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FIVE THEOREMS ON MACAULAY RINGS

M. HOCHSTER AND L. J. RATLIFF, JR.

The first three theorems concern localizations of a Noetherian ring such that the localization is a Macaulay ring, and the other two theorems give some necessary and sufficient conditions for certain Rees rings and form rings of a Noetherian ring to be locally Macaulay. Numerous consequences of the theorems are proved.

1. Introduction. Macaulay rings have been extensively studied, and numerous properties of such rings are known. It is the purpose of this paper to add to the knowledge of such rings (and Noetherian ring theory in general) by proving five theorems in this area, and to derive some consequences of these theorems.

Most of the results in §2 follow quite readily from the first theorem, Theorem 2.2. Among the corollaries of this theorem, are the following, where E is a finite module over a Noetherian ring R such that $E_Q \neq (0)$, for some $Q \in \operatorname{Spec} R$. The Macaulay locus of E (that is, the $Q \in \operatorname{Spec} R$ such that E_Q is Macaulay) is open in the VS topology (Definition 2.3) on $\operatorname{Spec} R$ (2.4.1). If $P \in \operatorname{Spec} R$ and height $P/(\operatorname{Ann} E) > 1$, then there exist infinitely many $p \in \operatorname{Spec} R$ such that $\operatorname{Ann} E \subset p \subset P$, height $P/p = 1$, and E_p is Macaulay (2.5.2). $\operatorname{Rad}(\operatorname{Ann} E) = \cap \{p \in \operatorname{Spec} R;\ E_p$ is Macaulay and p is a G-ideal$\}$ (that is, the G-ideals of the Macaulay locus are Zariski dense in the set of prime ideals containing $\operatorname{Ann} E$) (2.12). (An important special case is when $E = R$ is a Hilbert ring (2.17).) If E_Q is Macaulay and $P \in \operatorname{Spec} R$ is such that $Q \subset P$ and height $P/Q = d$, then, for $i = 0, 1, \cdots, d-1$, $Q = \cap \{p \in \operatorname{Spec} R;\ Q \subseteqq p \subset P$, E_p is Macaulay, height $p =$ height $Q + i$, height $P/p = d - i$, and $i =$ height $p/Q\}$ (2.19). Also these last intersectors can be adjusted to not being contained in a finite set of prime ideals which don't contain P (2.21), and to not containing a given element not in Q (2.23). Then this section is closed by proving a result, Proposition 2.26, which implies that, in a local ring (R, M), there exists a system of parameters $b_1, \cdots, b_a$ such that, if $P \neq M$ is a prime ideal in R such that $b_1, \cdots, b_j \in P$, for some $j < a$, then the images in R_P of $b_1, \cdots, b_j$ are an R_P-sequence and, if P is a minimal prime divisor of $(b_1, \cdots, b_j)R$, then, for $i = 0, 1, \cdots, j$, $(b_1, \cdots, b_i)R_P$ is a primary ideal (by (2.26.3) and (2.27.1)).

In §3, two theorems concerning a finitely generated ring A over a Noetherian ring R are proved. The first, Theorem 3.9, shows in particular that, if A is an integral domain and R satisfies the condition: $(0) = \cap \{M; M$ is a maximal ideal in R and R_M is Macaulay$\}$: then A

does. The second, Theorem 3.12, gives a necessary and sufficient condition for A to satisfy this condition, when R is semilocal and altitude $R > 1$. The theorem implies that most non-integral extensions of R do satisfy the condition and, if R is integrally closed, then every proper extension domain of R satisfies the condition (Corollary 3.13).

The last two theorems are proved in §4. The first of these and all its corollaries are closely related to (and/or are generalizations of) the results in [11]. Specifically, it is proved in [11] that the Rees ring $\mathscr{R}(A, Y)$ of a locally Macaulay ring A with respect to an ideal Y generated by an A-sequence is locally Macaulay, and some related results are obtained. In §4, a number of necessary and sufficient conditions are given for the Rees ring $\mathscr{R} = \mathscr{R}(R, B)$ to be locally Macaulay, where $(R; M_1, \cdots, M_e)$ is a semi-local ring such that height M_i = altitude $R = a(i = 1, \cdots, e)$ and B is an ideal in R such that $\operatorname{Rad} B = \cap M_i$. Namely, the following conditions are equivalent: (1) $\mathscr{R}$ is locally Macaulay; (2) The form ring $\mathscr{F}$ of R with respect to B is locally Macaulay; (3) $\mathscr{R}_{(\mathscr{R}\sim\cup\mathscr{M}_i)}$ (resp., $\mathscr{F}_{(\mathscr{F}\sim\cup\mathscr{N}_i)}$) is a Macaulay ring, where the $\mathscr{M}_i$, (resp., $\mathscr{N}_i$)$(i = 1, \cdots, e)$ are the maximal homogeneous ideals in $\mathscr{R}$ (resp., $\mathscr{F}$); (4) R is Macaulay and there exists a system of parameters $y_1, \cdots, y_a$ in R such that, for each $j = 1, \cdots, a$ and for all n, $(y_1, \cdots, y_j)R \cap B^n = \sum_1^j y_i B^{n-d_i}$, where $y_i \in B^{d_i}$, $\notin B^{d_i+1}$; (5) R is Macaulay and there exists a positive integer g and a system of parameters $z_1, \cdots, z_a$ contained in B^g such that, for each $j = 1, \cdots, a$ and for all $n \geqq g$, $((z_1, \cdots, z_j)R) \cap B^n = (z_1, \cdots, z_j)B^{n-g}$ (by (4.4), (4.6), and (4.11)). Further, if $\mathscr{R}$ is locally Macaulay, then following hold: (a) Given any finite number (say, s) of prime ideals P_i in R which have the same height (say, k), there exists an R-sequence $y_1, \cdots, y_k$ contained in $\cap P_i$ such that, for each $j = 1, \cdots, k$ and for all n, $(y_1, \cdots, y_j)R \cap B^n = \sum_1^j y_i B^{n-d_i}$ with d_i as in (4) (Corollary 4.5); (b) $\mathscr{R}(R, B^m)$ is locally Macaulay, for all $m > 0$ (Corollary 4.8.1); and, (c) For all $m > 0$, R and $R[B^m/b]$ are locally Macaulay for each nonzero-divisor $b \in B^m$ (by 4.8.2) and [11, Theorem 3.8]). Also, (a) – (c) hold when B is a power of an ideal generated by an R-sequence (Corollary 4.9). The last theorem (4.11) shows that (1) – (3) above are equivalent for an arbitrary ideal B in a Noetherian ring such that B is contained in the Jacobson radical of R, and each of (1) – (3) implies (c) for $m = 1$. Some related information is given in Propositions 4.10 and 4.12.

2. Macaulay localizations.

The terminology in this article is, in general, the same as that in [8]. However, to keep the article reasonably self-contained, a number of definitions will be given. In particular, if R is a ring and E is an R module, then the (ordered sequence of) elements $b_1, \cdots, b_n$ in R is an *E-sequence* in case: $(b_1, \cdots, b_n)E \neq E$; and, for $i = 1, \cdots n$, b_i isn't a zero-divisor on $E/(b_1, \cdots, b_{i-1})E$.

(As usual, the ideal generated by the empty set is defined to be the zero ideal, and $b \in R$ is a zerodivisor on an R-module E^* in case $bx = 0$, for some nonzero $x \in E^*$.) *Ass* $E = \{p \in \operatorname{Spec} R; (0): (x) = p$, for some nonzero $x \in E\}$, *Ann* $E = (0): E = \{r \in R; rE = (0)\}$, *Dim* $E =$ altitude $R/(\operatorname{Ann} E)$, and *Prof* E is the length of a maximal E-sequence. If R is a local (Noetherian) ring and E is a finite R-module, then E is a *Macaulay* R-module in case either $E = (0)$ or $E \neq (0)$ and $\operatorname{Dim} E = \operatorname{Prof} E$. (However, *in this paper, the statement that E is Macaulay will always mean* $E \neq (0)$ and E is Macaulay.) A local ring R is a *Macaulay local ring* in case R is a Macaulay R-module (that is, there exists a system of parameters in R which is an R-sequence). A Noetherian ring R is said to be a *locally Macaulay ring* in case R_P is a Macaulay local ring, for all prime ideals P in R; and R is said to be a *Macaulay ring* in case R is a locally Macaulay ring and height $M =$ altitude R, for all maximal M in R.

Many basic properties of Macaulay rings and Macaulay modules will be used implicitly throughout this paper. These properties can be found in [2, Chapter 0, §16.5], [5, Chapter 3], and [8, §25].

The following lemma is of basic importance in this section. (It should be noted that, in the lemma, $\mathscr{P}$ may be a finite set.)

LEMMA 2.1. *Let Q be a prime ideal in a Noetherian ring R, let $B_1, \cdots, B_h$ be ideals in R which are contained in Q, let $\mathfrak{P}_j$ be the set of prime divisors of B_j, and let $\mathscr{P}$ be the set of prime ideals p in R such that $Q \subset p$ and height $p/Q = 1$. Then there are at most a finite number of $p \in \mathscr{P}$ such that either p contains an ideal $P \in \bigcup_0^h \mathfrak{P}_j$ such that $P \notin Q$ or height $p >$ height $Q + 1$.*

Proof. It is known [6] that there are only finitely many $p \in \mathscr{P}$ such that height $p >$ height $Q + 1$. Also, at most a finite number of $p \in \mathscr{P}$ can contain a fixed ideal which isn't contained in Q (since R/Q is Noetherian). Hence, since $\bigcup_0^h \mathfrak{P}_j$ is a finite set, the lemma follows.

Most of the results in this section follow quite readily from the following theorem.

THEOREM 2.2. *Let R, Q, and $\mathscr{P}$ be as in* (2.1), *and let E be a finite R-module. If E_Q is Macaulay, then there are at most a finite number of $p \in \mathscr{P}$ such that E_p is not Macaulay.*

Proof. Let $\operatorname{Dim} E_Q = h$, and let $b_1, \cdots, b_h$ be elements in Q such that their images in R_Q are an E_Q-sequence and such that height $B_j = j (j = 0, 1, \cdots, h)$, where $B_j = (b_1, \cdots, b_j)R$. Then $\bigcup_0^h \operatorname{Ass}(E/B_jE)$

is a finite set of prime ideals in R, so, by (2.1) only finitely many ideals in $\mathscr{P}$ can contain an element in this set which isn't contained in Q or can have height greater than height $Q + 1$. Omitting these, it is readily checked that, for all other $p \in \mathscr{P}$, E_p is Macaulay.

