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THE DIOPHANTINE PROBLEM $Y^2 - X^3 = A$ IN A POLYNOMIAL RING

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Let $C[z]$ be the ring of polynomials in z with complex coefficients; we consider the equation $Y^2 - X^3 = A$, with $A \in C[z]$ given, and seek solutions of this with $X, Y \in C[z]$ i.e. we treat the equation as a "polynomial diophantine" problem. We show that when A is of degree 5 or 6 and has no multiple roots, then there are exactly 240 solutions (X, Y) to the problem with $\deg X \leqq 2$ and $\deg Y \leqq 3$.

It is possible that, A being of degree 6, solutions (X, Y) exist with $\deg X > 2$ or $\deg Y > 3$. We "normalize" the problem so as to remove these from our consideration, and give the following definitions: if A is any polynomial of degree d, we shall permit its *formal degree* to be any integer *divisible by* 6 and greater or equal to d. Given A of formal degree $6k$, we require the solutions X, Y of the equation to be of formal degrees $2k, 3k$ resp., i.e. $\deg X \leqq 2k$, $\deg Y \leqq 3k$. This problem will be called the *problem of order* k. The restriction on the degrees of X, Y causes no loss in generality, for if k is chosen large enough, it will exceed $1/2 \deg X$ and $1/3 \deg Y$. Furthermore, the classification by k has a natural geometric interpretation. We confine our attention to the problem of order 1. The order restriction enables us to projectivize the equation to an equation of degree $6k$, with $\deg A = 6k$, $\deg X = 2k$, $\deg Y = 3k$.

Suppose then that A has formal degree 6, and (X, Y) is a solution of proper formal degree, $\deg X \leqq 2$, $\deg Y \leqq 3$. The projective curve $K: w^3 - 3Xw + 2Y = 0$ has the z-discriminant $Y^2 - X^3 = A$, so the function $z: K \to S^2$ (proj. line) has its branches among the roots of A, for finite z. At $z = \infty$ we introduce $\tilde{z} = 1/z$, $\tilde{w} = w/z = \tilde{z}w$ and get

$$\tilde{z}^3 w^3 - 3\tilde{z}^3 X\left(\frac{1}{\tilde{z}}\right) w + 2\tilde{z}^3 Y\left(\frac{1}{\tilde{z}}\right) = 0 :$$

If $X = a_0 z^2 + \cdots$, $Y = b_0 z^3 + \cdots$, then

$$F = \tilde{w}^3 - 3(a_0 + a_1\tilde{z} + a_2\tilde{z}^2)\tilde{w} + 2(b_0 + b_1\tilde{z} + \cdots) = 0$$

and

$$\frac{\partial F}{\partial \tilde{w}} 3\tilde{w}^2 - 3(a_0 + \cdots) .$$

Now at $\tilde{z} = 0$ (i.e. $z = \infty$) z has a branch point if and only if $\partial F/\partial \tilde{w} = 0$;

i.e. we must have

$$\tilde{w}^3 - 3a_0\tilde{w} + 2b_0 = 0$$

and

$$3\tilde{w}^2 - 3a_0 = 0$$

which is true if and only if $\Delta = -a_0^3 + b_0^2 = 0$ i.e. if and only if $\deg A < 6$. Hence if $\deg A < 6$, we put a "formal root" of A at ∞ with multiplicity 6-$\deg A$.

We now assume the roots of A to be *distinct*. This entails $\deg A = 5$ or 6, with no multiple (finite) roots. The roots will be called $z_1, \cdots, z_6$. Note that if either X or Y were zero at z_ι, the other would also be, since A is zero there (for the case $z_\iota = \infty$ just imagine the projective form of $Y^2 - X^3 = A$; the statement then reads that $\deg A < 6$ and if $\deg Y < 3$ then $\deg X < 2$ and conversely). Hence A would have at least a *double* zero at z_ι, (or at ∞: $\deg A \leqq 4$) contrary to hypothesis. Hence $X, Y \neq 0$ at z_ι, and $\deg X = 2$ or $\deg Y = 3$. Away from a branch point we may write locally:

$$w_0 = \sqrt[3]{-Y + \sqrt{A}} + \sqrt[3]{-Y - \sqrt{A}}$$

$$w_1 = \omega\sqrt[3]{-Y + \sqrt{A}} + \omega^2\sqrt[3]{-Y - \sqrt{A}}$$

$$w_2 = \omega^2\sqrt[3]{-Y + \sqrt{A}} + \omega\sqrt[3]{-Y - \sqrt{A}}$$

for proper choice of the roots; as we go around $z_\iota, \sqrt{A}$ changes to $-\sqrt{A}$, and we get a root permutation $w_0 \leftrightarrow w_0$, $w_1 \leftrightarrow w_2$. Thus the branching number b_ι at z_ι is 1, and the total branching is 6, so the genus is $g = b/2 - r + 1 = 1$, i.e. K is a torus.

