EXPANDED RADICAL IDEALS AND SEMIREGULAR IDEALS

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Let $R = K[x_1, \cdots, x_n]$ be a polynomial ring, where K is a field of characteristic zero, and embed R in the polynomial ring $S = K[y_{ij}: 1 \leq i \leq n - 1, 1 \leq j \leq n]$ by mapping x_j to the minor of the matrix $[y_{ij}]$ obtained by deleting the jth column. Let I be a homogeneous radical ideal of R. It will be shown that if IS is radical, then I is semiregular, that is, R/I is Cohen-Macaulay. Several other related results will be established, in which the fact that certain expanded radical ideals remain radical either implies, or is implied by, the fact that certain other ideals are semiregular. Each one of these results has some connection with invariant theory.

We shall now give a more detailed summary of the main results of the paper. If R is a finitely generated K-algebra, where K is a field, we shall say that an ideal I of R is *absolutely* radical (over K) if for each extension field L of K, $L \otimes_K I$ is radical in $L \otimes_K R$. Of course, it suffices that $L \otimes_K I$ be radical when L is a maximal purely inseparable extension of K, or any larger field, e.g. an algebraic closure of K.

THEOREM 1. *Let K be a field, and let S be a graded Noetherian Cohen-Macaulay K-algebra which is flat over the K-subalgebra R generated by the forms $f_1, \cdots, f_m$, where $f_1, \cdots, f_m$ have positive degree. (We do not require that S be generated by its 1-forms.) Suppose that that $(f_1, \cdots, f_m)$ is an absolutely radical ideal of S. Then for every absolutely radical ideal I of R, the expanded ideal IS is absolutely radical in S.*

In particular, if S is a polynomial ring over K (the indeterminates need not be assigned degree 1), and $f_1, \cdots, f_m$ is an R-sequence of forms in S which generates an absolutely radical ideal of S, then S is flat over $K[f_1, \cdots, f_m]$ and the result holds.

We note that this theorem generalizes, in a certain direction, an invariant-theoretic result of Kostant, [6] Theorem 14, p. 388.

We point out that the hypothesis implies that $(f_1, \cdots, f_m)$ is a semiregular ideal of S, which plays an important role in the proof.

It is also worth noting that from the point of view of testing an ideal for being absolutely radical, an R-sequence of polynomial forms which generates an absolutely radical ideal is as good as a sequence of indeterminates, in the sense of the following:

COROLLARY. *Let $f_1, \cdots, f_m$ be an R-sequence of forms in the polynomial ring $K[x_1, \cdots, x_n]$ which generates an absolutely radical ideal. Let $g_1(y_1, \cdots, y_m, z_1, \cdots, z_p), \cdots, g_t(y_1, \cdots, y_m, z_1, \cdots, z_p)$ be polynomials in $K[y_1, \cdots, y_m, z_1, \cdots, z_p]$ which generate an absolutely radical ideal. Then $g_1(f_1, \cdots, f_m, z_1, \cdots, z_p), \cdots, g_t(f_1, \cdots, f_m, z_1, \cdots, z_p)$ generate an absolutely radical ideal of $K[x_1, \cdots, x_n, z_1, \cdots, z_p]$.*

The corollary follows from the last statement in Theorem 1 (applied to the R-sequence $f_1, \cdots, f_m, z_1, \cdots, z_p$ of $K[x_1, \cdots, x_m, z_1, \cdots, z_p]$).

THEOREM 2. *Let K be a field of characteristic zero, and let I be a homogeneous radical ideal of the polynomial ring $K[x_1, \cdots, x_n]$. Suppose $I = (f_1, \cdots, f_m)$. Let $[y_{ij}]$ be an $n-1$ by n matrix of new indeterminates, and let Δ_i be the $n-1$ by $n-1$ minor of $[y_{ij}]$ which results from deleting the ith column, $1 \leqq i \leqq n$. Then a sufficient condition for I to be semiregular ($\equiv$ perfect in this case) is that $f_1(\Delta_1, \cdots, \Delta_n), \cdots, f_m(\Delta_1, \cdots, \Delta_n)$ generate a radical ideal in $K[y_{ij}: i, j]$.*

In other words, it is sufficient that I expand to a radical ideal under the injective K-homomorphism $K[x_1, \cdots, x_n] \to K[y_{ij}: i, j]$ which takes x_i to $\Delta_i, 1 \leqq i \leqq n$.