DEFINITION 2.3. Let R be a Noetherian ring, and let $U \subseteqq \operatorname{Spec} R$. Then U is said to be *VS-open* in case the following condition holds: If $Q \in U$, then U contains all but finitely many prime ideals p in R such that $Q \subset p$ and height $p/Q = 1$. It is clear that the VS-open sets are the open sets of a topology on $\operatorname{Spec} R$ called the *very strong topology* (*VS topology*) on $\operatorname{Spec} R$.

A subset of $\operatorname{Spec} R$ which is either open or closed in the Zariski topology on $\operatorname{Spec} R$ is VS-open (for Zariski open, see (2.1)). Also, using (2.1) it is readily seen that the VS topology is Hausdorff and totally disconnected.

REMARK 2.4. 2.4.1. It follows from (2.2) that if $Q \in \operatorname{Spec} R$ is such that E_Q is Macaulay, then $\{p \in \operatorname{Spec} R;\ Q \subseteqq p$ and E_p is Macaulay$\}$ is VS-open. Thus, since, for each minimal prime ideal $Q \in \operatorname{Ass} E$, E_Q is Macaulay (since $\operatorname{Dim} E_Q = 0$), (2.2) *asserts that the Macaulay locus of E* (that is, the set of Q *in* Spec R such that E_Q is Macaulay) *is VS-open.*

2.4.2. In [3, pp. 162–163], the question of whether the Macaulay locus of R is always open in the Zariski topology was raised. An affirmative answer is given there for homomorphic images of locally Macaulay rings [3, (6.11.8)]. However, in [1, Proposition 3.5], an example is given to show that, in general, the answer is "no".

COROLLARY 2.5. *Let E be a finite module over a Noetherian ring R, let P be a prime ideal in R such that $\operatorname{Ann} E \subseteqq P$, and let height $P/(\operatorname{Ann} E) = h > 1$.*

2.5.1. *If Q is a prime ideal in R such that $Q \subset P$, height $P/Q > 1$, and E_Q is Macaulay, then, for all but a finite number of prime ideals p in R such that $Q \subset p \subset P$ and height $p/Q = 1$, E_p is Macaulay.*

2.5.2. *There exist infinitely many prime ideals p in R such that $\operatorname{Ann} E \subset p \subset P$, height $P/p = 1$, and E_p is Macaulay.*

Proof. On localizing at P, (2.5.1) follows immediately from (2.2), and (2.5.2) follows from successive applications of (2.5.1) with Q a minimal prime ideal in Ass E.

It follows from (2.19) below that in (2.5.2) infinitely many of the ideals p satisfy height $p/(\operatorname{Ann} E) = h - 1$.

Some further corollaries to (2.2) will be given below. However, before this, some topological remarks will be given. The following lemma sets the stage for these remarks. (In the lemma, R_c is the ring R_S, where $S = \{c^i; i \geqq 0\}$.)

LEMMA 2.6. *The following statements are equivalent for a prime ideal Q in a Noetherian ring R:*

2.6.1. *The set of P in Spec R containing Q is finite.*

2.6.2. *Depth $Q \leqq 1$ and R/Q is a semi-local ring.*

2.6.3. *There exists an element $c \in R$, $\notin Q$, such that QR_c is a maximal ideal in R_c.*

2.6.4. *There exists a finitely generated extension ring R' of R which contains a maximal ideal M such that $M \cap R = Q$.*

2.6.5. *Q is isolated in the VS topology on Spec R.*

Proof. (2.6.2) $\Leftrightarrow$ (2.6.4) follows from [8, (14.10)], and the rest of the proof is straightforward.

DEFINITION 2.7. A prime ideal in a Noetherian ring R which satisfies the equivalent conditions of (2.6) is a *G-ideal.*

It is easy to see that every prime ideal in R is an intersection of G-ideals. Also, the set of G-ideals is the least class of prime ideals in R which has this property. Concerning G-ideals in an arbitrary commutative ring with unit element, see [5, pp. 12–21].

DEFINITION 2.8. Let R be a Noetherian ring, and let $B \subseteqq \operatorname{Spec} R$. B is said to be *big* in case the following condition holds: If $Q \in B$ and Q isn't a G-ideal, then the set of $p \in B$ such that $Q \subset p$ and height $p/Q = 1$ is an infinite set.

REMARK 2.9. It is readily seen that VS-open sets are big. Also, if U is VS-open and B is big, then $U \cap B$ is big.

For an example of a big set which may not be VS-open, see (2.18) and the comment which follows it.

The notion of bigness is worth isolating because of the following lemma and corollaries.

LEMMA 2.10. *If R is Noetherian and $B \subseteqq \operatorname{Spec} R$ is big, then, for each $Q \in B$, $Q = \cap \{P \in B; Q \subseteqq P$ and P is a G-ideal$\}$.*

Proof. Suppose not, and let $Q \in B$ be maximal for which the lemma doesn't hold. Then Q isn't a G-ideal, so, since B is big, $\mathscr{P} = \{P \in B; Q \subset P$ and height $P/Q = 1\}$ is an infinite set, hence $Q = \cap \{P;$

$P \in \mathscr{P}\}$. By maximality of Q, a contradiction obtains, since each $P \in \mathscr{P}$ is the intersection of the G-ideals in B which contain it.

COROLLARY 2.11. *Let R be a Noetherian ring, Q a prime ideal in R, and E a finite R-module. If E_Q is Macaulay, then $Q = \cap \{p \in \mathrm{Spec}\, R;\ Q \subseteqq p,\ E_p$ is Macaulay, and p is a G-ideal$\}$.*

Proof. Let $\mathscr{P} = \{p \in \mathrm{Spec}\, R;\ Q \subseteqq p$ and R_p is Macaulay$\}$. Then $\mathscr{P}$ is VS-open (2.4.1), hence big (2.9), and so the corollary follows from (2.10).

The assertion in (2.11) that $Q = \cap p_\lambda$ says, geometrically, that the p_λ are Zariski dense in the locus of (set of prime ideals containing) Q. Thus, this section of the paper is, in two related senses, concerned with the fact that prime ideals Q such that E_Q is Macaulay are plentiful: openness in the VS topology, and density in various closed sets in the Zariski topology.

If R is a Noetherian ring of altitude one or a Noetherian domain of altitude two, then R_P is Macaulay, for all non-maximal prime ideals P in R—even if R isn't Macaulay. On the other hand, if R is a local ring with a prime ideal Q such that depth $Q \geqq 2$ and R_Q isn't Macaulay, then there are infinitely many prime ideals p in R such that $Q \subset p$ and, for each such p, R_p isn't Macaulay. Even so, the following corollary shows, with $R = E$, that, for an arbitrary Noetherian ring R of altitude greater than one, there are infinitely many prime ideals P in R such that depth $P \leqq 1$ and R_P is Macaulay. (However, as already noted, the Macaulay locus of R need not be Zariski-open [1, Proposition 3.5].)

COROLLARY 2.12. *If R is a Noetherian ring, and $E \neq (0)$ is a finite R-module, then $\mathrm{Rad}\,(\mathrm{Ann}\, E) = \cap \{p \in \mathrm{Spec}\, R;\ E_p$ is Macaulay and p is a G-ideal$\}$. Hence, $\{p \in \mathrm{Spec}\, R;\ R_p$ is Macaulay and p is a G-ideal$\}$ is Zariski dense in $\mathrm{Spec}\, R$.*

Proof. For each minimal prime ideal q in Ass E, E_q is Macaulay. Hence the corollary follows from (2.11), since $\mathrm{Rad}\,(\mathrm{Ann}\, E) = \cap \{q;\ q \in \mathrm{Ass}\, E\}$.

Two special cases of the following corollary will be given in (2.14) and (2.20).

COROLLARY 2.13. *Let I and J be ideals in a Noetherian ring R such that $I \subseteqq \mathrm{Rad}\, J$, and let E be a finite R-module. If E_Q/IE_Q is Macaulay, for each minimal prime divisor Q of J, then $\mathrm{Rad}\, J = \cap \{p;\ p$ is a G-ideal in R, $J \subseteqq p$, and E_p/IE_p is Macaulay$\}$.*

Proof. Reducing to the case $I = (0)$, the result follows from (2.11).

COROLLARY 2.14. *For each ideal J in a Noetherian ring R, Rad $J = \cap \{p; p$ is a G-ideal in $R, J \subseteqq p$, and R_p/JR_p is Macaulay$\}$.*

Proof. This follows from (2.13) with $I = J$ and $E = R$.

REMARK 2.15. 2.15.1. It follows from (2.14) and [5, Theorem 156] that, if c is a nonunit regular element in R, then Rad $cR = \cap \{p; p$ is a G-ideal in $R, c \in p$, and R_p is Macaulay$\}$. A similar statements holds for R-sequences of length greater than 1.

2.15.2. In the case $J = Q$ is prime and R_Q is Macaulay, (2.11) and (2.14) give an interesting comparison of Q expressed as an intersection of G-ideals in R. In particular, such a comparison holds for each minimal prime ideal in R; and also for each height one prime ideal in R, if R is an integral domain. Also, such a comparison holds for a radical ideal I in R such that R_p is Macaulay, for each prime divisor p of I.

DEFINITION 2.16. A *Hilbert ring* is a ring R such that each prime ideal in R is the intersection of the maximal ideals in R which contain it.

It is clear by the definition that a factor ring of a Hilbert ring is a Hilbert ring, and it is known [5, Theorem 31] that a finite extension ring of a Hilbert ring is a Hilbert ring.

COROLLARY 2.17. *If R is a Noetherian Hilbert ring and $(0) \neq E$ is a finite R-module, then Rad $(Ann\ E) = \cap \{M \in Spec\ R; M$ is a maximal ideal and E_M is Macaulay$\}$.*

Proof. The only G-ideals in R are the maximal ideals in R, hence the corollary follows from (2.12).

If P is a prime ideal in R and c is a non-nilpotent element in PR_P, then $(R_P)_c$ is a Hilbert ring [4, (10.5.8)]. Using this and (2.17), an alternate proof of (2.5.2) is readily obtained.

It will be seen in (3.3) that in some rings which are neither Hilbert, nor Macaulay, Rad R is the intersection of the maximal ideals M in R such that R_M is Macaulay.

To generalize (2.5.1) and to derive some further corollaries to (2.2), the following lemma is needed.

LEMMA 2.18. *Let $Q \subset P$ be prime ideals in a Noetherian ring R such that height $P/Q > 1$, and let $\mathscr{P} = \{p \in Spec\ R; Q \subset p \subset P\}$. Then* *there are infinitely many* $p \in \mathscr{P}$ *such that height* $p =$ *height* $Q + 1$ *and height* $P/p =$ *height* $P/Q - 1$.

Proof. In R_P/Q_P each nonzero non-unit has depth equal to height $P/Q - 1$ and has only a finite number of minimal prime divisors, at least one of which must have depth equal to height $P/Q - 1$. Hence, since at most a finite number of $p \in \mathscr{P}$ have height greater than height $Q + 1$ [6], the conclusion follows.

