PACIFIC JOURNAL OF MATHEMATICS
Vol. 43, No. 3, 1972

SLICES, MULTIPLICITY, AND LEBESGUE AREA

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For a large class of k dimensional surfaces, S, it is shown that the Lebesgue area of S can be approximated by the integral of the $k-1$ area of a family, F, of $k-1$ dimensional surfaces that cover S. The family F is regarded as being composed of the slices of the surface S. In addition, a topological characterization of a certain multiplicity function is given. This multiplicity function when integrated with respect to k dimensional Hausdorff measure, yields the Lebesgue area of f.

Suppose X is a smooth compact k dimensional manifold and let $f: X \to E^n$ be a continuous map into Euclidean n-space, $k \leqq n$, which has finite Lebesgue area. Let $u: E^n \to E^1$ be a Lipshitz function with Lipschitz constant no greater than one. In [7], it was shown that if $k = 2$ or if the $k+1$ Hausdorff measure of $f(X)$ is zero, then the Lebesgue area of f, $\mathscr{L}(f)$, can be approximated by the integral of the $k-1$ area of f restricted to the boundary of $\{x: u \circ f(x) < t\}$, provided that the function u has been chosen appropriately. Of course, the important element of this problem is to give a reasonable interpretation to the concept of the $k-1$ area of f restricted to the boundary of our open set. In [7], this was expressed in terms of the theory developed by H. Federer [4]. It is the purpose of this paper to show that a definition given by J. Cecconi in [1] can be used to obtain results similar to those found in [7].

During the development of this paper, we were able to provide a topological characterization of the multiplicity function which was shown, in [4], to yield the Lebesgue area when integrated with respect to k dimensional Hausdorff measure. It turns out that this characterization is not needed to prove the main theorem of this paper, but we include its proof because of its independent interest.

2. Slices and Cecconi area.

In this section we will give a definition of the $k-1$ area of f restricted to the boundary of an open set. This definition is a slight modification of the one given by Cecconi in [1]. The modification is desirable since our domain is taken to be a smooth oriented compact k-manifold, X. Our development relies heavily on the work of Federer [4] and consequently the notation of that paper will be used here without change. Thus, a continuous map $f: X \to E^n$ has a monotone-light factorization $f = \ell_f \circ m_f$ where the light factor ℓ_f is defined on the middle space M_f. Moreover, if

$k = 2$ or $H^{k+1}[f(X)] = 0$, (H^{k+1} denotes $k+1$ dimensional Hausdorff measure) then there is current-valued measure μ_f defined on the Borel sets of M_f whose total variation $\|\mu_f\|$ is equal to $\mathscr{L}(f)$. If T is a current, we will denote by:

$$\begin{aligned} &M(T), \text{ the mass of } T \\ &F(T), \text{ the flat norm of } T \\ &\partial T, \text{ the boundary of } T . \end{aligned}$$

Finally, L_k will denote Lebesgue measure on E^k, and $B(x, r)$ will be the closed ball with center x and radius r.

DEFINITION 2.1. Let U be an open set in X. Then, *the* $k-1$ *area of f restricted to the boundary of U*, $C(f, U)$ is defined as follows. Let $\{\pi_i\}$ be a sequence of open subsets of U whose boundaries are smooth manifolds. Assume also that every compact subset of U is eventually in every π_i. Let f_i be a sequence of smooth maps defined on X that converge uniformly to f. Then

$$C(f, U) = \inf \{\liminf_{i\to\infty} \mathscr{L}(f_i \mid \operatorname{bdry} \pi_i)\}$$

where the infimum is taken over all $\{\pi_i\}$ and $\{f_i\}$ as described above. Here, $\mathscr{L}(f \mid \operatorname{bdry} \pi_i)$, is used to denote the Lebesgue area of f_i restricted to the boundary of π_i.

DEFINITION 2.2. Let $u\colon E^n \to E^1$ be a Lipschitz function. Then $C(f; u, t)$ is defined to be $C(f, U_t)$ where U_t is the open set

$$\{x\colon u \circ f(x) < t\} .$$

LEMMA 2.3. *Let $u_i\colon E^n \to E^1$ $i = 0, 1, 2, \cdots$ be a sequence of Lipschitzian maps such that $u_1 \geqq u_2 \geqq \cdots$ and $\lim_{i\to\infty} u_i = u_0$. Then*

$$C(f; u_0, t) \leqq \liminf_{i\to\infty} C(f; u_i, t)$$

for every $t \in E^1$.

