightarrow Z^*(G)_S$ by

$$\xi(t) = \{x \in G: x^{\perp\perp} \in t\} \quad (t \in E)\ .$$

THEOREM 2.3. *The map $\xi: E \rightarrow Z^*(G)_S$ is surjective and the topology of $Z^*(G)_S$ is the quotient topology.*

Proof. Surjectivity is easy to check. We have

$$\xi^{-1}\bar{U}_a = \{t \in E: a^{\perp\perp} \notin t\}$$

so $\xi^{-1}\bar{U}_a$ is a compact open subset of E. The same is true of $\xi^{-1}\bar{V}_a$. Hence ξ is continuous. Since E is compact the quotient topology is compact. It is finer than the Hausdorff topology $Z^*(G)_S$, (because ξ is continuous) and the result follows.

Theorem 2.3 suggests that $Z^*(G)_S$ is a natural structure space of G. We shall see later that this is not true in the functorial sense (Proposition 5.2 *et seq.*)

THEOREM 2.4. *The following are equivalent.*
(1) *E and $Z^*(G)_S$ are homeomorphic.*
(2) *$\xi: E \rightarrow Z^*(G)_S$ is injective.*
(3) *$\mathfrak{B}$ is generated, as a Boolean algebra, by the set $\{a^{\perp\perp}: a \in G\}$.*
(4) *The subalgebra of $\mathfrak{B}$ generated by $\{a^{\perp\perp}: a \in G\}$ is complete.*

Proof. The equivalence of (1) and (2) is immediate from Theorem 2.3. The equivalence of (3) and (4) is an algebraic triviality.

(1) implies (3). Let $M \in \mathfrak{B}$ and define $U = \cup\{\bar{U}_x: x \in M\} \subset Z^*(G)$ and $V = \cup\{\bar{U}_x: x \in M^\perp\} \subset Z^*(G)$. The sets U, V are disjoint and open. Since E and $Z^*(G)_S$ are homeomorphic, $Z^*(G)_S$ is a Stone space and $\mathrm{cl}\, U$, $\mathrm{cl}\, V$ are disjoint and open. Since E is the Stone representation space of $\mathfrak{B}$ and, in $\mathfrak{B}$, $M = \vee\{x^{\perp\perp}: x \in M\}$ we have

$$\xi^{-1}\,\mathrm{cl}\, U = \{t \in E: M \notin t\} \quad \text{and} \quad \xi^{-1}\,\mathrm{cl}\, V = \{t \in E: M^\perp \notin t\}\ .$$

By compactness the compact open set cl U is a finite union of sets of the form $\bar{U}_a$, $\bar{V}_b$ and $\bar{U}_f \cap \bar{V}_g$, each of which is disjoint from V (and cl V). Now $x \in M^\perp$ if and only if U_x is disjoint from each $\bar{U}_a$, $\bar{V}_b$, $\bar{U}_f \cap \bar{V}_g$ and by Lemma 1.4 this happens if and only if $x^{\perp\perp} \subset a^\perp$ for each $\bar{U}_a$, $x^{\perp\perp} \subset b^{\perp\perp}$ for each $\bar{V}_b$ and $x^{\perp\perp} \subset (f^{\perp\perp} \cap g^\perp)^\perp$ for each $\bar{U}_f \cap \bar{V}_g$. It follows that M is the (finite) supremum of the $a^{\perp\perp}$, $b^\perp$ and $f^{\perp\perp} \cap g^\perp$. This demonstrates (3).

(3) implies (2). Let $s, t \in E$, $s \neq t$. Then there exists $M \in \mathfrak{B}$ such that $M \in s$ and $M^\perp \in t$. By (3), there exist finite subsets A, B, of G and C of $G \times G$ such that $0 \notin A$, B contains no weak unit and $M = \bigvee\{a^{\perp\perp}: a \in A\} \vee \bigvee\{b^\perp: b \in B\} \vee \bigvee\{f^{\perp\perp} \cap g^\perp: (f, g) \in C\}$.