It follows from (2.18) that the set of prime ideals p between two prime ideals $Q \subset P$ in a Noetherian ring R such that height $p =$ height $Q +$ height p/Q and height $P/p =$ height P/Q-height p/Q is big in Spec R_P (where Spec $R_P \subseteqq$ Spec R in a natural way). Using [8, Example 2, pp. 203–205] an example can be given in which the set isn't VS-open.

Up to now we've seen that if E_Q is Macaulay and Q isn't a G-ideal, then $Q = \cap p_\lambda$, where E_{p_λ} is Macaulay and either height $p_\lambda =$ height $Q + 1$ or where the p_λ are G-ideals. The intermediate cases are given in the following corollary.

COROLLARY 2.19. *Let* R *be a Noetherian ring, let* $Q \subset P$ *be prime ideals in* R *such that height* $P/Q = d$, *and let* E *be a finite* R*-module. If* E_Q *is Macaulay, then, for each* $i = 0, 1, \cdots, d - 1$, $Q = \cap \{p; p \in \mathscr{P}_i\}$, *where* $\mathscr{P}_i = \{p \in \text{Spec } R; Q \subseteqq p \subset P, E_p$ *is Macaulay, height* $p =$ *height* $Q + i$, *height* $P/p = d - i$, *and* $i =$ *height* $p/Q\}$.

Proof. This follows from successive applications of (2.5.1) and (2.18).

(2.19) Holds, with $E = R$, for each minimal prime ideal Q in R (and for all but a finite number of height one prime ideals Q in R) with P a maximal ideal in R such that $Q \subset P$. If, moreover, depth $Q = d < \infty$, then P may be chosen such that height $P/Q = d$.

(2.20) shows that in (2.19) the case $i = d$ can be included when R is a Hilbert ring.

COROLLARY 2.20. *Let* E *be a finite* R*-module and let* Q *be a prime ideal in a Noetherian Hilbert ring* R *such that* E_Q *is Macaulay. Then* $Q = \cap \{M; M \in \mathscr{S}\}$, *where* $\mathscr{S} = \{M; M$ *is a maximal ideal in* R, $Q \subseteqq M$, *and* E_M *is Macaulay*$\}$. *Moreover, if depth* $Q = d < \infty$, *then* $Q = \cap \{M; M \in \mathscr{S}$ *and height* $M =$ *height* $Q + d\}$.

Proof. The first statement follows from (2.13) with $I = (0)$ and $J = Q$. For the second statement, if $d = 0$, then the conclusion is obvious. If $d > 0$, then let $\mathscr{P} = \{p \in \text{Spec } R; Q \subset p$, height $p =$ height $Q + 1$, and depth $p = d - 1\}$. Then $\mathscr{P}$ is an infinite set (if $d > 1$, by (2.18); if $d = 1$, by [6], since R is Hilbert), hence $\mathscr{P}' = \{p \in \mathscr{P};\ E_p$ is Macaulay$\}$ is an infinite set, by (2.2). Therefore, $Q = \cap \{p;\ p \in \mathscr{P}'\}$. Since each $p \in \mathscr{P}'$ satisfies depth $p = d - 1$, the corollary follows by induction on d.

COROLLARY 2.21. *Let R and E be as in* (2.20), *let $Q \subset P$ be prime ideals in R such that height $P/Q > 1$ and E_Q is Macaulay, and let $N_1, \cdots, N_g$ be prime ideals in R such that $P \not\subseteq \cup N_j$. Then, for $i = 1, \cdots, d-1$, $Q = \cap \{p;\ p \in \mathscr{P}_i\}$, where $\mathscr{P}_i = \{p \in \operatorname{Spec} R;\ Q \subset p \subset P$, height $p =$ height $Q + i$, height $P/p =$ height $P/Q - i$, $i =$ height p/Q, $p \not\subseteq \cup N_j$, and E_p is Macaulay$\}$.*

Proof. If it can be proved that $Q = \cap \{p;\ p \in \mathscr{P}_1\}$, then the result for $i > 1$ readily follows from (2.19), so only the case $i = 1$ will be considered. Let $\mathscr{P}'$ be the finite set of prime ideals p in R such that $Q \subset p \subset P$ and height $p/Q = 1 <$ height $p -$ height Q [6]. Let $b_1, \cdots, b_h \in Q$ such that their images in R_Q are a maximal E_Q-sequence and such that height $(b_1, \cdots, b_j)R = j\ (j = 0, 1, \cdots, h)$. Let $\mathscr{D}_j$ be the set of prime ideals in Ass $E/(b_1, \cdots, b_j)E$ which don't contain P, and let $\mathscr{P}^* = (\bigcup_0^h \mathscr{D}_j) \cup \mathscr{P}' \cup \{N_1, \cdots, N_g\}$. Then $\mathscr{P}^*$ is a finite set of prime ideals in R and $P \not\subseteq \cup \{q;\ q \in \mathscr{P}^*\}$. Therefore, for each positive integer n, there exist $c_1, \cdots, c_n$ in P and not in any prime ideal in $\mathscr{P}^*$ such that no height one prime ideal in R/Q contains more than one c_k $(k = 1, \cdots, n)$, since each $(Q, c_k)R$ has only a finite number of minimal prime divisors. Hence, if p_k is a minimal prime divisor of $(Q, c_k)R$ which is contained in P and is such that height $P/p_k =$ height $P/Q - 1$, then E_{p_k} is Macaulay and $p_k \not\subseteq \cup N_i$. It follows that $Q = \cap \{p;\ p \in \mathscr{P}_1\}$.

To obtain another corollary to (2.2), the following lemma is needed.

LEMMA 2.22. *Let $\mathscr{S}$ be a set of prime ideals in a Noetherian ring R, let $I = \cap \{p;\ p \in \mathscr{S}\}$, let P be a minimal prime divisor of I, and let $c \in R$, $\notin P$. Then the following statements hold for $\mathscr{S}' = \{p \in \mathscr{S};\ c \notin p\}$:*

2.22.1. *$P = \cap \{p;\ p \in \mathscr{T}\}$, and $PR_c = \cap \{pR_c;\ p \in \mathscr{T}\}$, where $\mathscr{T} = \{p \in \mathscr{S}';\ P \subseteqq p\}$.*

2.22.2. *If $\mathscr{S}$ is big, then $\mathscr{S}'$ is big.*

2.22.3. *If $\mathscr{S}$ is VS-open, then $\mathscr{S}'$ is VS-open.*

Proof. For (2.22.1) let $K = \cap \{p \in \mathscr{S};\ c \in p$ and $P \subseteqq p\}$. Then $K \cap (\cap \mathscr{T}) = P$, hence $P = \cap \mathscr{T}$, and (2.22.1) follows from this. The last two stements follow from the definitions and the fact that R/p

is Noetherian, for each $p \in \mathscr{S}$.

COROLLARY 2.23. *Let* R, E, *and* Q *be as in* (2.20), *so* E_Q *is Macaulay, and let* $b \in R$, $\notin Q$. *Then for each maximal ideal* M *in* R *such that* $Q \subset M$, $Q = \cap \{p \in \mathscr{S};$ *height* $M/p = 1$ *and height* $p =$ *height* $Q +$ *height* $M/Q - 1\}$, *where* $\mathscr{S} = \{p \in \mathrm{Spec}\, R;\ Q \subseteqq p \subset M,\ b \notin p$, *and* E_p *is Macaulay*$\}$. *Moreover, if* R *is local, then* $\mathscr{S}$ *is VS-open.*

Proof. This is clear if height $M/Q = 1$, and, if height $M/Q > 1$, then the first statement follows from (2.19) and (2.22.1), and the last statement follows from (2.4.1) and (2.22.3).

REMARK 2.24. 2.24.1. In (2.23), if, moreover, R is a Hilbert ring, then $Q = \cap \{M; M$ is a maximal ideal in R, $Q \subset M$, $b \notin M$, and R_M is Macaulay$\}$ by (2.20) and (2.22.1).

2.24.2. If R satisfies the first chain condition for prime ideals [8, p. 123], then it follows from (2.23) that $Q = \cap \{p \in \mathrm{Spec}\, R;\ Q \subseteqq p,\ b \notin p$, depth $p = 1$, and R_p is Macaulay$\}$. Also, depth $p = 1$ if and only if height $p =$ altitude $R - 1$.

2.24.3. It follows immediately from (2.23) that, if P is a prime ideal in a Noetherian ring R and $b \in P$ is such that b is not in some minimal prime ideal $q \subset P$, then $q = \cap \{p \in \mathrm{Spec}\, R;\ q \subseteqq p \subset P,\ b \notin p$, height $P/p = 1$, height $p =$ height $P/q - 1$, and R_p is Macaulay$\}$.

One final corollary, (2.25) below, which pertains to rather recent research in local ring theory will be given. The following background information on the corollary should be noted: It is known that, if P is a prime ideal in an unmixed (resp., quasi-unmixed) local ring R, then R_P is unmixed (resp., quasi-unmixed) [7, Proposition 6] (resp., [10, Lemma 2.5]). It was recently shown that there exist quasiunmixed local rings which are not unmixed (by [1, Proposition 3.3] together with [12, Proposition 3.5]). Thus it seems natural to inquire if there exist prime ideals P of depth one in a quasi-unmixed local ring R such that R_P is unmixed. It follows from (2.25) below that the answer is yes.

COROLLARY 2.25. *Let* R *be a local ring such that altitude* $R = a > 1$. *Then* $\mathscr{S} = \{p \in \mathrm{Spec}\, R;\ R_p$ *is unmixed*$\}$ *is VS-open in* Spec R *and* $\{p \in \mathscr{S};$ *height* $p = a - 1\}$ *is an infinite set and is Zariski dense in* Spec R.

Proof. Tne first statement follows from (2.4.1) with $E = R$, since a Macaulay local ring is unmixed [8, (34.9)]. The second statement follows from [8, (34.9)] and (2.19) with Q a minimal prime ideal and P the maximal ideal in R.

This section will be closed with the following proposition and remarks. The proposition is closely related to a number of the results in this section, and it gives, in particular, some interesting supplementary information to (2.5.2) in the case $E = R$.