Proof. For $t \in E^1$ observe that the sets $V_{i,t} = \{x\colon u_i \circ f(x) < t\}$ $i = 1, 2, \cdots$, are nested and that their union is equal to $V_{0,t}$. For each positive i, select a smooth map f_i and an open set $\pi_i \subset V_{i,t}$ with smooth boundary such that

(i) $|f_i(x) - f(x)| < i^{-1}$ for all $x \in X$

(ii) $|\mathscr{L}(f_i \mid \operatorname{bdry} \pi_i) - C(f; u_i, t))| < i^{-1}$

(iii) dist (closure π_i, $X - V_{i,t}) < i^{-1}$.

Now, the sequences $\{\pi_i\}$ and $\{f_i\}$ will be admissible in the definition of $C(f; u_0, t)$. Hence,

$$C(f;\,u_0,\,t) \leqq \liminf_{i\to\infty} \mathscr{L}(f_i\,|\,\text{bdry}\,\pi_i) = \liminf_{i\to\infty} C(f;\,u_i,\,t)\ .$$

The following theorem was proved in [1], but for completeness, we will exhibit a different and perhaps simpler proof here.

THEOREM 2.4. *Let $u\colon E^n \to E^1$ be a Lipschitzian map with Lipschitz constant $K > 0$. Then*

$$K\mathscr{L}(f) \geqq \int_{-\infty}^{\infty} C(f;\,u,\,t)dL_1(t)\ .$$

Proof. It is easy to see using the techniques of Lemma 2.3 that $C(f;u,t)$ is lower-semicontinuous in t and, hence, L_1 integrable. Now select a sequence of C^∞ maps $\{f_i\}$ such that f_i converge to f uniformly on X and such that $\mathscr{L}(f_i) \to \mathscr{L}(f)$. Choose a sequence of C^∞ maps $\{u_i\}$ decreasing to u with the Lipschitz constant of u_i less or equal to $K + i^{-1}$. Fixing i, then with $\sigma_i = \sup_{x\in X} |u_i \circ f_i(x) - u_i \circ f(x)|$, and with $g_m(x) = u_i \circ f_m(x) + \sigma_i$, $\{g_m\}$ converges uniformly to $u_i \circ f$ on X and each g_m is smooth and greater than $u_i \circ f$. Thus, for every t,

$$V_{m,t} = \{x\colon g_m(x) < t\} \subset V_t = \{x\colon u_i \circ f(x) < t\}$$

and for every compact subset K of V_t, $V_{m,t}$ contains K for m sufficiently large. In addition, for almost every t, $V_{m,t}$ is a C^∞ manifold so the pairs f_m and $V_{m,t}$ approximate f and V_t for almost every t and

$$\begin{aligned} C(f;\,u_i,\,t) &\leqq \varliminf_{m\to\infty} C(f_m,\,V_{m,t}) \\ &= \varliminf_{m\to\infty} C(f_m;\,u_i,\,t-\sigma_m)\ . \end{aligned} \tag{1}$$

However, it is immediate that for smooth functions, f_m, on open sets with smooth boundaries, $V_{m,t}$, that

$$C(f_m,\,V_{m,t}) \leqq \mathscr{L}(f\,|_{\text{bdry}\,V_{m,t}})\ . \tag{2}$$

From [3, Theorem 6.18] with $N(y,f)$ denoting the number of points in $f^{-1}(y)$ (possibly ∞) follows

$$\mathscr{L}(f_m\,|_{\text{bdry}\,V_{m,t}}) = \int_{u_i^{-1}(t-\sigma_m)} N(y,f_m)dH^{k-1}(y)\ . \tag{3}$$

Combining (1), (2), and (3) and using Fatou's lemma gives

$$\int_{-\infty}^{\infty} C(f;\,u_i,\,t)d\,L_i \leqq \varliminf_{m\to\infty} \int_{-\infty}^{\infty}\int_{P_t} N(y,f_m)dH^{k-1}(y)d\,L_1(t) \tag{4}$$

with $P_t = u_i^{-1}(t-\sigma_m)$. However, [5, Theorem 3.2.12] gives

$$(5)\qquad \int_{-\infty}^{\infty}\int_{P_t} N(y, f_m)dH^{k-1}(y)dL_1(t) = \int_{E_n} N(y, f_m)\,|\operatorname{grad} u_i \,|\, Q_m|\,d\,H^k$$

where Q_m is the image of x under f_m. As the Lipschitz constant of u_i dominates the gradient of $u_i \,|\, Q_m$, (4) and (5) give

$$\begin{aligned}\int_{-\infty}^{\infty} C(f;\, u_i,\, t)d\, L_1 &\leqq \varliminf_{m\to\infty}\, (K + i^{-1})\int_{E^n} N(y, f_m)d\, H^k \\ &= \varliminf_{m\to\infty}\, (K + i^{-1})\, \mathscr{L}(f_m) \\ &= (K + i^{-1})\, \mathscr{L}(f)\, .\end{aligned}$$

