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ON c-REALCOMPACT SPACES AND LOCALLY BOUNDED NORMAL FUNCTIONS

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Let X be a completely regular Hausdorff space. Characterizations are obtained of the c-realcompact spaces lying between X and its Stone-Čech compactification βX, and of those spaces lying between X and its minimal c-realcompact extension uX. These results are used to derive several necessary and sufficient conditions for the equality $E(\upsilon X) = \upsilon E(X)$ to hold, where $E(X)$ is the absolute of X and υX is the Hewitt realcompactifiɔation of X.

A subset A of a topological space X is called regular closed provided $A = \mathrm{cl}_X \mathrm{int}_X A$. The complement of a regular closed set is said to be regular open. The symbol $R(X)$ will denote the family of all regular closed subsets of X. It is well known that $R(X)$ is a complete Boolean algebra under the algebraic operations which are set out explicitly in [11, Theorem 1.1]. In particular, we will recall for our needs here that $A \wedge B = \mathrm{cl}_X \mathrm{int}_X(A \cap B)$ defines the meet of A, B in $R(X)$ and if $(A_\alpha)_\alpha$ is a family of regular closed sets then $\mathrm{cl}_X \bigcup_\alpha A_\alpha$ is regular closed; it is the join of $(A_\alpha)_\alpha$ in $R(X)$. A filterbase in $R(X)$ is a subfamily $F \subseteq R(X)$ such that $\phi \notin F$ and $A \wedge B \in F$ for all $A, B \in F$. A filterbase F is called a filter if $A \in F$ and $B \in R(X)$ with $A \subseteq B$ implies $B \in F$. If X is dense in a space T then one may easily show that the map $A \to \mathrm{cl}_T A$ is a Boolean algebra isomorphism of $R(X)$ with $R(T)$.

For a real-valued function f on X the upper limit of f at $x \in X$ is defined as follows:

$$f^*(x) = \inf \{\sup \{f(y) \mid y \in U\} \mid U \in N(x)\} ,$$

where $N(x)$ is the neighborhood system at x. The lower limit of f at x is defined dually and is denoted by $f_*(x)$. Then f^* and f_* are extended real-valued functions on X and are respectively upper semicontinuous (usc) and lower semicontinuous (lsc). A real-valued function is called normal usc if $(f_*)^* = f$ at each point of X. Dually, f is normal lsc if $(f^*)_* = f$ at each point of X.

Dilworth initiated the study of bounded normal usc functions and used them to obtain the Dedekind-MacNeille completion of the lattice of bounded continuous functions on X. The following result from [1] which is valid in the present setting will be used below without specific reference.

THEOREM. *An usc function f on X is normal iff for each real* *number* λ, $\{x \mid f(x) > \lambda\}$ *is a union of regular closed sets.*

A corollary of the Theorem is that the characteristic function of a regular closed (open) set is normal usc (lsc).

Some new properties of normal functions have been given in [7]. We will recall two of these results.

LEMMA. *Two normal usc (lsc) functions which agree on a dense subset of X are equal everywhere on X.*

LEMMA. *If f is normal usc on X and X is dense in T then f has a unique (extended real-valued) normal usc extension $\bar{f}$ to T given by the expression*

$$\bar{f}(x) = \inf\{\sup\{f(y) \mid y \in U \cap X\} \mid U \in N(x)\},\ x \in T\ .$$

If f is bounded on X then $\bar{f}$ is bounded on T.

Unless stated otherwise, the space X will always be completely regular and Hausdorff. We refer the reader to [6] for general background, notation and terminology.

The authors would like to thank Professor John Mack for helpful correspondence on the subject treated herein.

1. The c-realcompact extensions of a space. In [2, p. 576] a space is called c-realcompact if for every point $p \in \beta X - X$ there exists a normal lsc function f on βX with $f > 0$ on X and $f(p) = 0$. By using the characterization of realcompact spaces in [4, p. 152] it is clear that every realcompact space is c-realcompact. Also, one can easily show that if $(Y_\alpha)_\alpha$ is a family of c-realcompact spaces with $X \subseteqq Y_\alpha \subseteqq \beta X$ for each α, then $\bigcap_\alpha Y_\alpha$ is c-realcompact.