PROPOSITION 2.26. *Let $P_1, \cdots, P_e$ be prime ideals in a Noetherian ring R which have the same height say height $P_i = a > 0$. Then there exist $b_1, \cdots, b_{a-1}$ in $\cap P_i$ such that, with $B_j = (b_1, \cdots, b_j)R$:*

2.26.1. *Height $B_j = j (j = 0, 1, \cdots, a-1)$.*

2.26.2. *If $Q \not\supseteq P_i (i = 1, \cdots, e)$ is a prime ideal in R such that $B_j \subseteq Q$, for some $j \geqq 1$, then the images of $b_1, \cdots, b_j$ in R_Q are an R_Q-sequence.*

2.26.3. *If P is a minimal prime divisor of B_k, for some k $(0 \leqq k \leqq a-1)$, then R_P is Macaulay and B_jR_P is a primary ideal $(j = 0, 1, \cdots, k)$.*

2.26.4. *If $a > 1$ and $\mathscr{P}_i$ is the infinite set of height $a-1$ prime ideals $p \subset P_i$ such that $B_{a-2} \subset p$ and R_p is Macaulay, then, for each $i = 1, \cdots, e$, at most a finite number of $p \in \mathscr{P}_i$ don't satisfy: B_jR_p is primary, for $j = 0, 1, \cdots, a-2$.*

Proof. The proposition is trivial for $a = 1$, so it may be assumed that $a > 1$. Let $\mathfrak{P}_0$ be the set of prime divisors q of zero in R such that $q \not\supseteq P_i (i = 1, \cdots, e)$, let $\mathfrak{E}_0$ be the set of minimal prime divisors of zero in R, and let $\mathfrak{A}_0$ be the set of height one prime ideals in R which contain more than one element in $\mathfrak{P}_0$. Then, $\mathfrak{A}_0 \cup \mathfrak{P}_0$ is a finite set of prime ideals, by (2.1). Thus, since $a > 1$, there exists $b_1 \in \cap P_i$ such that b_1 isn't in any ideal in $\mathfrak{A}_0 \cup \mathfrak{P}_0$. Then, with $B_1 = b_1R$, it is readily checked that, if $a = 2$, then (2.26.1)—(2.26.3) hold; and (2.26.4) holds by (2.1).

Assume $a > 2$ and $\mathfrak{P}_{k-1}, \mathfrak{E}_{k-1}, \mathfrak{A}_{k-1}$, and $b_1, \cdots, b_k$ have been defined $(1 \leqq k < a-1)$, and let $\mathfrak{P}_k$ be the set of prime divisors q of B_k such that $q \not\supseteq P_i (i = 1, \cdots, e)$. Assume (2.26.1) holds for $j = 0, 1 \cdots, k$, let $\mathfrak{E}_k$ be the set of minimal prime divisors of B_k, and let $\mathfrak{A}_k$ be the set of height $k+1$ prime ideals in R which contain an element in $\mathfrak{E}_k$ and also contain more than $k+1$ prime ideals in $\bigcup_0^k \mathfrak{P}_j$. Assume further that $b_1, \cdots, b_k$ have been chosen such that (2.26.2) holds for $j \leqq k$ and, for $0 \leqq h \leqq k$ and each $p \in \mathfrak{E}_h$, B_jR_p is a primary ideal $(j = 0, 1, \cdots, h)$. Then $U = (\bigcup_0^k \mathfrak{P}_j) \cup \mathfrak{A}_k$ is a finite set of prime ideals, by (2.1). Therefore, there exists b_{k+1} in $\cap P_i$ which is not in any prime ideal in U. Then it is easily checked that, if P is a minimal prime divisor of $B_{k+1} = (B_k, b_{k+1})R$, then R_P is Macaulay and B_jR_P is a primary ideal $(j = 0, 1, \cdots, k+1)$, and (2.26.1) and (2.26.2) hold for $j \leqq k+1$. Therefore, it follows that the desired elements $b_1, \cdots, b_{a-1}$ exist such that (2.26.1)—(2.26.3) hold.

To show that (2.26.4) holds, note that, if $p \in \mathscr{P}_i$, then there exists a minimal prime divisor q of B_{a-2} contained in p, and then $B_j R_q$ is primary ($j = 0, 1, \cdots, a-2$), by (2.26.3). Hence, since B_{a-2} has only a finite number of minimal prime divisors contained in P_i, (2.26.4) holds by (2.1).

REMARK 2.27. Let the notation be as in (2.26).

2.27.1. If the P_i are maximal, then (2.26.2) shows that there exist $b_1, \cdots, b_{a-1}$ in $\cap P_i$ such that, for all prime ideals $P \notin \{P_1, \cdots, P_e\}$ such that $B_j \subseteqq P$, for some j, the images in R_P of $b_1, \cdots, b_j$ is an R_P-sequence.

2.27.2. It follows from (2.1) and (2.26.3) that, if R is Hilbert and altitude $R = a$, then there are infinitely many maximal ideals M in R such that R_M is Macaulay and $B_j R_M$ is primary ($j = 0, 1, \cdots, a-1$).

2.27.3. If P is a prime ideal in R such that $a =$ height $P \geqq 2$, then (2.26.4) shows that there exist an infinite subset $\mathscr{P}'$ of the set of prime ideals $Q \subset P$ such that height $Q =$ height $P - 1$ and R_Q is Macaulay such that $Q \in \mathscr{P}'$ if and only if there exists a system of parameters $c_1, \cdots, c_{a-1}$ in R_Q such that $(c_1, \cdots, c_j)R_Q$ is a primary ideal ($j = 0, 1, \cdots, a-1$).

2.27.4. Let $Q_1, \cdots, Q_z$ be prime ideals in R such that no $P_i (i = 1, \cdots, e)$ is contained in $\bigcup Q_j$. Then the proof of (2.26) can be readily adapted to show that the elements $b_1, \cdots, b_{a-1}$ can be chosen to satisfy the further condition that no b_h is contained in $\cup Q_j$.

REMARK 2.28. It is natural to inquire if (2.5.2) holds on replacing Macaulay by Gorenstein (or, regular). The answer is no. For, let $R = A[X_1, \cdots, X_n]$, where (A, q) is a primary ring whose zero ideal isn't irreducible, and where the X_i are indeterminates. Then, for each prime ideal P in R, R_P isn't Gorenstein, since $qR \subseteqq P$ and R_{qR} isn't Gorenstein.

3. Condition (*) and affine rings.

In this section two theorems concerning a finitely generated ring A over a Noetherian ring R will be proved. The first, Theorem 3.9, shows, in particular, that if A is an integral domain, then condition (*) (Definition 3.2) is inherited by A, and the second, Theorem 3.12, shows that, if R is a semi-local domain and altitude $R > 1$, then "most" finitely generated integral domains over R which aren't integrally dependent on R satify condition (*).

Definition 3.1. For a ring R, let $\mathscr{M}(R)$ be the set of maximal ideals M in R such that R_M is Macaulay.

Definition 3.2. A ring R is said to satisfy *condition* (*) in case

$\text{Rad } R = \cap \{M; M \in \mathscr{M}(R)\}$.

It follows from (2.17) that a Noetherian Hilbert ring satisfies condition (*). The next lemma shows that there exist rings which satisfy condition (*) and yet are neither locally Macaulay nor Hilbert rings (since $R[X]$ is Hilbert if and only if R is Hilbert [5, Theorem 31]).

LEMMA 3.3. *If R is a Noetherian ring, then, for all $n \geqq 1$, the polynomial ring $R[X_1, \cdots, X_n]$ satisfies condition* (*).

Proof. It may clearly be assumed that $n = 1$. Let $P \in \mathscr{G} = \{P; P \text{ is a } G\text{-ideal in } R \text{ and } R_P \text{ is Macaulay}\}$. ($\mathscr{G}$ isn't empty by (2.12) with $E = R$.) If P is a maximal ideal in R, then let $\mathscr{P}(P)$ be the set of maximal ideals in $R[X]$ which contain $PR[X]$. Then $R[X]_N$ is Macaulay, for each $N \in \mathscr{P}(P)$, and $PR[X] = \cap \{N; N \in \mathscr{P}(P)\}$. On the other hand, if P isn't maximal, then let $M_1, \cdots, M_e$ be the maximal ideals in R which contain P. Let $b \in \cap M_j, \notin P$, and, for $i > 0$, let $P_i = (P, b^i X - 1)R[X]$. Then the P_i are distinct prime ideals, each $R[X]_{P_i}$ is Macaulay, $PR[X] = \cap \{P_i; i > 0\}$, and each P_i is a maximal ideal in $R[X]$. Hence, since $\text{Rad } R = \cap \{P; P \in \mathscr{G}\}$ (2.12), the lemma follows from $\text{Rad } R[X] = (\text{Rad } R) \cdot R[X] = \cap \{PR[X]; P \in \mathscr{G}\}$.

COROLLARY 3.4. *If I is an ideal in a Noetherian ring R and X is an indeterminate, then $\text{Rad } IR[X] = \cap \{M; IR[X] \subset M \text{ and } M/IR[X] \in \mathscr{M}(R[X]/IR[X])\}$. In particular, if b is a nonunit regular element in R, then $\text{Rad } bR[X] = \cap \{M; b \in M \in \mathscr{M}(R[X])\}$.*

Proof. This follows from $R[X]/IR[X] = (R/I)[X]$ and [5, Theorem 156].

On the other hand, of course, there are height one prime ideals in $R[X]$ which aren't even an intersection of maximal ideals; for example, $XR[X]$ when R is a local domain.

To prove the first theorem in this section, a number of lemmas will first be proved.

LEMMA 3.5. *Let R be a Noetherian ring.*

3.5.1. *If A is a Noetherian ring such that A is integrally dependent on R and $R \subseteqq A \subseteqq R_c$, for some nonzero-divisor $c \in R$, and if A satisfies condition* (*), *then R does.*

3.5.2. *If R satisfies condition* (*), *then, for all nonzero-divisors c in R and for each ring A such that $R \subseteqq A \subseteqq R_c$, A satisfies condition* (*).

3.5.3. *If R satisfies condition* (*), *then each free principal integral* *extension ring of R (that is, $R[X]/(f)$, where f is monic) satisfies condition* (*).

Proof. 3.5.1. *If A* satisfies condition (*), then $A_c = R_c$ does and $\cap \{MA_c; M \in \mathscr{M}(A)\} = (0)$, by (2.22.1). Hence, since A is integrally dependent on R, R satisfies condition (*). For (3.5.2), if R satisfies condition (*), then $\cap \{MR_c; M \in \mathscr{M}(R)\} = (0)$, by (2.22.1), so $\cap \{MR_c \cap A; M \in \mathscr{M}(R)\} = (0)$. Hence, since $R_c/MR_c = R/M$ and $R_M = (R_c)_{MR_c}$ (if $c \notin M$), A satisfies condition (*).

For (3.5.3), let $B = R[X]/(f)$. Then, for each maximal ideal M in R, $(\text{Rad}\, MB)^d \subseteqq MB$, where d is *the degree of f* (consider $R[X]/(M, f)$). Therefore, if $u \in \cap \{N; N \in \mathscr{M}(B)\}$, then, for each $M \in \mathscr{M}(R)$, $u \in \text{Rad}\, MB$ (by freeness), so $u^d \in \cap \{MB; M \in \mathscr{M}(R)\} =$ (by freeness) $(\cap \{M; M \in \mathscr{M}(R)\})B$, and so u^d is nilpotent. Thus $u \in \text{Rad}\, B$, hence B satisfies condition (*).