The result now follows from Lemma 2.3.

In [7] if was shown that with $\lambda(f;\, u,\, t) = \sum M\,[\partial\mu_f(V)]$, where the summation is taken over all components V of $\ell_f^{-1}(\{x\colon u(x) < t\})$, that an inequality holds which is similar to 2.2 where $C(f;\, u,\, t)$ is replaced by $\lambda(f;\, u,\, t)$. Moreover, it was also shown that if $k = 2$ or $H^{k+1}\,[f(X)] = 0$, then

$$\sup\left\{\int \lambda(f;\, u,\, t)dL_1(t)\right\} = \mathscr{L}(f)$$

where the supremum is taken over all Lipschitz functions $u\colon E^n \to E^1$ whose Lipschitz constants are no greater than one. We will show in Theorem 2.8 below that this result is valid with $\lambda(f;\, u,\, t)$ replaced by $C(f;\, u,\, t)$.

In the case $k = 2$, it was show that Cesari's definition of length [2, 20.2] also worked satisfactorily in this theory. In [1], Cecconi showed that $C\,(f;\, u,\, t)$ agreed with Cesari's definition. Thus, in Theorem 2.8, it will be only necessary to consider $k > 2$.

DEFINITION 2.5. Let X be a compact oriented k-manifold and suppose $f\colon X \to E^k$ is continuous. For each $z \in M_f$, let $\Delta(z,\, r)$ be the component of $\ell_f^{-1}\,[B(\ell_f(z),\, r)]$ that contains z. Consider the induced homomorphism on Čech cohomology groups,

$$H^k(E^k,\, E^k - B\,(\ell_f(z),\, r)) \xrightarrow{f^*} H^k(X,\, X - m_f^{-1}(\Delta(z,\, r)))\, .$$

We assume the generators of these groups chosen to agree with the orientations on X and E^k. Then, f^* maps a generator of one group onto a multiple of the second. Call this integer $d_f(z,\, r)$. Let

$$d_f(z) = \lim_{r\to 0+} d_f(z,\, r)$$

if this limit exists and is finite. If not, let $d_f(z) = \infty$.

DEFINITION 2.6. Let W be an open connected set in X and let

$f\colon X \to E^k$. Suppose $y \in E^k - f(\text{bdry } W)$, and choose $0 < r < 1$ so that $f(W) \subset B(y, r^{-1})$ and $f(\text{bdry } W) \cap B(y, r) = 0$. Then $d(f, W, y)$ is defined as in 2.5 when the following is considered:

$$H^k[B(y, r^{-1}), B(y, r^{-1}) - B(y, r)] \xrightarrow{f^*} H^k(W, \text{bdry } W) .$$

Observe, that

$$d(f, W, y) = \sum d_f(z) \tag{6}$$

where the summation is taken over all z in the set $\ell_f^{-1}(y) \cap m_f(W)$. This equation is valid if $y \in E^k - f(\text{bdry } W)$ and if each $d_f(z) < \infty$.

REMARK 2.7. Let $f\colon X \to E^n$ be a continuous map with

$$\mathscr{L}(f) < \infty .$$

Suppose that $p\colon E^n \to E^k$ is an orthogonal projection and consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "m_f"] \arrow[dr, "m_{p\circ f}"'] & M_f \arrow[r, "\ell_f"] \arrow[d, "h"] & E^n \arrow[d, "p"] \\ & M_{p\circ f} \arrow[r, "\ell_{p\circ f}"'] & E^k \end{tikzcd} .$$

