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G_δ-DIAGONALS AND METRIZATION THEOREMS

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The topological space X is said to have a *G_δ-diagonal* if the diagonal $\Delta = \{(x, x): x \in X\}$ is a G_δ-set in $X \times X$. It is easy to see that if X has a coarser metrizable topology, then X has a G_δ-diagonal. The main result is that a completely regular pseudocompact space with a regular G_δ-diagonal is metrizable.

A considerable amount of research has been done on the question of what topological properties imply metrizability in the presence of a G_δ-diagonal. For example, it is well-known that the existence of a G_δ-diagonal is sufficient for metrizability in any of the following classes of spaces:

compact Hausdorff spaces
linearly ordered spaces
paracompact p-spaces.

A question still open is whether a countably compact regular space with a G_δ-diagonal must be metrizable. A space X is said to have a *regular G_δ-diagonal* if the diagonal Δ is the intersection of countably many closures of open subsets of $X \times X$ (see [5]). It is known that a countably compact space with a regular G_δ-diagonal is metrizable [1].

2. The main result.

DEFINITION 2.1. A space X is *pseudocompact* if every real-valued continuous function on X is bounded.

Pseudocompact spaces were first defined and investigated by Hewitt in [3]. The following characterization of completely regular pseudocompact spaces may be found in [2], page 134.

LEMMA 2.2. *Let X be a completely regular space. X is pseudocompact if and only if for every sequence $G_1 \supset G_2 \supset \cdots \supset G_n \supset \cdots$ of nonvoid open subsets of X, $\bigcap_{n=1}^{\infty} \mathrm{cl}_X(G_n) \neq \emptyset$.*

LEMMA 2.3. *Let X be a completely regular pseudocompact space. Suppose $G_1 \supset G_2 \supset \cdots \supset G_n \supset \cdots$ is a sequence of open sets such that*

$$\bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \mathrm{cl}_X(G_n) = \{x\}$$

for a point x of X. Then the sets G_n form a local neighborhood *base at* x.

Proof. Let G be an open set containing x. Suppose

$$G_n \cap (X - G) \neq \emptyset$$

for every n. Choose H open such that $x \in H \subset \mathrm{cl}_X(H) \subset G$. Then, $(G_n \cap (X - \mathrm{cl}_X(H)))_{n=1}^{\infty}$ is a decreasing sequence of nonvoid open subsets of X. Thus, by Lemma 2.2, there is a point p of X such that $p \in \bigcap_{n=1}^{\infty} \mathrm{cl}_X(G_n \cap (X - \mathrm{cl}_X(H))$. But, p belongs to $\bigcap_{n=1}^{\infty} \mathrm{cl}_X G_n$, a contradiction! Therefore, there must be an integer n such that $G_n \subset G$.!!

DEFINITION 2.4. Let $\mathscr{G}$ be an open cover of X, $x \in X$, and $H \subset X$. Then,

$$\mathrm{st}(x, \mathscr{G}) = \bigcup \{G \in \mathscr{G} : x \in G\}$$

$$\mathrm{st}(R, \mathscr{G}) = \bigcup \{G \in \mathscr{G} : G \cap H \neq \emptyset\}.$$

The following result was announced by Moore in [4].

LEMMA 2.5. (*Moore's metrization theorem*) *A topological space is metrizable if*

(1) X *is Hausdorff, and*

(2) *There is a decreasing sequence* $\mathscr{G}_1 \supset \mathscr{G}_2 \supset \cdots \subset \mathscr{G}_n \supset \cdots$ *of open covers of* X *such that for every* x *in* X, *the sets* $\mathrm{st}(\mathrm{st}(x, \mathscr{G}_n), \mathscr{G}_n)$ *for* $n = 1, 2, 3, \cdots$ *form a local neighborhood base at* x.

Our main result appears below.