In (3.6), we shall utilize [3, (6.10.6)]. The result is essentially local, and passing from the language of preschemes to the language of commutative rings we find that it asserts the following:

Let R be a Noetherian ring, let E be a finite R-module, and let I be an ideal in R such that $q = \text{Rad}\, I$ is prime and $E_q \neq (0)$. Then there exists $r \in R$, $\notin q$ such that, for each prime ideal Q in R such that $q \subseteqq Q$ and $r \notin Q$, the following holds:

$$\text{Dim}\, E_Q = \text{Dim}\, E_q + \text{Dim}\, (R/I)_Q \text{ and } \text{Prof}\, E_Q = \text{Prof}\, E_q + \text{Prof}\, (R/I)_{Q/I}\,.$$

LEMMA 3.6. *Let q be a minimal prime ideal in a Noetherian ring R. Then there exists an element $r \in R$, $\notin q$ such that, for each prime ideal Q in R which contains q but not r, R_Q is Macaulay if and only if $(R/q)_{Q/q}$ is Macaulay.*

Proof. If $s \in R$, $\notin q$, then it clearly suffices to prove the lemma for R_s instead of R. Hence it may be assumed that q is nilpotent. Then, with $q = I$ and $R = E$ in Grothendieck's result quoted above, $\text{Dim}\, E_q = \text{Prof}\, E_q = 0$ (since q is nilpotent). Therefore, for each prime ideal Q in R such that $r \notin Q$, altitude $R_Q =$ altitude $(R/q)_{Q/q}$ and $\text{Prof}\, R_Q = \text{Prof}\, (R/q)_{Q/q}$. Clearly, then, for each such prime ideal Q in R, R_Q is Macaulay if and only if $(R/Q)_{Q/q}$ is Macaulay.

LEMMA 3.7. *A Noetherian ring R satisfies condition* (*) *if and only if, for each minimal prime ideal q in R, R/q satisfies condition* (*).

Proof. Let R satisfy condition (*) and let q be a minimal prime ideal in R. Choose $r \in R$ as in (3.6). Since $q = \cap \{M; q \subseteqq M \in \mathscr{M}(R)\}$, by hypothesis, $q = \cap \{M; q \subseteqq M \in \mathscr{M}(R)$ and $r \notin M\}$ (2.22.1). Hence, by (3.6), R/q satisfies condition (*).

Conversely, let $q_1, \cdots, q_g$ be the minimal prime ideals in R, and assume that each R/q_i satisfies condition (*). Then, for each $i = 1, \cdots, g$ and with r_i as in (3.6), $q_i = \cap \{M; q \subseteqq M, M/q_i \in \mathscr{M}(R/q_i)$, and $r_i \notin M\}$, by (2.22.1). Thus, by (3.6), $q_i = \cap \{M; q_i \subseteqq M \in \mathscr{M}(R)$ and $r_i \notin M\}$ $(i = 1, \cdots, g)$, hence R satisfies condition (*).

COROLLARY 3.8. *A Noetherian ring R satisfies condition* (*) *if and only if $R/(\mathrm{Rad}\, R)$ satisfies condition* (*).

Proof. Clear by (3.7).

Since a finitely generated ring over a Noetherian Hilbert ring is again such a ring, it follows from (2.17) that a finitely generated ring over a Noetherian Hilbert ring satisfies condition (*). The following theorem can be considered a generalization of this result.

THEOREM 3.9. *Let A be a finitely generated extension ring of a Noetherian ring R, and assume R satisfies condition* (*). *Then A satisfies condition* (*) *if, for each minimal prime ideal Q in A, at least one of the following conditions holds*:

3.9.1. *$Q \cap R$ is minimal.*

3.9.2. *$R/(Q \cap R)$ satisfies condition* (*).

3.9.3. *A/Q is not algebraic over $R/(Q \cap R)$.*

Proof. A/Q satisfies condition (*) if (3.9.3) holds, by (3.3). Also (3.9.1) implies (3.9.2), by (3.7). Therefore, since A satisfies condition (*) if each A/Q does (3.7), it suffices to prove: *If A is a finitely generated integral domain over R and R satisfies condition* (*), *then A satisfies condition* (*).

For this, there exist algebraically independent elements $X_1, \cdots, X_n$ in A over R, and elements $a_1, \cdots, a_k$ in A integral over $R_n = R[X_1, \cdots, X_n]$ such that $B = R_n[a_1, \cdots, a_k] \subseteqq A \subseteqq B[1/b]$, for some nonzero element $b \in B$. Therefore, by (3.5.2), it suffices to prove B satisfies condition (*). By (3.3), R_n satisfies condition (*), so it may be assumed that $B = R[a_1, \cdots, a_k]$. Then the coefficients of the minimal polynomial of a_1 over the quotient field of R are in a finite integral extension R_1 of R contained in the quotient field of R, and $R_1[a_1]$ is a free principal integral extension domain of R_1. Therefore, by (3.5.2) and (3.5.3), $R_1[a_1]$ satisfies condition (*), hence $R[a_1]$ does, by (3.5.1). Therefore, the theorem follows by induction on k.

COROLLARY 3.10. *Let R and A be as in* (3.9). *If A contains an indeterminate over R, say t, such that, for each minimal prime ideal Q in A, $Q \cap R[t] = (Q \cap R)R[t]$, then A satisfies condition* (*).

Proof. This follows from (3.3) and (3.9), since $Q \cap R[t] = (Q \cap R)R[t]$ implies $\operatorname{trd}(A/Q)/(R/(Q \cap R)) > 0$.

Condition (b) in (3.9) suggests a way to construct extensions for which R satisfies condition (*) and A does not. In fact, let R be a Noetherian ring which satisfies condition (*) and which has a prime ideal P such that R/P doesn't satisfy condition (*). (For example, let R_0 be a local domain such that altitude $R_0 = 1$, let $R = R_0[X]$, where X is an indeterminate, and let $P = XR$.) Let $A = (R/P) \oplus R$, and let $f: R \rightarrow A$ by $f(r) = (r + P, r)$. Then A is finitely generated over $f(R)$ by $(1 + P, 0)$, $f(R)$ satisfies condition (*) (since f is a monomorphism), but A doesn't satisfy condition (*), by (3.7), since $Q = (0, 1)A$ is a minimal prime ideal in A such that $A/Q = R/P$ doesn't satisfy condition (*).

The following lemma is needed to shorten the proof of (3.12) below.

LEMMA 3.11. *Let b and c be non-unit regular elements in a Noetherian ring R, let $y = c/b$, let $I = (y, b)R[y]$, and assume $b \notin \operatorname{Rad} cR$. If $I \neq R[y]$, then height $I = 2$.*

Proof. If $1 \notin I$, then height $I \leqq 2$. Suppose Q is a height one prime ideal in $R[y]$ such that $I \subseteqq Q$. Localizing at $Q \cap R$, it may be assumed that R is a local ring with maximal ideal M and $Q \cap R = M$. Since height $Q = 1$, there exist $s \in R[y]$, $\notin Q$, and $n > 0$ such that $sb^n \in yR[y]$. Therefore, with $s = \Sigma r_i y^i (r_i \in R)$, $r_0 b^n \in yR[y]$. Now $r_0 \notin M$, since $s \notin Q$, so r_0 is a unit in R and $b^n \in yR[y]$. But multiplying by a suitable power of b will clear of fractions on the right and will show that $b \in \operatorname{Rad} cR$, a contradiction. Thus height $I = 2$.

Following the proof of the next theorem, an example will be given to show that the assumption that altitude $R > 1$ is necessary. Before stating the theorem, it should be noted that there may exist height one maximal ideals in the integral closure of a Noetherian domain R, even if R is local and altitude $R > 1$; for example, see [8, Example 2, pp. 203-205].

THEOREM 3.12. *Let A be a finitely generated integral domain* *over a semi-local domain R, and let altitude $R > 1$. A satisfies condition (*) if and only if $A \not\subseteq R'_c$, where R' is the integral closure of R in the quotient field of A and c is either a unit in R' or depth $cR' = 0$.*

Proof. If $\operatorname{trd} A/R > 0$, then A satisfies condition (*), by (3.9), and A isn't contained in any quotient ring of R', so it may be assumed that A is algebraic over R. Then, since there is a finite integral extension ring B of R contained in A such that B and A have the same quotient field, it may be assumed that R and B have the same quotient field.

Assume first that $A \subseteq R'_c$, for some such element $c \in R'$. Then, since A is finitely generated over R, there exist $c_1, \cdots, c_k$ in R' such that $B = R[c, c_1, \cdots, c_k] \subseteq A[c] \subseteq B_c$. Since B_c has only a finite number of maximal ideals (since its integral closure R'_c does (every prime ideal in B which contains c has height one, since depth $cR' = 0$)), B_c doesn't satisfy condition (*). Therefore, since $(A[c])_c = B_c$, it follows from (3.5.2) that $A[c]$ doesn't, hence A doesn't satisfy condition (*), by (3.9).

Conversely, assume, for each such element $c \in R'$, $A \not\subseteq R'_c$. Then, there exists $x \in A$ such that $x \notin R'_{M'}$, for some maximal ideal M' in R' such that height $M' > 1$. Fix one such M'. By (3.5.1), by adjoining to R a finite number of elements from R', it may be assumed that R and R' have the same number of maximal ideals, so, in particular, $R'_{M'}$ is the integral closure of R_M, where $M = M' \cap R$. If $1/x \in R'_{M'}$, then $1/x \in M'R'_{M'} \cap R_M[1/x]$, so $R_M[x] = R_M[1/x, x]$ is a Noetherian Hilbert domain [4, (10.5.8)]. Let $\mathscr{P} = \{P \in \operatorname{Spec} R;\ P \subset M,\ R/P$ is local, depth $P = 1$, and R_P is Macaulay$\}$, let $x = b/c$ with b and c in M, and let $\mathscr{S} = \{N \in \mathscr{M}(R_M[x]);\ N = P(R_M)_c \cap R_M[x]$, for some $P \in \mathscr{P}\}$. (For each $P \in \mathscr{P}$, $P(R_M)_c \cap R_M[x]$ is maximal, since each $P(R_M)_c$ is maximal and $R_M[x]$ is Hilbert.) Then $\cap \{P;\ P \in \mathscr{P}\} = (0)$ (by 2.21) with $Q = (0)$ and $N_1, \cdots, N_g$ the other maximal ideals in R), so $\cap \{N;\ N \in \mathscr{S}\} = (0)$, by (2.22.1). Fix $N \in \mathscr{S}$, let $p = N \cap R[x]$, and let Q be a maximal ideal in $R[x]$ such that $p \subseteq Q$. Then $Q \cap R \subseteq M$, since $p \cap R \in \mathscr{P}$. Therefore, since N is maximal, it follows that $Q = p$ (since M' is lost in $R'[x]$ implies M is lost in $R[x]$). Hence, since $\cap \{p;\ p = N \cap R[x]$, for some $N \in \mathscr{S}\} = (0)$, $R[x]$ satisfies condition (*). Therefore, A satisfies condition (*), by (3.9).