Since $M \in s$ and $M \notin t$ all the $a^{\perp\perp}$, $b^\perp$ and $f^{\perp\perp} \cap g^\perp$ are in s and at least one is different from $\{0\}$ and G and is not in t. Suppose this is $f^{\perp\perp} \cap g^\perp$. Then $f^{\perp\perp} \notin t$, $g^\perp \notin t$ and, because s is prime either $f^{\perp\perp} \in s$ or $g^\perp \in s$. If $f^{\perp\perp} \in s$ we have $f \in \xi(s) \sim \xi(t)$ and if $g^\perp \in s$ we have $g \in \xi(t) \sim \xi(s)$. The cases when an $a^{\perp\perp}$ or a $b^\perp$ is in $s \sim t$ are handled similarly. This proves (2).

3. Comparison of topologies. The possibility that the topologies on $P^*(G)$ or $Z^*(G)$ coincide leads to some interesting results. We also recover some results of Speed [17].

THEOREM 3.1. *The following are equivalent.*

(1) *$Z(G)_h$ is a Hausdorff space.*

(2) *The identity map $Z(G)_h \to Z(G)^h$ is continuous.*

(3) *If $\theta \in Z(G)$ then θ is a minimal prime subgroup of G.*

(4) *For any $a, b \in G$ there exists $x \in G$ such that $a^{\perp\perp} \cap b^\perp = x^{\perp\perp}$.*

(*The equivalence of* (1), (3), (4) *is the essential content of Theorem* 3.2 *of* [17].)

Proof. (2) imples (1). By (2) $Z(G)_h = Z(G)_S$ is Hausdorff.

(1) implies (4). Let $W = \bar{U}_a \cap \bar{V}_b$. Then $W \subset Z(G)$ and W is a compact open subset of $Z(G)_S = Z(G)_h$. It follows that there exist $x_1, \cdots, x_n \in G^+$ such that $W = \bar{U}_{x_1} \cup \cdots \cup \bar{U}_{x_n} = \bar{U}_{x_1+\cdots+x_n}$ (compactness and the known base for the topology of $Z(G)_h$). Put $x = x_1 + \cdots + x_n$, then $x^{\perp\perp} = a^{\perp\perp} \cap b^\perp$ by Lemma 1.3.

(4) implies (3). Let $\theta \in Z(G)$. Take $b \in \theta \cap G^+$, $a \notin \theta$ (possible since $\theta \neq G$). Let $x \in G^+$ satisfy $x^{\perp\perp} = a^{\perp\perp} \cap b^\perp$ (using (4)). Then $(b+x)^{\perp\perp} = b^{\perp\perp} \vee x^{\perp\perp} = b^{\perp\perp} \vee x^{\perp\perp} = b^{\perp\perp} \vee (a^{\perp\perp} \cap b^\perp) \supset a^{\perp\perp}$. Because $a \notin \theta$, $x \notin \theta$. Thus if $b \in \theta$, $b^\perp \not\subset \theta$ and θ is a minimal prime subgroup, [8, Theorem 5.1].

(3) implies (2). If $b \in G$ and $\theta \in V_b$ then, because θ is minimal there exists $a \in b^\perp \sim \theta$. Then $\theta \in U_a \subset V_b$ and V_b is an open subset of $Z(G)_h$.

COROLLARY 3.2. *Theorem* 3.1 *is valid with* $Z^*(G)$ *replacing* $Z(G)$ *and* (4) *replaced by*
(4)′ *If* $a \in G, a^{\perp} = b^{\perp\perp}$ *for some* $b \in G$.

Note that condition (4) of Theorem 3.1 is satisfied if G is projectable (with $x = a - [b]a$) and, *a fortiori*, if G is conditionally σ-complete. The stronger condition (4)′ of Corollary 3.2 is satisfied if G is laterally complete or orthocomplete [8, 3] or if G is projectable and has a weak unit.

THEOREM 3.3. *The following are equivalent.*
(1) $P(G)_h$ *is Hausdorff.*
(2) *The identity map* $P(G)_h \rightarrow P(G)^h$ *is continuous.*
(3) *If* $\theta \in P(G)$, θ *is a minimal prime subgroup of* G.
(4) G *is hyper-archimedean.*

Proof. If (3) holds (2) follows from Theorem 3.1. (2) gives $P(G)_h = P(G)_S$ and (1) follows. If (1) holds and $\theta \in P(G)$ suppose $\psi \in P(G)$ and $\psi \subset \theta$. Since $\psi \in U_a$ whenever $\theta \in U_a$ and $P(G)_h$ is Hausdorff $\psi = \theta$ and θ is minimal. Finally the equivalence of (3) and (4) is given in [4, Théorème 6.1] or in [5]. (Either of these references contains the definition of hyperarchimedean, which we do not give here.)