LEMMA 1.1. *A space X is c-realcompact iff for every point $p \in \beta X - X$ there exists a decreasing sequence $(A_n)_{n \in N}$ in $R(\beta X)$ with $p \in \bigcap_n A_n$ while $\bigcap_n (A_n \cap X) = \varnothing$.*

Proof. Suppose that X is c-realcompact. Choose $p \in \beta X - X$ and let f be a normal lsc function with $f > 0$ on X and $f(p) = 0$. Then $B_n = \{x \mid f(x) < 1/n\}$ is a union of regular closed sets. Hence, $A_n = \mathrm{cl}_{\beta X} B_n$ is regular closed and $A_{n+1} \subseteqq A_n$ for each n. Since $A_n \subseteqq \{x \mid f(x) \leqq 1/n\}$ we have that $\bigcap_n (A_n \cap X) = \varnothing$ while $f(p) = 0$ implies that $p \in \bigcap_n A_n$. Conversely let $p \in \beta X - X$ and let $(A_n)_{n \in N}$ be a decreasing sequence of regular closed sets with the required properties. Define $f_n(x) = 1$ for $x \in \beta X - A_n$ and $f_n(x) = 0$ for $x \in A_n$. Then each f_n is normal lsc and hence $f = \sum 2^{-n} f_n$ is normal lsc with $f(p) = 0$

and $f > 0$ on X.

A Hausdorff space is called almost realcompact [5] if for every ultrafilter $\mathscr{F}$ of open sets with $\bigcap_{0 \in \mathscr{F}} \mathrm{cl}_X 0 = \varnothing$ there exists a sequence $(0_n)_n$ in $\mathscr{F}$ such that $\bigcap_n \mathrm{cl}_X 0_n = \varnothing$. For each such $\mathscr{F}$, the family $\mathscr{A} = \{\mathrm{cl}_X 0 \mid 0 \in \mathscr{F}\}$ is an ultrafilter in the Boolean algebra $R(X)$ and conversely, given an ultrafilter $\mathscr{A}$ in $R(X)$, the family $\mathscr{F} = \{V: V = \mathrm{int}_X V$ and $\mathrm{int}_X A \subseteq V$ for some $A \in \mathscr{A}\}$ is an ultrafilter of open sets in X. This information is enough to prove the following:

THEOREM 1.2. *A space X is almost realcompact iff every ultrafilter in $R(X)$ with the countable intersection property converges.*

The following result appears in [2, p. 577] but we have an alternate proof.

THEOREM 1.3. *Every completely regular Hausdorff almost realcompact space is c-realcompact.*

Proof. Let X be almost realcompact and choose $p \in \beta X - X$. Then there exists an ultrafilter $\mathscr{A}$ in $R(X)$ with $\{p\} = \bigcap_{A \in \mathscr{A}} \mathrm{cl}_{\beta X} A$. Thus, $\bigcap_{A \in \mathscr{A}} A = \varnothing$ and by Theorem 1.2 there is a countable subfamily $(A_n)_{n \in N}$ of $\mathscr{A}$ (which can be chosen to be decreasing) with $\bigcap_n A_n = \varnothing$. Then, $B_n = \mathrm{cl}_{\beta X} A_n \in R(\beta X)$ and $(B_n)_{n \in N}$ is decreasing with $p \in \bigcap_n B_n$. However, $\bigcap_n (B_n \cap X) = \bigcap_n A_n = \varnothing$ and by Lemma 1.1 we are done.

An example of an almost realcompact space which is not realcompact is given in [3, p. 350]. The space X constructed in the example given on page 240 of [9] is c-realcompact but not almost realcompact.

By a c-realcompact extension of X we will mean any c-realcompact space Y with $X \subseteq Y \subseteq \beta X$. Our immediate aim is to identify all c-realcompact extensions of X.

A real-valued function f is locally bounded at x provided f is bounded on some neighbourhood of x. Let $LN(X)$ denote the set of all normal usc functions which are locally bounded at each point of X. For $f \in LN(X)$, let $\bar{f}$ denote the unique (extended real-valued) normal extension of f to βX. Write $W_f = \{x \in \beta X \mid \bar{f}$ is locally bounded at $x\}$. Now for each x in X there is an open neighbourhood U of x on which f is bounded. If V is open in βX with $U = V \cap X$ then the definition of $\bar{f}$ shows that $\bar{f}$ is bounded on V. Hence, $X \subseteq W_f \subseteq \beta X$ and $\beta W_f = \beta X$. Further, W_f is open in βX so that W_f is locally compact.

