PACIFIC JOURNAL OF MATHEMATICS
Vol. 44, No. 1, 1973

A MEAN STIELTJES TYPE INTEGRAL

D. B. PRIEST

For the extended mean Stieltjes integral R. A. Stokes has shown that joint discontinuities of the functions involved can be ignored just as in the ordinary mean Stieltjes integral as considered by Porcelli and others. A Stieltjes type integral of a function with respect to a function pair has been defind by E. D. Roach, but existence of the integral depends upon the simultaneous continuity of two or more of the functions involved. In this paper a mean Stieltjes type integral of a function with respect to a function pair is defined which overcomes these limitations. Representation theorems for the integral are also given.

1. Definitions and notations. Unless otherwise stated or implied, functions considered in this paper are real valued and defined on the rectangular interval $A = [\alpha, \gamma] \times [\beta, \delta]$. Limits are of the σ or refinement type on partitions D of A [4, 10]. If the subinterval $d = [p, r] \times [q, s]$ of A is an element of the partition D, then the f-area of the function f is given by $f(d) = f(p, q) - f(r, q) + f(r, s) - f(p, s)$ [6]. The horizontal and vertical contour maps of f are given by $f[-, n]$ and $f[m, -]$, respectively, where $(m, n) \in A$. The function f is of bounded variation in the sense of Vitali provided the sums of the type $\sum_{d \in D} |f(d)|$ are uniformly bounded. $V_{[p,q;r,s]}(f)$ (or $V_d(f)$) will denote the variation of f over d. Hardy and Krause require additionally that at least one horizontal and at least one vertical contour map of f be of bounded variation [1, 2]. The function f is said to be

(1) totally nondecreasing provided $f(d) \geqq 0$ for each $d \in D$, $f[\alpha, -]$ is nondecreasing, and $f[-, \beta]$ is nondecreasing;

(2) partially nondecreasing provided $f(d) \geqq 0$ for each $d \in D$, $f[\alpha, -]$ is nondecreasing, and $f[-, \delta]$ is nonincreasing; and

(3) anchored provided $f(d) \geqq 0$ for each $d \in D$ and $f[\alpha, -] = f[-, \beta] = 0$ [2, 11].

The (g, h)-evaluation over $d \in D$, denoted by $(g, h)(d)$, of the function pair (g, h) is given by $(g(r, q) - g(p, q))(h(p, s) - h(p, q)) - (h(r, s) - h(p, s))(g(r, s) - g(r, q))$. The lower (g, h)-evaluation, denoted by $(g, h)(1, d)$ is given by an analogous expression wherein $g(r, s) - g(r, q)$ has been replaced by $g(p, s) - g(p, q)$. $N_D(g, h)$ and $L_D(g, h)$ denote the sums $\sum_{d \in D} (g, h)(d)$ and $\sum_{d \in D}(g, h)(1, d)$, respectively. If $E \geqq D$ (i.e., E is a refinement of D), then E/d denotes that portion of E which is a partition of d. Let $N'(d) = \sup \{N_E(g, h)\colon E$ is any partition of $d\}$. $L'(d)$ is defined similarly.

The function f is quasicontinuous [8] provided the limits $f(s, t^{\pm})$,

$f(s^{\pm}, t)$ and $f(s^{\pm}, t^{\pm})$ exist for each (s, t) in A [12, p. 3]. The boundary of d will be denoted by $\partial(d)$.

2. The integrator pair. In this section, unless otherwise stated, we assume that g is totally nondecreasing and h is partially nondecreasing. Under these conditions, it follows directly from the definitions that (1) $N'(d)$ and $L'(d)$ are finite for each $d \in D$; (2) sums of the type $L_D(g, h)$ are nondecreasing under refinement; and (3) $L'(d) = \sum L'(e)$ where the summation is over all e in E/d. Thus using techniques analogous to those developed in [12, pp. 10–14] for functions of bounded variation in the sense of Vitali we have the following two lemmas:

LEMMA 2.1. *Let $\varepsilon > 0$ and $d = [p, r] \times [q, s] \subseteqq A$. Then there exists a pairwise, disjoint collection $\{c_j\}_{1 \leqq j \leqq 4}$ of subintervals of d such that*

(1) each c_j shares a separate vertex with d and

(2) if c_j is a subinterval of c_j which also shares a vertex with d, then $L'(c_j) - L'(c'_j) < \varepsilon$.