THEOREM 3.4. *The following are equivalent.*
(1) $Z(G)^h$ *is Hausdorff.*
(2) $Z^*(G)^h = Z^*(G)_h$.
(3) *If* $a \in G$ *there exists* $b \in G$ *such that* $a^{\perp} = b^{\perp\perp}$
(*This result essentially contains Theorem* 3.1 *of* [15].)

Proof. (1) implies (3). Since $\bar{V}_x \cap \bar{V}_y = \bar{V}_{x+y}$ for $x, y \in G^+$, and $\bar{V}_x = \varnothing$ if and only if x is a weak unit of G, we deduce from (1) that G has a weak unit, u say. Thus $Z(G) = Z^*(G)$ and from (1) because $Z^*(G)^h = Z(G)_S$, $\bar{U}_a$ is a compact open subset of $Z^*(G))^h(a \in G)$. Hence there exist $y_1, \cdots, y_n \in G^+$ such that

$$\bar{U}_a = \bar{V}_{y_1} \cup \cdots \cup \bar{V}_{y_n} = \bar{V}_{y_1 \wedge \cdots \wedge y_n} .$$

Putting $b = y_1 \wedge \cdots \wedge y_n$ we have $b^{\perp\perp} = a^{\perp}$ by Lemma 1.3.

(3) implies (2). If $b^{\perp\perp} = a^{\perp}$ then $\bar{V}_a = \bar{U}_b$. Thus (3) gives

$$\{\bar{U}_a : a \in G\} = \{V_b : b \in G\}$$

and (2) follows.

(2) implies (1) is trivial.

COROLLARY 3.5. *If* G *has a weak unit the hull-kernel or dual hull-kernel topology on* $Z(G) = Z^*(G)$ *is Hausdorff if and only if both* *are equal.*

EXAMPLE 3.6. Let G be the cardinal sum of countably many copies of the integers. It is easy to verify condtion (4) of Theorem 3.1. However G has no weak unit so $Z^*(G)^h$ is not Hausdorff.

THEOREM 3.7. *$P(G)^h$ is Hausdorff if and only if every element of $P^*(G)$ is minimal.*

Proof. This theorem is really a corollary of Corollary 3.5.

4. **Other structure spaces.** First consider the uniform structure space $\beta(G)$ of G [11]. For the construction of $\beta(G)$ we refer to [11]. It is easy to verify that the proof of Theorem 2 of [11] extends *verbatim* to prove.

THEOREM 4.1 *The set of polar subgroups contained in any proper prime subgroup θ of G contains exactly one maximal uniform ideal $u(\theta)$; and the maps $u: P(G)_h \to \beta(G)$ and its restriction to $Z(G)_h$ are continuous onto a dense subset.*

Isbell mentions in [11] that $\beta(G)$ is coarse. Just how coarse can be seen from the following. For G the group of real continuous functions of compact support on locally compact Y, $\beta(G)$ is the Stone-Čech compactification of Y [11, p. 63]. If $G^* = Z \times G$ (G as above) with lexicographic ordering (Z first) then G^* has a strong unit, no pair of polar subgroups (I, J) of G^* is supplementary unless $\{I, J\} = \{G, \{0\}\}$. Thus $\{\{0\}\}$ is the only uniform ideal of the Boolean algebra of polar subgroups and $\beta(G^*)$ reduces to a single point.

We next consider the structure space $\kappa(G)$ introduced by Isbell and Morse [12]. This space is, in our notation, the quotient space of $P(G)_h$ by the smallest equivalence relation which identifies points of $P(G)_h$ whose closures intersect. Adapting parts of [12, p. 304] we see that the map $u: P(G)_h \to \beta(G)$ factors naturally as $u = ts$ with s the quotient map $s: P(G)_h \to \kappa(G)$ and t defined in the obvious way. All these maps are continuous.

5. **Homomorphisms.** The functorial character of our spaces $P^*(G)$ *and* $Z^*(G)$ with their various topologies is easy to establish.