LEMMA 1.4. *For each $f \in LN(X)$, W_f is c-realcompact.*

Proof. Consider $p \in \beta X - W_f$. Case 1: If $\bar{f}(p) = \infty$ then $B_n =$

$\{x \mid \bar{f}(x) > n\}$ is a union of regular closed sets and so $A_n = \mathrm{cl}_{\beta X} B_n$ is regular closed. Thus, $(A_n)_{n \in N}$ is a decreasing sequence of regular closed sets with $p \in \bigcap_n A_n$. However, $\bar{f}$ is locally bounded at each point of W_f and this implies that $\bigcap_n (A_n \cap W_f) = \emptyset$. By Lemma 1.1 we have that W_f is c-realcompact. Case 2: If $\bar{f}(p) < \infty$ then $\bar{f}$ (being usc) is locally bounded above at p. Thus, $\bar{f}$ is not locally bounded below at p which implies that $p \in \mathrm{cl}_{\beta X}\{x \mid \bar{f}(x) < \lambda\}$, for all real λ. The set $O_n = \{x \mid \bar{f}(x) \leqq n\}$ is an intersection of regular open sets and has nonvoid interior so that $A_n = \mathrm{cl}_{\beta X} \mathrm{int}_{\beta X} O_n$ is nonempty regular closed. Now $(A_n)_{n \in N}$ is decreasing and $p \in A_n$ for each n. Finally, $\bigcap_n (A_n \cap W_f) = \emptyset$ as before and we apply Lemma 1.1 to obtain the result.

LEMMA 1.5. *If Y is c-realcompact with $X \subseteqq Y \subseteqq \beta X$ then for each point $p \in \beta X - Y$ there is $f \in LN(X)$ with $Y \subseteqq W_f$ and $p \notin W_f$.*

Proof. By Lemma 1.1 there exists a decreasing sequence $(A_n)_{n \in N}$ in $R(\beta X)$ with $p \in \bigcap_n A_n$ while $\bigcap_n (A_n \cap Y) = \emptyset$. Define $g_n(x) = 1$ for $x \in A_n$ and $g_n(x) = 0$ for $x \in \beta X - A_n$. Then, g_n is normal usc on βX and so is $g = \sum 2^{-n} g_n$. There exists an order-preserving homeomorphism $h\colon [0, 1] \to [0, \infty]$ and it follows that $h \circ g = \bar{f}$ is normal usc. Now $\bar{f}$ is locally bounded at each point in Y and hence $\bar{f} \mid X = f \in LN(X)$. However, $\bar{f}(p) = +\infty$ so that $p \notin W_f$.

We now have the following situation: For any family $\mathscr{F}$ of locally bounded normal usc functions on X, the space $\bigcap_{f \in \mathscr{F}} W_f$ is c-realcompact. Conversely, Lemma 1.5 shows that any c-realcompact spaces Y with $X \subseteqq Y \subseteqq \beta X$ can be written in this form, where $\mathscr{F} = \{f \in LN(X) \mid Y \subseteqq W_f\}$. This proves the following analogue of the description of the realcompact spaces between X and βX in [6, 8B].

THEOREM 1.6. *The c-realcompact extensions of X are precisely the spaces $\bigcap_{f \in \mathscr{F}} W_f$, for some indexing set $\mathscr{F} \subseteqq LN(X)$.*

The following corollary also appears in [2, p. 578].

COROLLARY 1.7. (a) *There exists a minimal c-realcompact extension of X, namely $uX = \bigcap_{f \in LN(X)} W_f$.* (b) *$uX$ is the largest subspace of βX to which every function in $LN(X)$ has a locally bounded normal usc extension.*

The space uX will be called the c-realcompactification of X. Thus, X is c-realcompact iff $X = uX$. The equality $LN(X) = LN(uX)$ will indicate that every function f in $LN(X)$ can be uniquely extended to a function f^u in $LN(uX)$.