LEMMA 2.2. *Let $\varepsilon > 0$, $d = [p, r] \times [q, s] \subseteqq A$, and b be a subinterval of d missing the vertices of d such that $\partial(d) \cap \partial(b) \neq \emptyset$. Then there exists a proper subinterval b' of b such that if E is a partition of b', then $\sum L'(e) < \varepsilon$ where the summation is over all e in E which have no point in common with $\partial(d)$.*

LEMMA 2.3. *If $g[-, \delta]$ or $h[-, \beta]$ is continuous on $[\alpha, \gamma]$, then for each $\varepsilon > 0$, there is a partition D of A such that $0 \leqq N_E(g, h) - L_E(g, h) < \varepsilon$ where $E \geqq D$.*

Proof. Assume that $g[-, \delta]$ is continuous. If $h[-, \beta]$ is continuous, the proof is similar. Let $\varepsilon > 0$. Assume that $K = h(\alpha, \beta) - h(\gamma, \beta) \neq 0$; otherwise, the result is immediate. There exists a partition $D = \{d_i = [t_{i-1}, t_i] \times [\beta, \delta]\}_{1 \leqq i \leqq n}$ of A such that $g(d_i) < \varepsilon/K$. Let E be a refinement of D. Under the conditions on g and h, it follows routinely that

$$N_E(g, h) - L_E(g, h) \leqq \sum_{i=1}^{n} (h(t_{i-1}, \beta) - h(t_i, \beta))(g(d_i)) < \varepsilon .$$

That the difference $N_E(g, h) - L_E(g, h)$ is nonnegative follows readily from the definitions.

LEMMA 2.4. *If f is a function of bounded variation in the sense of Vitali and each of its horizontal contour maps is continuous, then the variation function $V(x) = V_{[\alpha, \beta; x, \delta]}(f)$ for each x in $[\alpha, \gamma]$, is* *continuous.*

The following theorem parallels a known result concerning functions of bounded variation in Vitali's sense [9, p. 250].

THEOREM 2.5. *If the function g is of bounded variation in the sense of Hardy-Krause, then there exist totally nondecreasing functions g_1 and g_2 and partially nondecreasing functions h_1 and h_2 such that $g = g_2 - g_1 = h_2 - h_1$. Moreover, if the horizontal contour maps of g are continuous, then $g_1[-, \delta]$, $g_2[-, \delta]$, $h_1[-, \beta]$, and $h_2[-, \beta]$ are continuous.*

Proof. It is known that if g is of bounded variation in the sense of Hardy-Krause, then $g[\alpha, -]$, $g[-, \beta]$, and $g[-, \delta]$ are of bounded variation in the usual sense [2, p. 385]. Thus let g_1 be the function such that

$$g_1(x, y) = V_{[\alpha,\beta;x,y]}(g) + V_\alpha^x(g[-, \beta]) + V_\beta^y([\alpha, -])$$

for all (x, y) in A. Let h_1 be such that

$$h_1(x, y) = V_{[\alpha,\beta;x,y]}(g) - V_{[\alpha,\beta;x,\delta]}(g) + V_\beta^y(g[\alpha, -]) - V_\alpha^x(g[-, \delta])$$

for all (x, y) in A. Additionally let $g_2 = g + g_1$ and $h_2 = g + h_1$. The theorem now follows from the additivity of the variation functions on subintervals of a given interval [5, p. 107] and Lemma 2.4.

3. The integral. For the remainder of this paper, $F(d)$ will denote $1/4(f(p, q) + f(r, q) + f(r, s) + f(p, s))$, the mean evaluation of f at the four vertices of d in the partition D. [4, p. 274 and 12, p. 5]

DEFINITION 3.1. The function f is (g, h)-integrable provided the σ-limit of sums of the type $S_D(f, g, h) = \sum_{d \in D} (F(d))((g, h)(d))$ exists. The integral will be denoted by $\int_A f\,dg\,dh$.