PROPOSITION 5.1. *Let $T: G_1 \to G_2$ be a lattice group homomorphism and define $T^*\theta = T^{-1}(\theta)$ ($\theta \in P^*(G_2)$). Then T^* maps into $P^*(G_1)$ and is continuous for any of the hull-kernel, dual hull-kernel or strong topologies of $P^*(G_2)$ to the corresponding topology of $P^*(G_1)$.*

Proof. Verification of this is immediate.

The space $Z^*(G)$ needs a little more.

PROPOSITION 5.2. *Let $T: G_1 \rightarrow G_2$ be a lattice group homomorphism of lattice groups and define $T^\# = \zeta T^*|_{Z^*(G_2)}$, then $T^\#$ is continuous from $Z^*(G_2)^h$ to $Z^*(G_1)^h$.*

Proof. This is obvious using Propositions 5.1 and 2.2.

We leave open the problem of finding reasonable sufficient conditions for $T^\#$ to be continuous for the dual hull-kernel or strong topologies. Some uninteresting special cases can be constructed out of the coincidences of the topologies using Theorem 3.1 and 3.4.

6. **Applications to $C(X)$.** Suppose X is a completely regular Hausdorff space and $C(X)$ is the lattice group of real continuous functions on X. It is natural to ask for the relationship between $P^*(C(X))$ and X. In general we can say very little, $P^*(G)$ is always totally disconnected while, if $G = C(X)$, X can be as topologically simple as a unit interval. (We exclude the trivial situation when X is finite.) We shall consider two cases, the nice one when X is compact and a more extreme case when X is a P-space (definition later).

In connection with archimedian lattice groups, which always have representations as sub lattice groups of $D(X)$ (the lattice of extended real-valued continuous functions which are finit on a dense open set) we can characterise $P^*(G)$ and $Z^*(G)$ by properties of *all* such representations. Details of this will appear in another paper.

EXAMPLE 6.1. $G = C(X)$ with X compact Hausdorff. In this case $G \notin Z^*(G)$. For $f \in G$ we will write $z(f) = \{t \in X: f(t) = 0\}$ (the usual notation for the zero set of f is $Z(f)$ [10] but it seems desirable not to use Z with two different meanings here). If $\theta \in P(G)$ define $z(\theta) = \bigcap \{z(f): f \in \theta\}$. For each $\theta \in P(G)$, $z(\theta)$ is a singleton. To see this let K be a closed, and hence compact subset of X. If, for each $t \in K$, some element of θ is nonzero at t, a routine compactness argument produces a positive $f \in \theta$ such that f is bounded away from zero on K. If this were the case for X we would have $\theta = C(X)$ contrary to hypothesis. This proves $z(\theta)$ nonempty. If $s, t \in X$ and $s \neq t$, we can find $f, g \in C(X)$ such that $f \wedge g = 0$, $s \notin z(f)$ and $t \notin z(g)$. Since θ is prime one of f or g is in θ and s and t are not both in $z(\theta)$. we thus have a map $z: P(G) \rightarrow X$ which is clearly surjective. In fact z is continuous on $P(G)_h$ and on $P(G)^h$, and *a fortiori* on $P(G)_S$. For continuity on $P(G)_h$ suppose U is a neighborhood of $z(\theta)$, then there exists $f \in C(X)$ such that $f(\theta) = 1$ and $\sigma(f) \subset U$ ($\sigma(f)$ is the support of f). Clearly $\theta \in U_f$ and for any $t \notin \sigma(f)$ there is a $g \in C(X)$ such that $g(t) = 1$ and $\sigma(g) \cap \sigma(f) = \varnothing$. Since $|f| \wedge |g| = 0$ we have $g \in \psi(\psi \in U_f)$ and hence $t \notin z(\psi)$ $(\psi \in U_f)$. It follows that $z(\psi) \in U$ $(\psi \in U_f)$. Continuity of z on $P(G)^h$ is easier. Take U an open neighborhood of $z(\theta)$ and put $K = X \sim U$. As we have seen, since K is compact, there exists $g \in \theta$ such that $K \cap z(g) = \varnothing$. Then $\theta \in V_g \subset U$.