Recall that $LN(X)$ is a lattice-ordered ring [8]. The order in $LN(X)$ is defined pointwise. However $LN(X)$ is not closed under pointwise ring operations and we will therefore outline the definition of sum for f and g in $LN(X)$: Let $\bar{f}, \bar{g}$ denote their normal extensions to βX. There exists a dense G_δ set $W \subseteq \beta X$ on which $\bar{f}$ and $\bar{g}$ are both continuous and real-valued [7]. If $U = W \cap W_f \cap W_g$ then $(\bar{f}|U + \bar{g}|U) = u$ is continuous on U. Let $h = \bar{u}|X$. It follows that $\bar{u}$ is locally bounded on $W_f \cap W_g$ so that $h \in LN(X)$ defines the sum of f and g uniquely. The other operations are defined analogously and under these $LN(X)$ becomes a ring. It is evident from these definitions that the following result holds.

THEOREM 1.8. *The mapping $f \rightarrow f^u$ is an isomorphism of the lattice-ordered rings $LN(X)$ and $LN(uX)$.*

The next theorem gives the analogue of 8.6 in [6].

THEOREM 1.9. *Let $X \subseteq T \subseteq \beta X$. The following conditions are equivalent.*

(a) $X \subseteq T \subseteq uX$.

(b) *Given a decreasing sequence $(A_n)_{n \in N}$ in $R(X)$, then*

$$\mathrm{cl}_T \bigcap_n A_n = \bigcap_n \mathrm{cl}_T A_n .$$

Proof. (a) implies (b): Assume that there is a decreasing sequence $(A_n)_{n \in N}$ in $R(X)$ and a point $p \in \bigcap_n \mathrm{cl}_T A_n - \mathrm{cl}_T \bigcap_n A_n$. Let $B_n = \mathrm{cl}_{\beta X} A_n$ so that $A_n = B_n \cap X$ and $\mathrm{cl}_T A_n = B_n \cap T$. There is a regular closed set B in βX with $p \in \mathrm{int}_{\beta X} B$ and $B \cap \mathrm{cl}_T \bigcap_n A_n = \varnothing$. If $K_n = B_n \wedge B$ for each n then $(K_n)_{n \in N}$ is a decreasing sequence in $R(\beta X)$ with $p \in \bigcap_n K_n$ while $\bigcap_n (K_n \cap X) = \varnothing$. Thus, as in Lemma 1.1, we can find a nonnegative normal lsc function f on βX with $f > 0$ on X and $f(p) = 0$. Now $X \subseteq \mathrm{coz}\, f = \{x \in \beta X | f(x) > 0\}$, which is c-realcompact. Hence $uX \subseteq \mathrm{coz}\, f$ and $p \notin uX$. This contradicts (a). (b) implies (a): Suppose that $p \in T$ and $p \notin uX$. Since uX is c-realcompact we have a decreasing sequence $(B_n)_{n \in N}$ in $R(\beta X)$ with $p \in \bigcap_n B_n$ while $\bigcap_n (B_n \cap uX) = \varnothing$. Then $(B_n \cap X)_{n \in N}$ is a decreasing sequence in $R(X)$ for which (b) is false.

A topological space X is called weak *cb* if each lsc function which is locally bounded on X is bounded above by a continuous function Mack and Johnson defined weak *cb* spaces and gave some of their properties in [9]. In particular, if X is weak *cb* then for each $f \in LN(X)$ there exists $g \in C(X)$ with $f \leq g$ on X.

LEMMA 1.10. *If X is weak cb and $X \subseteq Y \subseteq uX$ then Y is weak cb.*

Proof. For any $f \in LN(Y)$ we have $h = f \mid X$ is in $LN(X)$. Thus there exists $g \in C(X)$ with $h \leqq g$ and so, if $g^{\upsilon} \in C(\upsilon X)$ denotes the continuous extension of g to υX we have $f \leqq g^{\upsilon} \mid Y$.

It is known that X is weak *cb* implies υX is weak *cb* [9, p. 239] but that the converse fails. However, we do have the following:

THEOREM 1.11. *X is weak cb iff uX is weak cb.*

Proof. The proof is a trivial consequence of the Lemma and the fact that $LN(X) = LN(uX)$.

The following result is stated without proof in [2, p. 578].

COROLLARY 1.12. *If X is weak cb then $uX = \upsilon X$.*

Proof. uX is c-realcompact and by Theorem 1.11 it is weak *cb* which implies that it must be realcompact [2, p. 576].