THEOREM 2.6. *Let* X *be a completely regular pseudocompact space. If* X *has a regular* G_δ*-diagonal, then* X *is metrizable.*

Proof. $\Delta = \{(x, x) : x \in X\}$. Then, there is a decreasing sequence $G_1 \supset G_2 \supset \cdots \supset G_n \supset \cdots$ of open subsets of $X \times X$ such that

$$\Delta = \bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \mathrm{cl}_{X \times X}(G_n) .$$

For each x in X, choose a sequence $(g_n(x))$ of open subsets of X such that $(x, x) \in g_n(x) \times g_n(x) \subset G_n$ for each n. Then, for each n let

$$\mathscr{G}_n = \bigcup_{k \geqq n} \{g_k(x) : x \in X\} .$$

Then, $\mathscr{G}_1 \supset \mathscr{G}_2 \supset \cdots \supset \mathscr{G}_n \supset \cdots$ is a decreasing sequence of open covers of X.

(i) For x in X, $\bigcap_{n=1}^{\infty} \mathrm{cl}_X(\mathrm{st}(x, \mathscr{G}_n)) = \{x\}$. Let $y \neq x$. Then, there is an integer n such that $(x, y) \notin \mathrm{cl}_{X\times X}(G_m)$ for $m \geqq n$. Then, there are neighborhoods U and V of x and y respectively such that $(U \times V) \cap G_m = \varnothing$ for $m \geqq n$. Suppose that $V \cap \mathrm{st}(x, \mathscr{G}_n) \neq \varnothing$. Then, there is an integer $k \geqq n$ and a point z of X such that x is in $g_k(z)$ and $V \cap g_k(z) \neq \varnothing$. Then,

$$\varnothing = (U \times V) \cap G_k \supset (U \times V) \cap (g_k(z) \times g_k(z)) \neq \varnothing .$$

Contradiction! Thus, it must be that

$$V \cap \mathrm{st}(x, \mathscr{G}_n) = \varnothing \quad \text{and} \quad y \notin \mathrm{cl}_X(\mathrm{st}(x, \mathscr{G}_n)) .$$

(ii) We conclude by Lemma 2.3 that $(\mathrm{st}(x, \mathscr{G}_n))$ forms a local base at x, for each x in X.

(iii) For x in X, $\bigcap_{n=1}^\infty \mathrm{cl}_X(\mathrm{st}(\mathrm{st}(x, \mathscr{G}_n), \mathscr{G}_n)) = \{x\}$. Let $y \neq x$. Then, there is an integer n such that $m \geqq n$ implies that

$$(x, y) \notin \mathrm{cl}_{X\times X}(G_m) .$$

Then, there are neighborhoods U and V of x and y respectively such that $(U \times V) \cap G_m = \varnothing$ for $m \geqq n$. There are integers k and j such that $\mathrm{st}(x, \mathscr{G}_k) \subset U$ and $\mathrm{st}(y, \mathscr{G}_j) \subset V$. Let $m = \max\{n, k, j\}$. Then, $(\mathrm{st}(x, \mathscr{G}_m) \times \mathrm{st}(y, \mathscr{G}_m)) \cap G_m \subset (U \times V) \cap G_m = \varnothing$. Suppose

$$\mathrm{st}(y, \mathscr{G}_m) \cap \mathrm{st}(\mathrm{st}(x, \mathscr{G}_m), \mathscr{G}_m) \neq \varnothing .$$

Then, there is an integer $k \geqq m$ and a point z of X such that

$$g_k(z) \cap \mathrm{st}(x, \mathscr{G}_m) \neq \varnothing$$

and $\mathrm{st}(y, \mathscr{G}_m) \cap g_k(z) \neq \varnothing$. Then,

$$(\mathrm{st}(x, \mathscr{G}_m) \times \mathrm{st}(y, \mathscr{G}_m)) \cap (g_k(z) \times g_k(z)) \neq \varnothing .$$

Contradiction! Thus, it must be that $\mathrm{st}(y, \mathscr{G}_m) \cap \mathrm{st}(\mathrm{st}(x, \mathscr{G}_m), \mathscr{G}_m) = \varnothing$ and hence $y \notin \mathrm{cl}_X(\mathrm{st}(\mathrm{st}(x, \mathscr{G}_m), \mathscr{G}_m))$.

(iv) We conclude by Lemma 2.3 that $(\mathrm{st}(\mathrm{st}(x, \mathscr{G}_m), \mathscr{G}_m))$ forms a local base at x, for each x in X.