It is illuminating to compare this example with Theorem 4.1. By [11, p. 63] $\beta(C(X)) = X$ for compact Hausdorff X and Isbell's map u coincides with our z. Even so we have a stronger result, with less effort, than Theorem 4.1 will yield directly.

We also note that if X is a Stone space then z is injective on $Z(G) = Z^*(C(X))$. A good way to see this to check that if $\theta \in Z(G)$, then $\theta = \{f \in G: z(\theta) \notin \sigma(f)\}$. Under these conditions we also have $Z(G)_S$ a retract of $P(G)_S$. This is trivial and indeed trivial in any case when $Z^*(G)_h$ is Hausdorff.

EXAMPLE 6.2. $G = C(X)$ with X a P-space. P-spaces are treated in [10, 4J]. They are spaces in which every zero-set is open. Since this ensures that the characteristic function of every zero-set (and of every cozero-set) is in G, condition (4) of Corollary 3.2 is satisfied and every z-prime subgroup of G is minimal. It is easy now to check that if $\theta \in Z(G)$ the set $\{z(f): f \in \theta\}$ is a z-filter on X [10, Chapter 2]. Further each such filter is prime. Suppose $u, v \in C(X)$ and $z(u) \cup z(v) = z(w)$ for some $w \in \theta$. Then

$$z(|u| \wedge |v|) = z(w) \text{ and } |u| \wedge |v| = \bigvee\{n|w| \wedge |u| \wedge |v|: n = 1, 2, \cdots\}.$$

Hence $|u| \wedge |v| \in w^{\perp\perp} \subset \theta$. Because θ is prime u or v is in θ and we are done. Conversely if F is a prime z-filter on X the set $\{f \in C(X): z(f) \in F\}$ is a z-prime subgroup of G different from G itself. (This last because X is a P-space and $\phi \notin F$). Since every prime z-filter on X is contained in a z-ultrafilter which in turn is prime, and since all z-prime subgroups of $C(X)$ are minimal it follows that the prime z-filters on X are precisely the z-ultrafilters on X. Further we have just exhibited a natural bijection between $Z^*(G) = Z(G)$ and this set of z-ultrafilters. By [10, Chapter 6] the set of z-ultrafilters on X is the Stone-Čech compactification, $\beta(X)$, of X. Let us denote our bijection $Z(G) \to \beta(X)$ by z. We check that z is continuous when $Z(G)$ has its (one) natural topology. By [10, p. 87] a base for the closed sets of $\beta(X)$ is the set of sets $\bar{F} = \{p \in \beta(X): F \in p\}$ where F is a zero set in X. For each such F, if f denotes the characteristic function of F we have $f \in G$ and $z^{-1}(\bar{F}) = V_f$. This gives continuity of z and hence $Z^*(C(X))$ and $\beta(X)$ are naturally homeomorphic.

Final comment 6.3. For F-spaces X, as we just saw, the space $Z^*(C(X))_S$ is a very natural object. Of course every discrete space is a P-space and every P-space is basically disconnected, which makes the class of P-spaces look rather small. However, there are P-spaces with no isolated points [10, 13P].

Viewing 6.1 and 6.2 a little differently we see that for any completely regular X we can concoct two compact spaces $P(C^*(X))$ and $Z((C^*(X))$ (C^* denoting bunded continuous functions) and continuous surjections $P(C^*(X))_S \to \beta(X)$ and $Z(C^*(X))_S \to \beta(X)$. For $Z(C^*(X))$, at least, we can realise the map as relating some prime z-filters on X to z-prime subgroups of $C^*(X)$. This correspondence needs further clarification.

Another way to approach this is to compare $C(X)$, or $C^*(X)$, with $CP(C(X))$. In the case of compact X the natural surjection $z: P(C(X)) \to X$ described in §6.1 is not an injection. Consequently it gives rise to a proper embedding of $C(X)$ into $CP(C(X))$. A second natural candidate for comparison with $CP(C(X))$ is the second dual $C^{**}(X)$ of $C(X)$ since this is also properly larger than $C(X)$. The referee has raised the possibility of eqality of $C^{**}(X)$ and $CP(C(X))$ (and is even willing to conjecture it). Besides the Banach space duals we may also consider one of the possible order duals, relax the requirement that X be compact and ask similar questions.

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UNIVERSITY OF TEXAS