THEOREM 1.13. *The following conditions are equivalent:*
(a) *X is pseudocompact*
(b) *Every function in $LN(X)$ is bounded*
(c) *$uX = \beta X$.*

Proof. (a) implies (b): Each pseudocompact space is weak *cb* [9] and hence by the corollary we have $uX = \upsilon X$. However, $\upsilon X = \beta X$ [6, 8A] which shows that $LN(X) = LN(\beta X)$ and we are done, since every locally bounded function on a compact space is bounded.
(b) implies (c): It is known [8] that every bounded normal usc function has a bounded extension to βX which shows that $uX = \beta X$.
(c) implies (a): If $uX = \beta X$ then $\upsilon X = \beta X$ and we may apply [6, 8A] to conclude that X is pseudocompact.

COROLLARY 1.14. *A space is compact iff it is both pseudocompact and c-realcompact.*

2. The equality $E(\upsilon X) = \upsilon E(X)$. The following facts and notation appear in [11] and will be recalled for our future use. For every space X, there is an extremally disconnected space $E(X)$ which can be mapped onto X by a perfect irreducible mapping. The space $E(X)$ is unique up to homeomorphism and is called the absolute of X. Let S denote the Stone space of the Boolean algebra $R(\beta X)$ and let

$\lambda: R(\beta X) \to B(S)$ be the canonical Boolean algebra isomorphism of $R(\beta X)$ with the Boolean algebra $B(S)$ of open-and-closed subsets of S. There exists a perfect irreducible mapping $k: S \to \beta X$ and we have $k[\lambda(A)] = A$ for all A in $R(\beta X)$. Finally, we may identify $E(\beta X)$ with S and $E(X)$ with the dense subspace $k^{\leftarrow}[X]$ of S.

It is well-known that the equality $E(\beta X) = \beta E(X)$ holds for every X. The search for necessary and sufficient conditions under which the corresponding equality $E(\upsilon X) = \upsilon E(X)$ holds may be considered to be one of the main motivations of the present paper. In fact the equality does hold for a large class of spaces as we now proceed to show.

A zero set Z in X is a subset of the form $Z = f^{\leftarrow}(0)$ for some $f \in C(X)$. The symbol $\mathscr{Z}(X)$ will denote collection of all zero sets in X. The following result is easily established.

LEMMA 2.1. *In an extremally disconnected space, every zero set is the intersection of a decreasing sequence of open-and-closed sets and conversely.*

LEMMA 2.2. *Given a decreasing sequence* $(U_n)_{n \in N}$ *of open-and-closed subsets of* $E(\beta X)$ *then* $E(\upsilon X) = \upsilon E(X)$ *iff* $\bigcap_n(U_n \cap E(X)) = \varnothing$ *implies* $\bigcap_n(U_n \cap E(\upsilon X)) = \varnothing$.

Proof. By the introductory remarks in this section, we may identify $E(\upsilon X)$ with $k^{\leftarrow}[\upsilon X]$ so that $E(X) \subseteqq E(\upsilon X) \subseteqq E(\beta X)$. Also by [6, 8.13], $E(\upsilon X)$ is realcompact and hence $\upsilon E(X) \subseteqq E(\upsilon X)$. Now $E(X)$ is C^*-embedded in $\beta E(X)(= E(\beta X))$ and hence $Z \in \mathscr{Z}(E(X))$ iff $Z = Z' \cap E(X)$ for some $Z' \in \mathscr{Z}(E(\beta X))$. Thus, Lemma 2.1 shows that $Z \in \mathscr{Z}(\upsilon E(X))$ iff there is a decreasing sequence $(U_n)_{n \in N}$ of open-and-closed subsets of $E(\beta X)$ with $Z = \bigcap_n(U_n \cap \upsilon E(X))$. By [6, 8.8], $\upsilon E(X)$ can be characterized as that realcompact space Y, $E(X) \subseteqq Y \subseteqq \beta E(X)$ such that $Z \in \mathscr{Z}(Y)$, Z nonvoid, implies $Z \cap E(X) \neq \varnothing$. Thus, $\upsilon E(X) = E(\upsilon X)$ iff given a decreasing sequence $(U_n)_{n \in N}$ of open-and-closed subsets of $E(\beta X)$ we have $\bigcap_n(U_n \cap E(\upsilon X)) \neq \varnothing$ implies $\bigcap_n(U_n \cap E(X)) \neq \varnothing$. The contrapositive of the latter statement gives the result.