It follows from [4, 3.8] that with $C_p = \{z\colon h^{-1}(z)$ is a non-degenerate continuum$\}$,

$$\|\mu_f\|(h^{-1}(C_p)) = 0 \quad \text{and} \quad L_k(\ell_{p\circ f}(C_p)) = 0 \tag{7}$$

for almost all $p\colon E^n \to E^k$. For such projections, it is easily seen that the current valued measure corresponding to $p \circ f$ is $h_\sharp(p_\sharp \circ \mu_f)$. Thus, it follows from [4, 2.1 and 4.1] that for any Borel set $E \subset M_f$,

$$h_\sharp(p_\sharp \circ \mu)\,[h(E)](w_k) = \int_{E^k} \sum_{A(y)} d_{p\circ f}(z)\, d\, L_k(y) \tag{8}$$

where $A(y) = \{z\colon z \in \ell_{p\circ f}^{-1}(y) \cap h(E)\}$ and where $w_k = p^\sharp(dx_1 \wedge \cdots \wedge dx_k)$. However, in view of (7), it follows that for L_k almost all $y \in E^k$,

$$\sum_{A(y)} d_{p\circ f}(z) = \sum_{B(y)} d_{p\circ f}\,[h(z)] \tag{9}$$

where $B(y) = \{z\colon z \in (p \circ \ell_f)^{-1}(y) \cap E\}$. Observe that if E is an open connected set, $W = m_f^{-1}(E)$, and if $L_k[p \circ f(\text{bdry } W)] = 0$, then (6) implies

$$d(p \circ f, W, y) = \sum_{A(y)} d_{p \circ f}(z) \tag{10}$$

for L_k almost all $y \in E^k$.

THEOREM 2.8. *Let $f: X \to E^n$ be a continuous map with finite Lebesgue area and let $k > 2$ with $H^{k+1}(f(X)) = 0$ then*

$$\sup_{u \in U} \int_{-\infty}^{\infty} C(t; f, u)\, d\mathscr{L}^1 = \mathscr{L}(f, X)$$

where U is the set of all real valued Lipschitzian maps on E^n with constant less or equal to one.

Proof. In [7], it was shown that for every $\varepsilon > 0$, there is a function $u: E^n \to E^1$ with Lipshitz constant one such that

$$\int_{-\infty}^{\infty} \lambda(f; u, t)\, dL_1(t) > \mathscr{L}(f) - \varepsilon\,. \tag{11}$$

The function u was obtained in the following manner: a certain family of closed disjoint n-balls, $B_i = B_i(y_i, r_i)$ with center y_i and radius r_i, was produced and u was defined by

$$u(x) = \sum_{i=1}^{\infty} u_i(x)$$

where $u_i(x) = -\operatorname{dist}(x, E^n - B_i)$. Thus,

$$\lambda(f; u, t) = \sum_{i=1}^{\infty} \lambda(f; u_i, t),\ \ 0 < t < \infty\ ,$$

and the same equality holds with λ replaced by C. At each point y_i there is a k-dimensional plane P_i containing y_i that describes the essential tangential behavior of the set $f(X)$. Let $p_i: E^n \to P_i$ be the orthogonal projection. Let $Z(t)$ be the set of components of $\ell_f^{-1}[\{x: u(x) < t\}]$. In order to establish (11) it was shown in the proof of [7, Theorem 3.3] that

$$\int \sum_{i=1}^{\infty} \sum_{V \in Z_j(t)} M[\partial P_{i\#}\mu_f(V)]\, dL_1(t) > \mathscr{L}(f) - \varepsilon\,.$$

Thus, in order to prove our theorem, it will suffice to show that for almost every t,

$$C(f, W) \geqq M[\partial P_{\#}\mu_f(V)]\,. \tag{12}$$

Here it is understood that V is a component of $\ell_f^{-1}(B)$ where B is an n-ball of radius t in E^n whose center is at y and that $p: E^n \to P$

is the orthogonal projection where P is an approximate tangent k-plane at y as desctibed in (11) of [7]. Also, $W = m_f^{-1}(V)$ and for simplicity, take $y = 0$.