We should also prove that K is irreducible; but if K were reducible, factoring as $(w - \alpha)(w^2 + \alpha w + \beta)$ (where α, β are polynomials in z by Gauss's lemma) i.e., we have $3X = \alpha^2 - \beta$ and $2Y = -\alpha\beta$, and $A = Y^2 - X^3 = 4\beta^3 + 15\alpha^2\beta^2 + 12\alpha^4\beta - 4\alpha^6 = -(\alpha^2 - 4\beta)(2\alpha^2 + \beta)^2$. It is easy to see that $\deg \alpha \leqq 1$, $\deg \beta \leqq 2$, and hence $\deg(\alpha^2 - 4\beta) \leqq 2$. Since $\deg A \geqq 5$ we see that $\deg(2\alpha^2 + \beta) \geqq 1$, whence A has double roots, contrary to hypothesis.

Thus, any solution X, Y gives us an elliptic curve K represented as a 3-sheeted branched covering of S^2 with branch points at z_ι, where $z\colon K \to S^2$ is an elliptic function of degree 3. Furthermore, w is also a function on K, and its poles are among those of z, and of order $\leqq$ the order of the z-poles: for expanding w_ι at $z = \infty$ we get

$$w_\iota = \omega^\iota\sqrt[3]{-b_0z^3 + \cdots + \sqrt{(b_0^2 - a_0^3)z^6 + \cdots}} + \omega^{2\iota}\sqrt[3]{\text{etc.}}$$

i.e.

$$w_\iota = \left(\omega^\iota \sqrt[3]{-b_0 + \sqrt{\varDelta}} + \omega^{2\iota} \sqrt[3]{-b_0 - \sqrt{\varDelta}}\right)z + \text{lower powers of } z$$

i.e. the order of w is $\leqq$ order of z at all places $z = \infty$. (Clearly w has no other poles). Note also that the sum Σw_ι of the three values of w over any z is zero.

Now suppose conversely that we are given a branched covering of S^2 with 6 simple branch points at the roots of A; we then have an elliptic curve K and a meromorphic function $z: K \to S^2$ with 3 poles (one of which is double if a branch point is at ∞) at places k_1, k_2, k_3. Now the set of meromorphic functions w on K whose poles are among the k_ι form a vector space V of dimension 3. Given any such w, the sum $w_0 + w_1 + w_2$ of its 3 values over any z gives us a function which is:

(1) finite for finite z

(2) of order $\leqq$ the order of z at $z = \infty$

(3) symmetric in the sheets, so rational in z.

Hence Σw_ι must be *linear* in z: $\Sigma w_\iota = a_w z + b_w$, where a_w and b_w are constants depending on w. Note that a_w and b_w are clearly *complex-linear* in w, i.e. a, b: $V \to C$ are linear maps. Furthermore, since both $w = 1$ and $w = z$ are in V we have a and b are linearly independent: for

$$a(1) = 0 \qquad a(z) = 3$$

$$b(1) = 3 \qquad b(z) = 0$$

and so $a_w = 0$, $b_w = 0$ defines a one dimensional subspace of V i.e. a $w \neq 0$, defined up to a constant multiple, of degree $\leqq 3$, with its poles among those of z, and with $\Sigma w_\iota = 0$. Hence w satisfies some equation

$$w^3 - 3Pw + 2Q = 0, \text{ with } P \;\&\; Q \text{ rational in } z\,;$$

but

$$-3P = w_1w_2 + w_2w_3 + w_3w_1 \text{ is \textit{finite} for } z \text{ finite}\,;$$

hence P is a polynomial; also its degree is $\leqq 2$ since the order of w_ι is $\leqq$ that of z at ∞. Likewise Q is a polynomial of degree $\leqq 3$ in z. Finally w is not rational in z since if it were, it would actually be linear, $w = az + b$, and then