Therefore, assume x and $1/x \notin R'_{M'}$, and let $x = b/c$ with b and $c \in M'$. Then $bR' : cR'$ and $cR' : bR'$ have no common prime divisors, since, for each height one prime ideal p in R', b/c or $c/b \in R'_p$. Let $d \in bR' : cR'$ such that d isn't in any prime divisor of $cR' : bR'$, and let $e \in cR' : bR'$ such that $dc = be$. Then $x = d/e$, so $(d, e)R' \subseteq M'$ (since $M'R'[x]$ is a depth one prime ideal [17, Corollary, p. 20]), so it may be assumed that $b = d$ and $c = e$ and, by (3.5.1), that b and c are in R. Then $b \notin \operatorname{Rad} cR$, since, for each prime divisor p of $cR' : bR'$, p is a prime divisor of cR', hence $p \cap R$ is a prime divisor of cR [8, (33.11)] and $b \in p \cap R$. Let $y = 1/x = c/b$, and fix $j > 0$. Then bc is not in any minimal prime divisor of $(y - b^j)R[y]$; for, if Q is a height one prime ideal in R such that $(bc, y - b^j)R[y] \subseteqq Q$, then b or c and $y - b^j$ are in Q, and this implies the contradiction $(b, y)R[y] \subseteqq Q$ (3.11). Thus, for all $j \geqq 1$, there exists a minimal prime divisor p_j of $(c - b^{j+1})R$ contained in M such that $bc \notin p_j$. Fix j, and let $\mathscr{P}_j = \{P \in \mathscr{P}; bc \notin P \text{ and } p_j \subseteqq P\}$, where $\mathscr{P}$ is as in the preceding paragraph. Then it follows from (2.21) and (2.23) (with $P = M$, $Q = p_j$, and $N_1, \cdots, N_g$ the other maximal ideals in R) that $p_j = \cap \{P; P \in \mathscr{P}_j\}$, so $p_j R_c \cap R[x] = \cap \{N; N \in \mathscr{S}'\}$, where $\mathscr{S}' = \{N; N = PR_c \cap R[x]$, for some $P \in \mathscr{P}_j\}$ (2.22.1). Now, if $N \in \mathscr{S}'$ then N is a maximal ideal in $R[x]$ (since $b^j x - 1 \in N$ and $R/P \subseteqq R[x]/N \subseteqq R_c/PR_c =$ the quotient field of R/P). Since $\cap \{p_j R_c \cap R[x]; j \geqq 1\} = (0)$, $R[x]$ satisfies condition (*). Hence A satisfies condition (*), by (3.9).

A necessary and sufficient condition for A to satisfy condition (*) was just given in (3.12), assuming $a =$ altitude $R > 1$. If $a = 1$, then the condition isn't necessary. For, let R be a discrete valuation ring whose maximal ideal is generated by c, and let $A = R_c$. Then A is finitely generated over R, $A \subseteqq R_c = R'_c$, and depth $cR' = 0$, but A satisfies condition (*), since A is a field. On the other hand, the condition is sufficient when $a = 1$. For, if $A \nsubseteqq R'_c$, for all nonzero $c \in R'$, then A isn't contained in the quotient field of R' (since $a = 1$ and R' is quasi-semi-local), so A is transcendental over R. Therefore A is finitely generated over a Noetherian ring of altitude greater than one, hence A satisfies condition (*), by (3.12).

COROLLARY 3.13. *Let R be a semi-local domain such that altitude $R > 1$, let S be the integral closure of R in its quotient field, and let A be a finitely generated integral domain over R such that A isn't integral over R. Then A satisfies condition (*) in each of the following cases:*

3.13.1. *$R = S$.*
3.13.2. *S is quasi-local.*
3.13.3. *S has no height one maximal ideals.*
3.13.4. *R satisfies the second chain condition for prime ideals.*

Proof. (3.13.1) follows from (3.12), since the integral closure R' of R in the quotient field of A has no height one maximal ideals [8, (10.14)]. Clearly, (3.13.2) implies (3.13.3), and (3.13.4) implies (3.13.3)

[12, Theorem 3.1 and Proposition 3.5]. Finally, (3.13.3) implies that R' has no height one maximal ideals [8, (10.14)], so (3.13.3) follows from (3.12).

4. **Rees rings and R-sequences.** Let $B = (b_1, \cdots, b_w)R$ be an ideal in a semi-local ring $(R; M_1, \cdots, M_e)$ such that $B \subseteqq J = \cap M_i$, let t be an indeterminate, and let $u = 1/t$. The *Rees ring* $\mathscr{R} = \mathscr{R}(R, B)$ *of R with respect to B* is defined to be the subring $\mathscr{R} = R[tb_1, \cdots, tb_w, u]$ of $R[t, u]$.

The following remark summarizes the basic facts on Rees rings which will be used in what follows.

REMARK 4.1. The elements in $\mathscr{R}$ are finite sums $\sum_{-m}^{n} c_i t^i$, where $c_i \in B^i$ (with the convention that $B^i = R$, if $i \leqq 0$). Thus, $\mathscr{R}$ is a graded Noetherian ring, u isn't a divisor of zero in $\mathscr{R}$, and $u^i\mathscr{R} \cap R = B^i$, for all $i \geqq 0$. For a homogeneous ideal H in $\mathscr{R}$ and $-\infty < n < \infty$, let $[H]_n$ denote the set of elements $r \in R$ such that $rt^n \in H$. Then, if also K is a homogeneous ideal in $\mathscr{R}$, then $[H + K]_n = [H]_n + [K]_n$, and $H \subseteqq K$ if and only if $[H]_n \subseteqq [K]_n$, for all n. Also, $B^n \supseteqq [H]_n \supseteqq [H]_{n+1} \supseteqq B[H]_n$, for all n [16, p. 11]. In particular, for an ideal C in R, let $C^* = CR[t, u] \cap \mathscr{R}$, so C^* and $(C^*, u)\mathscr{R}$ are homogeneous ideals in $\mathscr{R}$, and $[(C^*, u)\mathscr{R}]_n = (C \cap B^n) + B^{n+1}$. It follows that $B^* = (b_1t, \cdots, b_wt)\mathscr{R}$, and $\mathscr{M}_i = (M_i^* u)\mathscr{R} = (M_i, u, B^*)\mathscr{R}$ are the maximal homogeneous ideals in $\mathscr{R} (i = 1, \cdots, e)$. Also, it follows easily from the definition that, if $C = \cap Q_j$ is a normal decomposition of C, where Q_j is P_j-primary, then P_j^* is prime, Q_j^* is P_j^*-primary, and $C^* = \cap Q_j^*$ is a normal decomposition of C^* [15, Theorem 1.5]. Further, height C^* = height C, height $\mathscr{M}_i$ = height $M_i + 1$, and altitude $\mathscr{R}$ = altitude $R + 1$ [13, Remark 3.7].

Most of the results in this section follow from the following basic lemma.

LEMMA 4.2. *Let $(R; M_1, \cdots, M_e)$ be a semi-local ring such that height M_i = altitude $R = a (i = 1, \cdots, e)$. Let B be an ideal in R such that Rad $B = J = \cap M_i$, and let $\mathscr{R} = \mathscr{R}(R, B)$ be the Rees ring of R with respect to B. Assume each $\mathscr{R}_{((M_i^*, u)\mathscr{R})}$ is Macaulay, and let $P_1, \cdots, P_s$ be homogeneneous prime ideals in $\mathscr{R}$. Assume height $P_v = k (v = 1, \cdots, s)$ and either $u \in \cap P_v$ or $u \notin \cup P_v$. Then there exist homogeneous elements $x_1, \cdots, x_j$ in $\cap P_v$ such that every permutation of $x_1, \cdots, x_j, u$ is an $\mathscr{R}$-sequence and $j = k$ (if $u \notin \cup P_v$) or $j = k - 1$ (if $u \in \cap P_v$).*

Proof. Every prime divisor of each homogeneous ideal in $\mathscr{R}$ is homogeneous, hence is contained in $\mathscr{M}_i = (M_i^*, u)\mathscr{R}$, for some $i = 1$,

$\cdots, e$, and $\mathscr{R}_S$ is a Macaulay semi-local ring, where $S = \mathscr{R} \sim \bigcup \mathscr{M}_i$. Therefore, if $x_1, \cdots, x_j, u$ is an $\mathscr{R}$-sequence of homogeneous elements contained in $(J^*, u)\mathscr{R}$, then every permutation of it is an $\mathscr{R}$-sequence. Thus it suffices to prove the existence of homogeneous elements $x_1, \cdots, x_j$ in $\cap P_v \cap J^*$ such that some permutation of $x_1, \cdots, x_j, u$ is an $\mathscr{R}$-sequence and $j = k$ (if $u \notin \bigcup P_v$) or $j = k - 1$ (if $u \in \cap P_v$).

Since this is clear if $k = 0$, let $k > 0$, and assume the lemma holds for such finite sets of homogeneous prime ideals in $\mathscr{R}$ of height less than k. For $v = 1, \cdots, s$, let Q_v be a homogeneous prime ideal in $\mathscr{R}$ such that $Q_v \subset P_v$, height $Q_v = k - 1$, and either $u \in \cap Q_v$ or $u \notin \bigcup Q_v$. By induction, let $x_1, \cdots, x_m$ be homogeneous elements in $\cap Q_v \cap J^*$ such that $x_1, \cdots, x_m, u$ is an $\mathscr{R}$-sequence and $m =$ height Q_v (if $u \notin \bigcup Q_v$) or $m + 1 =$ height Q_v (if $u \in \cap Q_v$). Let $p_1, \cdots, p_f$ be the prime divisors of $(x_1, \cdots, x_m, u)\mathscr{R}$, let $\mathscr{P} = \{p_1, \cdots, p_f\}$, and fix $p \in \mathscr{P}$. Then: $(u, J)\mathscr{R} \subseteqq p$, since $u \in p$ and Rad $B = J$ imply $P \cap R = M_i$, for some $i = 1, \cdots, e$; and, p is homogeneous, so $p \subseteqq \mathscr{M}_i$, for some $i = 1, \cdots, e$. Therefore, since each $\mathscr{R}_{\mathscr{M}_i}$ is Macaulay, height $p = m + 1$. If one the P_v is in $\mathscr{P}$, then $u \in P_v$ (hence $u \in \cap P_v$) and $u \notin Q_v$ (hence $u \notin \bigcup Q_v$), and so every P_v is in $\mathscr{P}$, hence the lemma holds. Thus it may be assumed that no P_v is $\mathscr{P}$. Therefore, no $\mathscr{M}_i$ is in $\mathscr{P}$; for, if one $\mathscr{M}_i \in \mathscr{P}$, then $m = a$ and so $\mathscr{P} \subseteqq \{\mathscr{M}_1, \cdots, \mathscr{M}_e\}$, hence, since $k \geqq m + 1$, it follows that all $P_v \in \mathscr{P}$; contradiction.