To this end, we will consider only those t for which

$$H^k\,[\{y\colon \operatorname{dist}(y, 0) = t\} \cap f(x)] = 0 \,. \tag{13}$$

From [5, Theorem 2.10.25] it follows that this will be true for almost all t. From the definition of $C(f, W)$ it follows that there is a sequence of regions $\pi_m \subset W$ and Lipshitzian maps $f_m\colon X \to E^n$ such that

$$\lim_{m\to\infty} \mathscr{L}(f_m \mid \operatorname{bdry} \pi_m) = C(f, W) \,.$$

Let T_m denote the integral k-current $f_{m\#}(\pi_m)$ and observe that

$$M(\partial T_m) \leqq \mathscr{L}(f_m \mid \operatorname{bdry} \pi_m) \tag{14}$$

since $\mathscr{L}(f_m \mid \operatorname{bdry} \pi_m)$ can be expressed as the integral of an elementary counting function, [3, Theorem 6.18]. Without loss of generality, we may assume that $C(f, W) < \infty$, and therefore, there is a constant $K > 0$ such that

$$M(\partial T_m) \leqq K \,, \quad m = 1, 2, \cdots \,. \tag{15}$$

If the orthogonal projection $p\colon E^n \to P$ does not satisfy the conditions of Remark 2.7, select a projection $p^*\colon E^n \to P$ that does. Let $S_m = p^*_\#(T_m)$ and observe that (15) implies that $M(\partial S_m)$ is a bounded sequence. since S_m is an integral k-current in E^k, the isoperimetric inequality [5, Theorem 4.5.9(32)] is applicable and we can conclude that $N(S_m)$ is a bounded sequence. Hence, by the compactness theorem for integral currents [5, Theorem 4.2.17] there is an integral current S and a subsequence of the S_m such that $F(S_m - S) \to 0$. But for k-currents in E^k, the flat norm agrees with the mass norm and thus

$$M(S_m - S) \longrightarrow 0 \,. \tag{16}$$

Since S is an integral k-current in E^k, there is an integer valued density function $s\colon E^k \to E^1$ such that for each C^∞ differential k-form φ with compact support,

$$S(\varphi) = \int s \cdot \varphi \,.$$

The density function s_m associated with S_m is $s_m(y) = d(p^* \circ f_m, \pi_m, y)$ and (16) implies

$$\int_{E^k} |s_m - s| \, dL_k \longrightarrow 0 \, .$$

In view of (13), it follows [6, p. 131] that as $m \to \infty$,

$$s_m(y) \longrightarrow d(p^* \circ f, \, W, \, y)$$

for L_k almost all y. Thus,

$$s(y) = d(p^* \circ f, \, W, \, y) \quad \text{for } L_k \text{ almost all } y \, .$$

Consequently, by Remark 2.7,

$$S = h_\#(p_\#^* \circ \mu_f)[h(V)] = p_\#^*[\mu_f(V)] \, .$$

Let λ^* be the Lipschitz constant of p^*. Then, from (14) and the lower semi-continuity of mass,

$$\begin{aligned} C(f, \text{bdry } W) \geqq \limsup M(\partial T_m) &\geqq (\lambda^*)^{-k} \operatorname{lim\,sum} M(\partial S_m) \\ &\geqq (\lambda^*)^{-k} M(\partial S) \\ &\geqq (\lambda^*)^{-k} M(\partial p_\#^* \mu_f(V)] \, . \end{aligned}$$

Now, in order to establish (12), note that a sequence of projections $p_m^*\colon E^n \to P$ can be selected that satisfy the conditions of 2.7 and that converge to the orthogonal projection p. Then, $\lambda_m^* \to 1$ and

$$\liminf_{m \to \infty} M[\partial p_{m\#}^* \, \mu_f(V)] \geqq M[\partial P_\# \mu_f(V)] \, .$$

This completes the proof of the theorem.

3. Multiplicity and topological degree. Let $f\colon X \to E^n$ have finite Lebesgue area and suppose that $k = 2$ or $H^{k+1}[f(X)] = 0$. Then, it follows from [4, 2.1] that there is a Hausdorff k-rectifiable set $R \subset E^n$ and a Baire function v defined on M_f, such that for $\|\mu_f\|$ almost all $z \in M_f$, $v(z)$ is a simple k-vector that lies in the approximate tangent k-plane to R at $\ell_f(z)$. For H^k almost all $y \in R$, let $\tau(y)$ be a simple k-vector of unit norm that lies in the approximate tangent plane to R at y. It can be assumed that τ is a H^k measurable function. Further, for $\|\mu_f\|$ almost all $z \in M_f$, $|v(z)|$ is an integer and

$$\|\mu_f\|(A) = \int_{M_h} |v(z)| \, dH^k(z)$$

for every Borel set $A \subset M_f$. The following theorem shows that $|v(z)|$ can be described topologically.

THEOREM 3.1. *For almost all projections* $p\colon E^n \to E^k$,

$$| d_{p\circ f}[h(z)] | = | v(z) |$$

for $\| \mu_f \|$ *almost all* $z \in M_f$.