$$\Sigma w_\iota = 3w = 3az + 3b = 0, \quad \text{i.e.} \quad w \equiv 0\,.$$

Hence $w^3 - 3Pw + 2Q = 0$ is irreducible, and thus *defines* the curve K. Because of this, we must have the branch points as roots of the discriminant $Q^2 - P^3$ ($\neq 0$); i.e. $A \mid Q^2 - P^3$; $\deg Q^2 - P^3 \leqq 6$, and is <6 if and only if as we have seen previously, ∞ is a branch point of K; in the latter case we also have $\deg A = 5$, and so in every case we have $\deg (Q^2 - P^3) = \deg A$, i.e. $A = k(Q^2 - P^3)$ for some constant $k \neq 0$. If now we replace w by $w/\alpha (\alpha \in \boldsymbol{C})$, we replace P by P/α^2 and Q by Q/α^3 and $Q^2 - P^3$ by $(Q^2 - P^3)/\alpha^6$; Hence we may choose a scale factor α, determined up to a 6th root of unity, and a rescaled w such that $Q^2 - P^3 = A$, i.e. (P, Q) is a solution. Thus we have shown that any 3 sheeted covering of S^2 with simple branches at $A = 0$ gives us exactly 6 solutions to the problem (These 6 solutions are distinct since two could be equal if and only if P or $Q \equiv 0$, which is impossible). Furthermore, if we have two different such branched coverings K_1, K_2, then the corresponding solutions (P_1, Q_1), (P_2, Q_2) must be distinct, since the data (P_ι, Q_ι) actually *define* K.

Thus the only remaining problem is to enumerate the different coverings possible.

We choose a base point $q \in S^2$, distinct from the roots z_ι, and loops p_ι ($\iota = 1, \cdots, 6$) encircling the roots z_ι acting as free generators of the fundamental group $\pi_1(S^2 - \bigcup_j z_j)$, subject only to the relation $p_1 \cdots p_6 =$ identity. Choosing a numbering 1, 2, 3 of the sheets over q, each p_ι determines a permutation π_ι (in S_3) of the sheets, and these completely determine the surface. Since the branches are all simple, these permutations must be *transpositions*: (12), (23) or (31). Also not all the π_ι can be equal, for then two sheets over q would remain unconnected from the third. If we choose $\pi_1, \cdots \pi_5$ arbitrarily then π_6 is determined by $\pi_1 \pi_2 \cdots \pi_6 = e$. Note however that $\pi_1, \cdots \pi_5$ may not be chosen all equal, since π_6 would also be same by virtue of the relation. Hence we may choose $\pi_1, \cdots \pi_5$ in $3^5 - 3$ ways, obtaining all possible coverings of the required nature. Two such choices π_ι, π'_ι give the same covering if and only if they differ by a renumbering of the sheets over q, i.e. if and only if $\pi'_\iota = g \pi_\iota g^{-1}$ for some $g \in S_3$. Since at least two different transpositions occur among the π_ι, conjugation by the elements of S_3 produces exactly 6 different equivalent choices of π_ι; hence the total number of different surfaces is $(3^5 - 3)/6 = (3^4 - 1)/2 = 40$. Remembering that to each such surface there are 6 solutions, we have:

THEOREM. *If A is a polynomial of degree* 5 *or* 6 *without multiple roots, then there are exactly* 240 *distinct solutions of the equation $Y^2 - X^3 = A$ in polynomials X, Y for which* $\deg X \leqq 2$, $\deg Y \leqq 3$.

It should be pointed out that, in principle at least, the determination of the solutions (X, Y) for a given A could be solved by classical elimination theory. For example, if $X = a_0 z^2 + a_1 z + a_2$ and

$Y = b_0z^3 + b_1z^2 + b_2z + b_3$ is a solution to $Y^2 - X^3 = A = \alpha_0z^6 + \cdots + \alpha_6$, then treating the a_ι and b_j as unknowns, formal manipulation and the equating of coefficients gives us 7 polynomial equations in 7 unknowns which presumably (assuming independence) gives a finite set of solutions for the unknowns a_ι, b_j. This also shows us that the a_ι and b_j are algebraic over the field of the α_k. In practice, however, this elimination would probably not be computationally feasible.

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