(v) By Moore's Metrization Theorem (Lemma 2.5), X is metrizable.!!

COROLLARY 2.7. *If X is a completely regular pseudocompact space with a coarser metric topology, then X is metrizable.*

Proof. If X has a coarser metric topology, so does $X \times X$.!!

EXAMPLE 2.8. The space $E \cap [0, 1]$ of [2], problem 3J is pseudocompact, Hausdorff, and has a coarser metric topology. Since the space is not completely regular, it is not metrizable.

EXAMPLE 2.9. The space Ψ of [2], Problem 5I is pseudocompact, completely regular, and the diagonal in $\Psi \times \Psi$ is a G_δ-set. But, Ψ is not metrizable.

3. Some remarks on the countably compact case.

DEFINITION 3.1. A space X is countably compact if every countable family of closed sets with the finite intersection property has nonempty intersection.

PROPOSITION 3.2. *If X is countably compact, regular, with a G_δ-diagonal, then X is first countable.*

Proof. Suppose $\Delta = \bigcap_n G_n$ where the sets $G_1, G_2, \cdots, G_n, \cdots$ are open subsets of $X \times X$. For x in X, choose a sequence $(g_n(x))$ of open subsets of X which contain x such that for each n,

$$\mathrm{cl}_X\,(g_{n+1}(x)) \subset g_n(x) \qquad \text{and} \qquad g_n(x) \times g_n(x) \subset G_n \,.$$

Note that $\bigcap_{n=1}^{\infty} \mathrm{cl}_X\,(g_n(x)) = \{x\}$. Now, suppose G is an open subset of X which contains x. If it is true that no set $g_n(x)$ is contained in G, then $(\mathrm{cl}_X\,(g_n(x)) \cap (X-G))_n$ is a countable collection of closed sets with the finite intersection property. Thus, since X is countably compact, $(\bigcap_{n=1}^{\infty} \mathrm{cl}_X\,(g_n(x))) \cap (X-G) \neq \varnothing$. Contradiction! Hence, there must exist an integer n such that $g_n(x) \subset G$. This shows that $(g_n(x))_n$ forms a neighborhood base at x and hence X is first countable.!!

PROPOSITION 3.3. *If X is countably compact, regular, with a G_δ-diagonal, then $X \times X$ is countably compact, regular, and has a G_δ-diagonal.*

Proof. It is well-known that regularity is productive and that countable compactness is countably productive in the presence of first countability. Now, suppose that $\Delta = \bigcap_n G_n$ with the sets G_n open in $X \times X$. Let $\Delta' = \{((x, y), (x, y)) \colon x, y \in X\}$. For each n, let

$$g_n(x, y) = g_n(x) \times g_n(y)$$

where the sets $g_n(x)$ are as in Proposition 3.2. Let

$$H_n = \bigcup_{(x,y) \in X \times X} (g_n(x, y) \times g_n(x, y)) \,.$$

Claim: $\Delta' = \bigcap_{n=1}^{\infty} H_n$. Clearly, $\Delta' \subset \bigcap_{n=1}^{\infty} H_n$. Suppose $(x_1, y_1) \neq (x_2, y_2)$.

Case I. $x_1 \neq x_2$. Then, there is an integer n such that

$(x_1, x_2) \notin G_n$. Suppose $((x_1, y_1), (x_2, y_2)) \in H_n$. Then, there is a pair (x, y) in $X \times X$ such that $(x_1, y_1) \in g_n(x, y)$ and $(x_2, y_2) \in g_n(x, y)$. Then, $x_1 \in g_n(x)$ and $x_2 \in g_n(x)$ which implies that $(x_1, x_2) \in G_n$. Contradiction!

Case II. $y_1 \neq y_2$. Similar argument to that of Case I. Thus, $X \times X$ has a G_δ-diagonal.!!

PROPOSITION 3.4. *Every countably compact, regular, space with a G_δ-diagonal is metrizable if and only if every countably compact, regular, space with a G_δ-diagonal is normal.*

Proof. If X has a G_δ-diagonal and $X \times X$ is normal, then X has a regular G_δ-diagonal.!!

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