Proof. Choose $p: E^n \to E^k$ as in 2.7 and define

$$\psi(z) = d_{p\circ f}[h(z)] \, | p[\tau(\ell_f(z))] |$$

for $\| \mu_f \|$ almost all $z \in M_f$. Let D be the set where $v(z) \neq 0$ and A any Borel subset of D. Let

$$F(y) = \sum d_{p\circ f}[h(z)]$$

where the summation is taken over all $z \in \ell_f^{-1}(y) \cap A$. An application of [5, Theorem 13.2.22] yields

$$\begin{aligned} \int_R F(y) \, | p[\tau(y)] | \, dH^k(y) &= \int_{E^k} \sum_{y \in p^{-1}(w) \cap R} F(y) dL_k(w) \\ &= \int_{E^k} \sum_{B(w)} d_{p\circ f}[h(z)] \, dL_k(w) \end{aligned}$$

where $B(w) = \{z \colon z \in A \cap (p \circ \ell_f)^{-1}(w)\}$. However, [4, 2.2] implies

$$\begin{aligned} \int_R F(y) \, | p[\tau(y)] | \, dH^k(y) &= \int_{E^n} \sum \psi(z) dH^k(y) \\ &= \int_A \psi(z) dH^k(z) \end{aligned}$$

where $C(y) = \{z \colon z \in \ell_f^{-1}(y) \cap A\}$. By appealing to 2.7, it is clear that

$$h_\#(p_\# \circ \mu_f)[h(A)](w_k) = \int_A \psi(z) dH^k(z) \,, \tag{17}$$

where w_k is the orienting unit k form for E^k. However,

$$\begin{aligned} h_\#(p_\# \circ \mu_f)[h(A)](w_k) &= p_\# \circ \mu_f[h^{-1}(h(A))](w_k) \\ &= \mu_f[h^{-1}(h(A))] \, p^\# w_k \\ &= \int_A p^\# w_k[\ell_f(z)] \cdot v(z) dH^k \,. \end{aligned}$$

Combining this with (17) yields

$$\int_A \psi(z) dH^k = \int_A p^\# w_k[\ell_f(z)] \cdot v(z) dH^k \,;$$

and since A is arbitrary,

$$\psi(z) = p^\# w_k[\ell_f(z)] \cdot v(z) \,, \tag{18}$$

H^k almost everywhere in D. As $\| \mu_f \| (M_f - D) = 0$ and $\| \mu_f \|$ is absolutely continuous with respect to H^k in D, (18) and the definition of $\psi(z)$ gives

$$d_{p\circ f}[h(z)]\,|\,p[\tau(\ell_g(z))]\,| = p^{\#}w_k[\ell_f(z)] \cdot v(z)\,, \tag{19}$$

$\|\mu_f\|$ almost everywhere. As $v(z)$ and $\tau(\ell_f(z))$ are parallel k-vectors,

$$|\,p^{\#}w_k[\ell_f(z)] \cdot v(z)\,| = |\,v(z)\,|\,|\,p[\tau(\ell_f(z))]\,|\,. \tag{20}$$

The result follows from (19) and (20) provided $|\,p[\tau(\ell_f(z))]\,| \neq 0$, $\|\mu_f\|$ almost everywhere for almost all p.

To this end, observe that for H^k almost all $y \in R$, $\tau(y)$ exists and, thus, for almost all p, $p[\tau(y)] \neq 0$. However, the set of pairs (y, p) so that $y \in R$ and $p[\tau(y)] = 0$ is a Borel set. Thus, Fubini's theorem gives for almost all p, $p[\tau(y)] \neq 0$ for H^k almost all $y \in R$. Further, if $B \subset D \subset M_f$ and $H^k[\ell_f(B)] = 0$ then [4, 2.2] gives $\|\mu_f\|(B) = 0$. So for almost all p, $p[\tau(\ell_f(z))] \neq 0$ for $\|\mu_f\|$ almost all $z \in M_f$, and the result follows.

REMARK 3.2. It is interesting to note that an application of Fubini's theorem gives the following conclusion to Theorem 3.1: for $\|\mu_f\|$ almost $z \in M_f$,

$$|\,d_{p\circ f}[h(z)]\,| = |\,v(z)\,| \quad \text{for almost every } p\,.$$

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Received August 25, 1971. Research supported in part by NSF grant #GP 7505.

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