REMARKS. It follows immediately from the definition that the integral is linear in each of its three components i.e., the integrand f and each of its integrator function g and h. Also, if each of g and h is a function of a single variable, then $\int_A f\,dg\,dh$ agrees readily with one or the other of the extended mean Stieltjes integrals $\pm\, m(\sigma)\int_A f\,dk$ where $k = gh$. For in this case, $\lim_\sigma S_D(f, g, h) = \lim_\sigma \pm \sum_{d \in D} F(d)k(d) = \pm\, m(\sigma)\int_A f\,dk$ [12, p. 14].

THEOREM 3.2. *If f is quasicontinuous, g and h are of bounded variation in the sense of Hardy-Krause, and either the horizontal contour maps of g or the horizontal contour maps of h are continuous, then f is (g, h)-integrable.*

Proof. Let $\varepsilon > 0$. In view of Theorem 2.5 and the remarks preceding this theorem, it suffices to consider a totally nondecreasing g and a partially nondecreasing h such that either $g[-, \delta]$ or $h[-, \beta]$ is continuous. There is a partition $A' = \{A_1, \cdots, A_N\}$ of A such that (1) $|L_E(g, h) - L_F(g, h)| < \varepsilon$ where $E, F \geqq A'$ and (2) if P and Q are in the interior of A_i or in the interior of a particular edge of A_i for some $i \leqq N$, then $|f(P) - f(Q)| < \varepsilon$ [12, p. 4]. By Lemma 2.1 and 2.2 there exists a refinement B of A' such that $B/A_i = \{c_{ij}, e_{ij}, I_i\}_{1 \leqq j \leqq 4}$ where each c_{ij} has the property of c_j in Lemma 2.1, each e_{ij} has the property of b' in Lemma 2.2 and $I_i = A_i \backslash \bigcup_{j=1}^4 (c_{ij} \cup e_{ij})$, the complement of $\bigcup_{j=1}^4 (c_{ij} \cup e_{ij})$ relative to A_i. Suppose $G \geqq B$ and let $C_{ij} = \{e \in G/c_{ij} \mid e$ shares a vertex with $A_i\}$, $E_{ij} = \{e \in G/e_{ij} \mid \partial(e) \cap \partial(A_i) \neq \emptyset\}$, $C'_{ij} = (G/c_{ij}) \backslash C_{ij}$, and $E'_{ij} = (G/e_{ij}) \backslash E_{ij}$. Then

$$\Big|\sum_{b \in B} (F(b))((g, h)(1, b)) - \sum_{e \in G} (F(e))((g, h)(1, e))\Big| < M\big|L_B(g, h) - L_G(g, h)\big|$$
$$+ \varepsilon\Big[\sum_{i=1}^N \sum_{j=1}^4 (L_{C_{ij}}(g, h) + L_{E_{ij}}(g, h))\Big] + \varepsilon \sum_{i=1}^N L_{G/I_i}(g, h)$$
$$+ 2M\Big[\sum_{i=1}^N \sum_{j=1}^4 \sum_{e \in E'_{ij}} L'(e) + \sum_{i=1}^N \sum_{j=1}^4 \sum_{e \in C'_{ij}} L'(e)\Big] < \varepsilon K$$

where M is a positive bound on f [12, p. 4] and $K = M + L'(A) + 16MN$. Now by Lemma 2.3 there is a partition B' of A such that if $G \geqq B'$, $0 \leqq N_G(g, h) - L_G(g, h) < \varepsilon/2M$. Suppose D refines B and B'. It now follows routinely that $|S_E(f, g, h) - S_F(f, g, h)| < \varepsilon(2K + 1)$, where $E, F \geqq D$. It is well known that this Cauchy condition insures the σ-limit since $S_D(f, g, h)$ is a function of subdivisions [4, Th. 2.11, p. 266] i.e., f is (g, h)-integrable.

4. Representations for the integral. We now develop certain representation theorems for the integral, establishing relationships between it, the ordinary mean Stieltjes integral $\left(m(\sigma)\int_\alpha^\gamma f\, dg \text{ [3, 10]}\right)$, and the extended mean Stieltjes integral. We shall say that f is factorable into functions f_1 and f_2 provided there exist functions f_1 and f_2 such that $f(x, y) = f_1(x) f_2(y)$ for each (x, y) in A.