For $h = 1, \cdots, f$, let $I_h = \bigcap_{j \neq h} p_j \cap B^* \cap P_1 \cap \cdots \cap P_s$. Then I_h is homogeneous, and $I_h \nsubseteq p_h$; for, $I_h \subseteqq p_h$ implies $B^* \subseteqq p_h$, hence $(u, J, B^*)\mathscr{R} \subseteqq p_h$, and so $p_h = \mathscr{M}_i$, for some $i = 1, \cdots, e$; contradiction. Therefore, there exists a homogeneous element $z_h \in I_h$, $\notin p_h$; say $z_h = r_h t^{d_h}(d_h > 0$, since, for $n \leqq 0$, $[I_h]_n \subseteqq [B^*]_n = B \subseteqq J \subseteqq [p_h]_n)$. Thus, with $D_h = \pi_{j \neq h} d_j$, $x_{m+1} = \sum z_h^{D_h}$ is a homogeneous element in $\cap P_v \cap J^*$ and not in $p_1, \cdots, p_f$ hence $x_1, \cdots, x_m, u, x_{m+1}$ is an $\mathscr{R}$-sequence.

REMARK 4.3. The homogeneous elements $x_h (h = 1, \cdots, j)$ in (4.2) must have positive degree, since u, x_h is an $\mathscr{R}$-sequence and every homogeneous nonunit of nonpositive degree is in some prime divisor of $u\mathscr{R}$ (since (Rad $u\mathscr{R}) \cap R =$ Rad $(u\mathscr{R} \cap R) =$ Rad $B = J$).

The following two definitions are needed for (4.4) below.

A set of elements $y_1, \cdots, y_a$ in the Jacobson radical J of a semi-local ring R is a *system of parameters* in R in case Rad $(y_1, \cdots, y_a)R = J$ and $a =$ altitude R [8, p. 77].

If B is an ideal in a ring R, and $x \in R$, then the *degree of x with respect to B*, denoted $d_B(x)$, is the largest integer n such that $x \in B^n$, if such n exists. If $x \in B^n$, for all n, then $d_B(x) = \infty$.

THEOREM 4.4. *Let $(R; M_1, \cdots, M_e)$, J, B, a, and $\mathscr{R}$ be as in* (4.2). *Then $\mathscr{R}$ is locally Macaulay if and only if R is Macaulay and there* *exists a system of parameters* $y_1, \cdots, y_a$ *in* R *such that, for each* $j = 1, \cdots, a$ *and for all* $n \geqq 1$, $(y_1, \cdots, y_j)R \cap B^n = \sum_1^j y_i B^{n-d_i}$, *where* $d_i = d_B(y_i)(i = 1, \cdots, a)$.

Proof. If $a = 0$, then altitude $\mathscr{R} = 1$. Therefore $\mathscr{R}$ is locally Macaulay (since the prime divisors of (0) in $\mathscr{R}$ are the ideal q^* with q a prime ideal in R, and so have height 0). Hence the theorem is trivially true upon defining $\sum_1^a y_i B^{n-d_i}$ to be zero when are no y_i. Therefore, let $a > 0$.

If $\mathscr{R}$ is locally Macaulay, then R is Macaulay [11, Theorem 3.8]. Also, by (4.2), there exist homogeneous elements $x_1, \cdots, x_a \in J^*$ such that $x_1, \cdots, x_a, u$ is an $\mathscr{R}$-sequence. For $j = 1, \cdots, a$, let $X_j = (x_1, \cdots, x_j)\mathscr{R}$, let $x_i = y_i t^{d_i}$ with $y_i \in B^{d_i}$, and let $Y_j = (y_1, \cdots, y_j)R$. Then, since $x_1, \cdots, x_j, u$ is an $\mathscr{R}$-sequence $(j = 1, \cdots, a)$, $d_B(y_i) = d_i$. Also, $Y_j^* = Y_j R[t, u] \cap \mathscr{R} = X_j$ (hence Rad $Y_a = J$), since $R[t, u] = \mathscr{R}[1/u]$ implies $Y_j^* = X_j \colon u^k \mathscr{R}$, for all large k. Hence $Y_j \cap B^n = [Y_j^*]_n = [X_j]_n = \sum_1^j y_i B^{n-d_i}$, for $j = 1, \cdots, a$ and for all n.

Conversely, assume R is Macaulay and such a system of parameters $y_1, \cdots, y_a$ exists in R, let $Y_j = (y_1, \cdots, y_j)R(j = 1, \cdots, a)$, and let $x_i = y_i t^{d_i}$. To prove $\mathscr{R}$ is locally Macaulay, let N be a maximal ideal in $\mathscr{R}$.

(i) If $N = \mathscr{M}_i = (M_i^*, u)\mathscr{R}$, for some $i = 1, \cdots, e$, then $\mathscr{M}_i$ is a minimal prime divisor of $(Y_a^*, u)\mathscr{R}$. Also, $Y_j^* = (x_1, \cdots, x_j)\mathscr{R}$, since $[Y_j^*]_n = [(x_1, \cdots, x_j)\mathscr{R}]_n$, for all n (for $n \geqq 1$, by hypothesis; for $n < 1$, this is clear). Hence, since the prime divisors of the Y_j^* are the ideals P^* with P a prime divisor of Y_j, and since $u \notin P^*$, $x_1, \cdots, x_a$, u is an $\mathscr{R}$-sequence, and so $\mathscr{R}_N$ is Macaulay.

(ii) If $N \neq \mathscr{M}_i(i = 1, \cdots, e)$, then either $u \notin N$, or $tb \notin N$, for some nonzero-divisor $b \in B$. (If $(u, B^*)\mathscr{R} \subseteqq N$, then, since Rad $B = J$, $N = \mathscr{M}_i$, for some i; and, since $a > 0$ and R is Macaulay, B can be generated by nonzero divisors [9, Lemma 10, p. 229].) If $u \notin N$, then $\mathscr{R}_N$ is Macaulay, since it is a quotient ring of $\mathscr{R}[1/u] = R[t, u]$, and $R[t, u]$ is locally Macaulay, since R is Macaulay. If $tb \notin N$, then let $A = R[B/b]$ denote the R-subalgebra of $R[1/b]$ generated by the elements c/b with $c \in B$. Let $\mathscr{S} = \mathscr{R}[1/tb]$. Then $\mathscr{S} = A[tb, 1/tb]$, and $P' = N\mathscr{S} \cap A$ is a prime ideal in A. Since N is maximal and isn't homogeneous, $P'\mathscr{S} \subset N\mathscr{S}$ and $P = P'\mathscr{S} \cap \mathscr{R}$ is a homogeneous prime ideal (as in [13, Remark 3.11]). Also, height $N/P = 1$, since $N\mathscr{S} \cap A = P\mathscr{S} \cap A = P'$. Since P is homogeneous, (i) and (4.2) imply there exists an $\mathscr{R}$-sequence of $s =$ height P homogeneous elements in P, say $x_1, \cdots, x_s$ (possibly one x_h is u). Since N isn't homogeneous, it follows that N isn't a prime divisor of $(x_1, \cdots, x_s)\mathscr{R}$. Also, since R is Macaulay and height $N/P = 1$, height $N =$ height $P + 1$ [8, (34.8) and (25.10)]. Hence $\mathscr{R}_N$ is Macaulay.

The next result is a considerable strengthening of [11, Corollary 3.6]. It also shows the unexpected result that every prime ideal P in R contains a prime sequence of height P elements which can be extended to a maximal prime sequence which has the property described in (4.4).

COROLLARY 4.5. (cf. [11, Corollary 3.6].) *Let* $(R; M_1, \cdots, M_e)$, J, B, a, *and* $\mathscr{R}$ *be as in* (4.2), *assume* $\mathscr{R}$ *is locally Macaulay, and let* $P_1, \cdots, P_s$ *be prime ideals in* R. *If height* $P_v = k (v = 1, \cdots, s)$, *then there exists an* R-*sequence* $y_1, \cdots, y_a$ *such that* $y_1, \cdots, y_k$ *are in* $\cap P_v$ *and, for each* $j = 1, \cdots, a$, *for each permutation* π *of* $\{1, \cdots, a\}$, *for all positive integers* $f_1, \cdots, f_k$, *and, for all* n, $(y_{\pi 1}^{f_1}, \cdots, y_{\pi j}^{f_j}) R \cap B^n = \sum_1^j y_{\pi i}^{f_i} B^{n-d_i f_i}$, *where* $d_i = d_B(y_{\pi i})$.

Proof. By (4.2), let $x_1, \cdots, x_k$ be homogeneous elements in $\cap P_v^*$ such that each permutation of $x_1, \cdots, x_k, u$ is an $\mathscr{R}$-sequence. By the proof of (4.2), there exist homogeneous elements $x_{k+1}, \cdots, x_a$ in $\mathscr{R}$ such that each permutation of $x_1, \cdots, x_a, u$ is an $\mathscr{R}$-sequence. Hence, for each $j = 1, \cdots, a$, $x_{\pi 1}^{f_1}, \cdots, x_{\pi j}^{f_j}, u$ is an $\mathscr{R}$-sequence. Let $x_{\pi i} = y_{\pi i} t^{d_i}$ with $y_{\pi i} \in B^{d_i} (i = 1, \cdots, a)$, and, for $j = 1, \cdots, a$, let $Y_j = (y_{\pi 1}^{f_1}, \cdots, y_{\pi j}^{f_j}) R$ and $X_j = (x_{\pi 1}^{f_1}, \cdots, x_{\pi 1}^{f_j}) \mathscr{R}$. Then $Y_j * = X_j$, since $X_j : u\mathscr{R} = X_j$. Hence, since the prime divisors of Y_j^* are the ideals P^* with P a prime divisor of Y_j, it follows that $y_{\pi 1}^{f_1}, \cdots, y_{\pi a}^{f_a}$ is an R-sequence and, for $j = 1, \cdots, a$, $Y_j \cap B^n = [Y_j *]_n = [X_j]_n = \sum_1^j y_{\pi i}^{f_i} B^{n-d_i f_i}$.

The following corollary gives another necessary and sufficient condition for $\mathscr{R}$ to be locally Macaulay.