We are now in a position to prove two main results which together provide an answer to the problem stated in the introduction to this section.

THEOREM 2.3. *Given a decreasing sequence* $(A_n)_{n \in N}$ *in* $R(\beta X)$ *then* $E(\upsilon X) = \upsilon E(X)$ *iff* $\bigcap_n(A_n \cap X) = \varnothing$ *implies* $\bigcap_n(A_n \cap \upsilon X) = \varnothing$.

Proof. For every decreasing sequence $(A_n)_{n\in N}$ in $R(\beta X)$ there exists a decreasing sequence $(U_n)_{n\in N}$ of open-and-closed subsets of $E(\beta X)$ with $U_n = \lambda(A_n)$ for each n, where λ is the Boolean algebra isomorphism defined above. Then, by Lemma 2.2, $E(\upsilon X) = \upsilon E(X)$ iff $\bigcap_n(\lambda(A_n) \cap k^{\leftarrow}[\upsilon X]) \neq \emptyset$ implies $\bigcap_n(\lambda(A_n) \cap k^{\leftarrow}[X]) \neq \emptyset$. Also, since k is onto we have $E(\upsilon X) = \upsilon E(X)$ iff $k[\bigcap_n\lambda(A_n)] \cap \upsilon X \neq \emptyset$ implies $k[\bigcap_n\lambda(A_n)] \cap X \neq \emptyset$. Thus, the theorem will hold if $k[\bigcap_n\lambda(A_n)] = \bigcap_n A_n$. To show this, notice that $k[\bigcap_n\lambda(A_n)] \subseteqq \bigcap_n k[\lambda(A_n)] = \bigcap_n A_n$. Conversely, take $p \in \bigcap_n A_n$ and consider $\mathscr{N} = \{F \in R(\beta X) \mid p \in \operatorname{int}_{\beta X} F\} \cup (A_n)_{n\in N}$. It is clear that $\mathscr{N}$ is a filterbase in $R(\beta X)$ and as such may be embedded in an ultrafilter $\mathscr{A}$ in $R(\beta X)$. Now $\mathscr{A}$ is a point in $E(\beta X)$ and $\mathscr{A} \in \lambda(F)$ for every $F \in \mathscr{N}$. By the definition of k [11] we have $k(\mathscr{A}) = p$, since $\mathscr{A}$ converges to p. Thus, $\bigcap_n A_n \subseteqq k[\bigcap_n\lambda(A_n)]$ and the proof is complete.

THEOREM 2.4. *The following statements are equivalent.*
(a) $E(\upsilon X) = \upsilon E(X)$.
(b) *Given a decreasing sequence* $(A_n)_{n\in N}$ *in* $R(X)$ *then*

$$\operatorname{cl}_{\upsilon X} \bigcap_n A_n = \bigcap_n \operatorname{cl}_{\upsilon X} A_n$$

(c) $uX = \upsilon X$
(d) $LN(X) = LN(\upsilon X)$.

Proof. (b) iff (c): This follows from Theorem 1.9. (c) iff (d): Apply Theorem 1.8. It is easy to show that (b) implies the condition in Theorem 2.3 and the method used in the first part of the proof of Theorem 1.9 shows that the converse holds. Hence, (b) iff (a) which completes the proof.

It may be mentioned that Theorems 2.3 and 2.4 give conditions which are both necessary and sufficient for the equality $LN(X) = LN(\upsilon X)$ to hold. This answers more fully the question raised in [9, p. 237] where only necessary conditions were given.

In the remainder of this section we will give an internal characterization of those spaces for which any one of the conditions of Theorem 2.4 is satisfied.

Recall that a family $\mathscr{F}$ of subsets of X has the countable intersection property (CIP) if $\bigcap_n F_n \neq \emptyset$ for each sequence $(F_n)_{n\in N}$ of sets drawn from $\mathscr{F}$.

Recall that an ultrafilter $\mathscr{A}$ in $R(X)$ converges to $p \in \beta X$ if $\{p\} = \bigcap_{A\in\mathscr{A}} \operatorname{cl}_{\beta X} A$.

LEMMA 2.5. *The c-realcompactification* uX *has the following description:* $uX = \{p \in \beta X \mid$ *each ultrafilter in* $R(X)$ *converging to* p *has CIP*$\}$.