THEOREM 4.1. *Suppose f is quasicontinuous and g and h are of bounded variation in the sense of Vitali and factorable into nonconstant functions g_1, g_2, and h_1, h_2 respectively. If $\mu = h_1 g_2$ and $\nu = g_1 h_2$ have* *no common discontinuities, then f is (g, h)-integrable and* $\int_A f\,dg\,dh = m(\sigma)\int_A f\mu\,d\nu - m(\sigma)\int_A f\nu\,d\mu$.

Proof. Since g and h are of bounded variation in Vitali's sense and factorable into nonconstant functions, μ and ν are of bounded variation in Vitali's sense [5, p. 107]. It also follows readily that $f\mu$ and $f\nu$ are quasicontinuous [5, p. 40]. Thus each of $m(\sigma)\int_A f\mu\,d\nu$ and $m(\sigma)\int_A f\nu\,d\mu$ exists [12, Th. 3.1, p. 15]. Since $S_D(f, g, h) = \sum_{d\in D}(F(d))(\mu(p, q))(\nu(d)) - \sum_{d\in D}(F(d))(\nu(r, s)(\mu(d))$, the result follows by showing that the σ-limits of the sums on the right are $m(\sigma)\int_A f\mu\,d\nu$ and $m(\sigma)\int_A f\nu\,d\mu$ respectively. An outline of the proof of the first of these is given. Suppose $V = V_A(\nu) > 0$. Let J, K and L denote positive bounds on f, g_2 and h_1 respectively and let $\varepsilon > 0$. There is a partition $A' = \{A_1, \cdots, A_N\}$ of A such that if P and Q are in the interior of A_i for some i, $|\mu(P) - \mu(Q)| < \varepsilon/4JV$; and if $E \geqq A'$, $|m(\sigma)\int_A f\mu\,d\nu - \sum_{e\in E} FM(e)\nu(e)| < \varepsilon/4$ [12, Th. 3.1, p. 15]. Under the continuity conditions on μ and ν, there is a refinement B of A' such that $B/A_i = \{c_{ij}, e_{ij}, I_i\}_{1\leqq j\leqq 4}$ and such that either (1) $V_b(\nu) < \varepsilon/64JKLN$ (for $b = c_{ij}$ or e_{ij}) or (2) $|\mu(u, v) - \mu(s, t)| < \varepsilon/8JV$ where $E \geqq B$ and $(s, t) \in [u, w] \times [v, x] \in E/b$ (for $b = c_{ij}$ or e_{ij}). Thus if $E \geqq B$ and b ranges over all c_{ij} and e_{ij},

$$\begin{aligned}\left| m(\sigma)\int_A f\mu\,d\nu - \sum_{e\in E}(F(e))(\mu(u, v))(\nu(e)) \right| &< \frac{\varepsilon}{4} \\ &+ \sum_{e\in E/b} |FM(e) - F(e)(\mu(u, v))|\,|\nu(e)| \\ &+ \sum_{i=1}^{N}\sum_{e\in E/I_i} |FM(e) - F(e)(\mu(u, v))|\,|\nu(e)| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon\,.\end{aligned}$$

Similarly one can see that $\lim_\sigma \sum_{d\in D}(F(d))(\nu(r, s))(\mu(d)) = m(\sigma)\int_A f\nu\,d\mu$. Thus $\lim_\sigma S(f, g, h) = m(\sigma)\int_A f\mu\,d\nu - m(\sigma)\int_A f\nu\,d\mu$.

REMARK. If in the above theorem we were to allow f to be a nonconstant factorable function and require only that g_i and $h_i (i = 1, 2)$ share no discontinuities from the right or from the left, then we can show by a similar argument that the integral $\int_A f\,dg\,dh$ can be represented by

$$\left(m(\sigma)\int_\alpha^\gamma f_1h_1dg_1\right)\left(m(\sigma)\int_\beta^\delta f_2g_2dh_2\right) - \left(m(\sigma)\int_\alpha^\gamma f_1g_1dh_1\right)\left(m(\sigma)\int_\beta^\delta f_2h_2dg_2\right).$$