COROLLARY 4.6. *With the notation of* (4.2), $\mathscr{R}$ *is locally Macaulay if and only if there exists a positive integer* g *and a system of parameters* $z_1, \cdots, z_a$ *contained in* B^g *such that, for each* $j = 1, \cdots, a$, *for each* (*or, for some*) *permutation* π *of* $\{1, \cdots, a\}$, *and for all* $n \geqq g$, $(z_{\pi 1}, \cdots, z_{\pi j}) R \cap B^n = (z_{\pi 1}, \cdots, z_{\pi j}) B^{n-g}$.

Proof. If $\mathscr{R}$ is locally Macaulay, then let $y_1, \cdots, y_a$ be elements in J as in (4.5). Say $d_i = d_B(y_i)$, so $d_i > 0$, by (4.3). Let $D_i = \pi_{j \neq i} d_j$, let $g = d_i D_i$, and let $z_i = y_i^{D_i} (i = 1, \cdots, a)$. Then the conclusion follows from (4.5). The converse follows from (4.4).

(4.7) and (4.8.1) below are known when R is Macaulay and B is generated by an R-sequence [11, Corollary 3.9 and p. 406]. (4.8.2) is new even for the R-sequence case but follows easily from (4.7) and (4.8.1); and (4.9) follows from (4.5) – (4.8). The basis of the proof of (4.7) is that, if b is a nonzero-divisor in a locally Macaulay ring R, then R/bR is locally Macaulay.

COROLLARY 4.7. (cf. [11, Corollary 3.9].) *With the notation of* (4.2), *if* $\mathscr{R}$ *is locally Macaulay, then, for each nonzero-divisor* $b \in B$, $R[b_1/b, \cdots, b_w/b]$ *is locally Macaulay, where* $B = (b_1, \cdots, b_w)R$.

Proof. The proof is the same as the proof of [11, Corollary 3.9].

COROLLARY 4.8. (cf. [11, p. 406 *and Corollary* 3.9].) *With the notation of* (4.2), *if* $\mathscr{R}$ *is locally Macaulay, then the following statements hold, for all positive integers* m:

4.8.1. $\mathscr{R}(R, B^m)$ *is locally Macaulay.*

4.8.2. *For each nonzero-divisor* $b \in B^m$, $R[\beta_1/b, \cdots, \beta_z/b]$ *is locally Macaulay, where* $B^m = (\beta_1, \cdots, \beta_z)R$.

Proof. 4.8.1. If $\mathscr{R}$ is locally Macaulay, then R is Macaulay and there exists an R-sequence $y_1, \cdots, y_a$ contained in J such that, for all $j = 1, \cdots, a$ and all positive integers m and n, $(y_1^m, \cdots, y_j^m)R \cap B^n = \sum_1^j y_i^m B^{n-d_i m}$, where $d_B(y_i) = d_i$ ((4.4) and (4.5)). Then, with $n = mh$ $(h \geqq 1)$, it follows from (4.4) that $\mathscr{R}(R, B^m)$ is locally Macaulay, if $d_{B^m}(y_i^m) = d_i$. But this holds, since $x_1^m, \cdots, x_a^m, u$ is an $\mathscr{R}$-sequence (as in (i) in the proof of (4.4)), where $x_i = y_i t^{d_i}$. (4.8.2) follows from (4.8.1) and (4.7).

Applying the last three corollaries to the case when R is Macaulay and B is a power of the ideal generated by an R-sequence, the following corollary is obtained.

COROLLARY 4.9. *With the notation of* (4.2), *if* $B = Y^n$, *where* Y *is generated by an* R*-sequence* (*such that* $\text{Rad}\, Y = J$) *and* $n > 0$, *then* (4.5) – (4.8) *hold.*

Proof. This follows from (4.5) – (4.8), since R is Macaulay (since $\text{Rad}\ Y = J$), hence $\mathscr{R}$ is locally Macaulay [11, p. 406].

The following proposition has the status of folklore—and may even appear somewhere in the literature. It will be used in (4.11) to prove a number of necessary and sufficient conditions for $\mathscr{R}$ to be locally Macaulay. Also, the relationship, noted below, between Rees rings and form rings together with (4.10) shows that much of the material in this section really isn't so special.

PROPOSITION 4.10. *Let* R *be a Noetherian ring, and let* S *be a finitely generated positively graded* R*-algebra such that* $S_0 = R$. *Then* S *is locally Macaulay if and only if, for each maximal ideal* M *in* R, $S_{(M+S_+)}$ *is locally Macaulay, where* S_+ *is the ideal in* S *generated* *by the forms of positive degree.*

Proof. The condition is clearly necessary. Conversely, let Q be a prime ideal in S, and let M be a maximal ideal in R such that $Q \cap R \subseteqq M$. Then S_Q is a localization of $R_M \otimes_R S$, so it may be assumed that R is a local ring with maximal ideal M. Let $R(X) = R[X]_{MR[X]}$, where X is an indeterminate, and let R^* be the completion of $R(X)$. We can replace S by $R^* \otimes_R S$, so it may be assumed that R is a complete local ring with an infinite residue field. Then S is a homomorphic image of a regular ring, so the Macaulay locus of S is Zariski open [3, (6.11.3)]. Suppose that the non-Macaulay locus of S isn't empty, and let I be its defining radical ideal. Then it suffices to show that I is homogeneous, for then $I \subseteqq M + S_+$ which contradicts the hypothesis.

If $a \in R$, $\notin M$, then there exists an R-automorphism of S which takes each form F of degree d to $a^d F$. Therefore, let $\sum_0^d F_i \in I$ (where each F_i is a form of degree i) and choose units $a_0, \cdots, a_d$ in R with distinct residue classes modulo $M(R/M$ is infinite). Then, since clearly I is invariant under every automorphism on S, $\sum_{i=0}^d a_j^i F_i$ is in $I(0 \leqq j \leqq d)$. But Det $(a_j^i) = \pm \pi_{i<j}(a_i - a_j) \in R$, $\notin M$, hence is a unit in R. Therefore each $F_i \in I$, as desired.

If B is an ideal in a Noetherian ring R, then, as in [15, Theorem 2.1], the form ring $\mathscr{F} = \mathscr{F}(R, B)$ of R with respect to B is (isomorphic to) $\mathscr{R}/u\mathscr{R}$, and the B-form ideal C' of an ideal C in R is (isomorphic to) $(C^*, u)\mathscr{R}/u\mathscr{R}$. This fact is used in (4.11) below.

If $M_1, \cdots, M_e$ are special maximal ideals in a Noetherian ring R such that each R_{M_i} is Macaulay, then it isn't true, in general, that R is locally Macaulay. However, this is true for $\mathscr{R}$ and $\mathscr{F}$, as is shown by the following theorem.

THEOREM 4.11. *Let B be an ideal in a Noetherian ring R such that B is contained in the Jacobson radical of R, let $\mathscr{R} = \mathscr{R}(R, B)$, and let $\mathscr{F} = \mathscr{F}(R, B)$ be the form ring of R with respect to B. Then statements* (4.11.1)—(4.11.4) *below are equivalent and each implies* (4.11.5).

4.11.1. *$\mathscr{R}$ is locally Macaulay.*

4.11.2. *$\mathscr{R}_{\mathscr{M}}$ is Macaulay, for all maximal homogeneous ideals $\mathscr{M}$ in $\mathscr{R}$.*

4.11.3. *$\mathscr{F}$ is locally Macaulay.*

4.11.4. *$\mathscr{F}_{\mathscr{N}}$ is Macaulay, for all maximal homogeneous ideals $\mathscr{N}$ in $\mathscr{F}$.*

4.11.5. *R and all rings $R[b_1/b, \cdots b_w/b]$ are locally Macaulay, where b is a nonzero-divisor in $B = (b_1, \cdots, b_w)R$.*

Proof. Clearly (4.11.1) implies (4.11.2), and (4.11.3) implies (4.11.4). Also, (4.11.1) implies (4.11.3), and (4.11.2) implies (4.11.4), since $\mathscr{F} \cong \mathscr{R}/u\mathscr{R}$ (as in [15, Theorem 2.1]). Further, (4.11.4) implies (4.11.3), by (4.10). Now (4.11.3) implies $\mathscr{R}_P$ is Macaulay, for all prime ideals P in $\mathscr{R}$ such that $u \in P$. Thus, if M is a maximal ideal in R, then $\mathscr{M} = (M^*, u)\mathscr{R}$ is a maximal ideal in $\mathscr{R}$ (since $B \subseteq M$), so, since $\mathscr{R}_{\mathscr{M}}$ is Macaulay, $\mathscr{R}_{M^*} = R[u]_{MR[u]}$ is Macaulay. It follows that R is locally Macaulay, and so $R[t, u]$ is locally Macaulay. From this it follows that (4.11.3) implies (4.11.1), and so (4.11.1)—(4.11.4) are equivalent and each implies that R is locally Macaulay. Finally, since $R[b_1/b, \cdots, b_w/b][tb, 1/tb] = \mathscr{R}[1/tb]$ and tb is transcendental over R, (4.11.1) implies (4.11.5).

This paper will be closed with the following result which gives two equivalences of (4.11.5).

PROPOSITION 4.12. *Let B be an ideal in a Noetherian ring R such that B is contained in the Jacobson radical of R, and let $\mathscr{R} = \mathscr{R}(R, B)$. Then the following statements are equivalent:*

4.12.1. *R and all rings $R[b_1/b, \cdots, b_w/b]$ are locally Macaulay, where $B = (b_1, \cdots, b_w)R$ and b is a nonzero-divisor in B.*

4.12.2. *For each prime ideal P in $\mathscr{R}$ such that $(u, B^*) \nsubseteq P$, $\mathscr{R}_P$ is Macaulay.*

4.12.3. *For each homogeneous prime ideal P in $\mathscr{R}$ such that $(u, B^*)\mathscr{R} \nsubseteq P$, $\mathscr{R}_P$ is Macaulay.*

Proof. If K is a ring and X an indeterminate, then K is locally Macaulay if and only if $K[X, 1/X]$ is locally Macaulay. The equivalence of (4.12.1) and (4.12.2) follows from this and the facts $\mathscr{R}[1/u] = R[t, u]$ and $\mathscr{R}[1/tb] = A[tb, 1/tb]$, where $A = R[b_1/b, \cdots, b_w/b]$. Clearly (4.12.2) implies (4.12.3). Also, $R[t, u]_{MR[t,u]} = \mathscr{R}_{M^*}$ (where M is a maximal ideal in R), and, if P' is a prime ideal in A, then $P = P'\mathscr{R}[1/tb] \cap \mathscr{R}$ is homogeneous (as in [13, Remark 3.11]) and $A[tb]_{P'A[tb]} = \mathscr{R}_P$. Therefore, (4.12.3) implies (4.12.1).

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UNIVERSITY OF MINNESOTA
AND
UNIVERSITY OF CALIFORNIA, RIVERSIDE