Proof. Let $p \in \beta X$ be such that every ultrafilter in $R(X)$ converging to p has CIP and let T be c-realcompact with $X \subseteqq T \subseteqq \beta X$. If $p \notin T$ then there is a decreasing sequence $(A_n)_{n \in N}$ in $R(\beta X)$ with $p \in \bigcap_n A_n$ and $\bigcap_n (A_n \cap T) = \varnothing$. The family $\{F \in R(\beta X) \mid p \in \mathrm{int}_{\beta X} F\} \cup (A_n)_{n \in N}$ is a filterbase in $R(\beta X)$ which can be embedded in an ultrafilter $\mathscr{A}$ in $R(\beta X)$. Then, $\{U \cap X \mid U \in \mathscr{A}\}$ is an ultrafilter in $R(X)$ which converges to p but fails to have CIP, contrary to assumption. Thus p is in every c-realcompact space containing X. Conversely, given $p \in \beta X$, suppose there is an ultrafilter $\mathscr{A}$ in $R(X)$ converging to p and a sequence $(A_n)_{n \in N}$ in $\mathscr{A}$ with $\bigcap_n A_n = \varnothing$. Let $B_n = \mathrm{cl}\,[\bigcap_{i=1}^n \mathrm{int}_X A_i]$. Then $(B_n)_{n \in R}$ is decreasing in $R(\beta X)$ with $\bigcap_n (B_n \cap X) = \varnothing$ and $p \in \bigcap_n B_n$. The second part of the proof of Theorem 1.9 may be used to conclude that $p \notin uX$. Thus, the proof is complete.

Mandelker [10] defines a family $\mathscr{F}$ of subsets of X to be stable provided each $f \in C(X)$ is bounded on some member of $\mathscr{F}$.

LEMMA 2.6. *An ultrafilter $\mathscr{A}$ in $R(X)$ is stable iff $\mathscr{A}$ converges to a point of υX.*

Proof. We modify the proof of Theorem 5.1 in [10]. Let $\mathscr{A}$ be an ultrafilter in $R(X)$ and set $\{p\} = \bigcap_{A \in \mathscr{A}} \mathrm{cl}_{\beta X} A$. If $p \in \beta X - \upsilon X$ there exists $f \in C(X)$ with $f^*(p) = \infty$ where f^* is the extension to βX of $f\colon X \to \boldsymbol{R}^*$ and $\boldsymbol{R}^*$ is the one-point compactification of $\boldsymbol{R}$. The set $\{x \in \beta X \mid f^*(x) \in \boldsymbol{R}^* - (-n, n)\}$ is a neighbourhood of p for each $n > 0$ and hence meets every A in $\mathscr{A}$. It follows that f is unbounded on every member of $\mathscr{A}$ so that $\mathscr{A}$ is not stable. Conversely, suppose $p \in \upsilon X$ and choose $f \in C(X)$. Now $f^*(p)$ is finite and hence there exists $B \in R(\beta X)$ with $p \in \mathrm{int}_{\beta X} B$ and f^* bounded on B. However, $\mathrm{int}_{\beta X} B \cap \mathrm{int}_{\beta X} A \neq \varnothing$, for all A in $\mathscr{A}$ and the maximality of $\mathscr{A}$ implies that $B \cap X$ is in $\mathscr{A}$. Thus f is bounded on $B \cap X$ and $\mathscr{A}$ is stable.

THEOREM 2.7. *For any space X, the following are equivalent:*
(a) $uX = \upsilon X$
(b) *every stable ultrafilter in $R(X)$ has CIP.*

Proof. Let $uX = \upsilon X$ and suppose that $\mathscr{A}$ is a stable ultrafilter in $R(X)$. By Lemma 2.6 we have $\bigcap_{A \in \mathscr{A}} \mathrm{cl}_{\beta X} A = \{p\}$ where $p \in \upsilon X$. Thus, $p \in uX$ and by Lemma ultrafilter $\mathscr{A}$ has CIP. Conversely, if $p \in \upsilon X$ and $\mathscr{A}$ is any ultrafilter in $R(X)$ converging to p then $\mathscr{A}$ is stable by Lemma 2.6 and has CIP by assumption. Thus, Lemma 2.5 implies that $p \in uX$ and we are finished.

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