THEOREM 4.2. *Suppose f is quasicontinuous and g and h are of bounded variation in the sense of Hardy-Krause. If the contour maps of g [resp. h] are continuous, then there exists a function μ of bounded variation such that $\int_A f\,dg\,dh = m(\sigma)\int_A f\,d\mu$.*

Proof. By Theorem 3.2 $\int_A f\,dg\,dh$ exists. Let μ be such that for each $(x, y) \in A$,

$$\mu(x, y) = m(\sigma)\int_0^x h[-, y]dg[-, y] + m(\sigma)\int_0^y g[x, -]dh[x, -] - (gh)(x, y).$$

By Theorem 2.5, it suffices to consider g and h as in Theorem 3.2. Under these conditions and known integration by parts formulas [7, Th. 2.1, p. 61] it follows that $\sum_{d\in D}\mu(d)$ is equal to

$$\begin{aligned}&\sum_{d\in D}(g(r, q) - g(p, q))(h(\lambda, s) - h(\lambda, q))\\ &- \sum_{d\in D}(h(r, \theta) - h(p, \theta))(g(r, s) - g(r, q))\\ &- \sum_{d\in D}(h(\lambda', s) - h(p, \theta'))(g(d))\end{aligned}$$

where $d = [p, r] \times [q, s]$, $\lambda, \lambda' \in [p, r]$, and $\theta, \theta' \in [q, s]$. Whereupon it follows that μ is of bounded variation in the sense of Vitali and hence that $m(\sigma)\int_A f\,d\mu$ exists [12, Th. 3.1, p. 15]. Since the last term of $\sum_{d\in D}\mu(d)$ tends to zero under refinement, it follows that $\lim_\sigma \sum_{d\in D} F(d)\mu(d) = \int_A f\,dg\,dh$ i.e., $m(\sigma)\int_A f\,d\mu = \int_A f\,dg\,dh$.

THEOREM 4.3. *Suppose f is quasicontinuous and g and h are of bounded variation in the sense of Hardy-Krause. If the horizontal and vertical contour maps of h [resp. g] are continuous, then there exist function sequences (h_{1_p}), (h_{2_p}), (k_{1_p}), and (k_{2_p}) such that*

$$\int_A f\,dg\,dh = \sum_{p=1}^{\infty}\left(\int_A f\,dh_{1_p}\,dk_{1_p}\right) - \sum_{p=1}^{\infty}\left(\int_A f\,dh_{2_p}\,dk_{2_p}\right).$$

Proof. By Theorem 4.2 there is a function μ such that $\int_A f\,dg\,dh = m(\sigma)\int_A f\,d\mu$. As in that theorem, it can be shown that μ is bounded variation in the sense of Hardy-Krause. By Theorem 2.5, μ is the difference of two nodecreasing functions F_1 and F_2. By a theorem of Jolly [6, Th. 1, p. 317] there exist anchored nondecreasing functions G_1 and G_2, nondecreasing functions g_1 and g_2 on $[\alpha, \gamma]$, and nondecreasing functions h_1 and h_2 on $[\beta, \delta]$ such that $F_i = G_i + g_i + h_i$ (for $i = 1, 2$). Thus $\int_A f\, dg\, dh = m(\sigma)\int_A f\, dG_1 - m(\sigma)\int_A f\, dG_2$. By a theorem of Stokes [12, Th. 4.3, p. 34] there exist function sequences (h_{1_p}) and (h_{2_p}) each term of which is defined on $[\alpha, \gamma]$ and function sequences (k_{1_p}) and (k_{2_p}) each term of which is defined on $[\beta, \delta]$ such that

$$\sum_{p=1}^{n}\int_A f\, d(h_{1_p}k_{1_p}) \longrightarrow \int_A f\, dG_1$$

and

$$\sum_{p=1}^{n}\int_A f\, d(h_{2_p}k_{2_p}) \longrightarrow \int_A f\, dG_2$$

as $n \to \infty$. Since each h_{i_p} is a function of x and each k_{i_p} is a function of y, the theorem now follows from the remarks preceding Theorem 3.2.

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Received August 30, 1971 and in revised form March 24, 1972.

UNIVERSITY OF MISSISSIPPI
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HARDING COLLEGE