The proof of Theorem 2 depends on some invariant-theoretic results and on the following:

LEMMA. *Let K be a field, and let I be a homogeneous radical ideal in a reduced graded Noetherian Cohen-Macaulay K-algebra R. Let $x_1, \cdots, x_n$ be any elements of R such that the radical of $(x_1, \cdots, x_n)$ is the irrelevant maximal ideal M of R. Let $d = \dim R/I$, and let t_{ij} be $n(d-1)$ new indeterminates over R. Then I is semiregular if and only if for each $i, 1 \leqq i \leqq d-1$, the ideal*

$$ I + (t_{11}x_1 + \cdots + t_{1n}x_n, \cdots, t_{i1}x_1 + \cdots + t_{in}x_n) $$

is radical in $R[t_{ij}: i, j]$.

The condition given in Theorem 2 is not necessary. However, one might hope that some special classes of perfect radical homogeneous ideals in $K[x_1, \cdots, x_n]$ would expand to radical ideals in $K[y_{ij}]$. This is true at least for the ideals in $K[x_1, \cdots, x_n]$ which are generated by linear combinations of $x_1, \cdots, x_n$.

Let us say that a K-vector subspace V of a K-algebra R is *linearly radical* (over K) if for any field $L \supset K$ and for each finite set of elements $f_1, \cdots, f_m \in L \bigotimes_K V$, $(f_1, \cdots, f_m)$ is radical ideal in $L \bigotimes_K R$. A trivial example is obtained by letting V be the space of 1-forms in $K[x_1, \cdots, x_n]$. Less trivially, we have:

THEOREM 3. (a) *Let $[y_{ij}]$ be an r by s matrix of indeterminates over the field K, where $r \leq s$, and let V be the subspace of $K[y_{ij}: i, j]$ spanned by the r by r minors of $[y_{ij}]$. If $s = r + 1$ then V is linearly radical. If $r = 2$, it is at least true that any subject of the 2 by 2 minors generates a radical ideal.*

(b) *Let Λ be a partially ordered set and suppose that for each $\lambda \in \Lambda$ the set $\{u \in \Lambda: \mu \leq \lambda\}$ is finite and totally ordered. Let $\{U_\lambda: \lambda \in \Lambda\}$ be a family of matrices of indeterminates, all entries distinct, indexed by Λ, and suppose that (1) if λ is minimal in Λ then U_λ is a one-rowed matrix, and (2) for each $\lambda \in \Lambda$, if $\lambda_1 < \lambda_2 < \cdots < \lambda_n = \lambda$ are the elements of $\{\mu \in \Lambda: \mu \leq \lambda\}$, then the product matrix $U_{\lambda_1} \cdots U_{\lambda_n}$ is defined. Let R be the polynomial ring over K generated by the entries of all the U_λ, and let V be the K-vector subspace of R spanned by all the entries of all the product matrices $U_{\lambda_1} \cdots U_{\lambda_n}$ of the form just described. Then V is linearly radical.*

The author does not know whether (a) holds for arbitrary $r \leq s$. If it did, it would give a new proof in characteristic zero of the fact that certain ideals $\mathfrak{A}(B)$ described in [3] are radical, while (b) gives another proof in characteristic zero that the ideals $I_{H,n}$ described in [4] and [5] are radical. In each case, the fact that the ideals in the class are radical is the key point in the proof that certain rings of invariants are Cohen-Macaulay. This is discussed further in §4.

2. Proof of Theorem 1. We need some preliminary observations and results. Let S be a Noetherian graded K-algebra and M the irrelevant maximal ideal. Then we can always choose a system of parameters for S consisting of forms, i.e. if $\dim S = d$, we can choose forms $f_1, \cdots, f_d$ in M such that the ideal they generate is primary to M, in which case S is a finitely generated graded module over the polynomial subring $K[f_1, \cdots, f_d]$. Moreover, if graded $M = g$ (the grade of a proper ideal is the length of the longest R-sequence contained in it), then there is an R-sequence of forms in M of length g. These assertions follow by standard arguments from the easy observation:

(2.1) If S is a Noetherian graded K-algebra, I is a homogeneous ideal of S, and $P_1, \cdots, P_n$ are primes of S such that for each i,, $I \not\subset P_i$, then there is a form f in I such that $f \notin \bigcup_i P_i$.

We note that by the analogue in the graded case of Theorem (25.16) of [7] that S is Cohen-Macaulay if and only if for every system of parameters consisting of forms $f_1, \cdots, f_d$, S is a finitely generated free graded module over the polynomial ring $K[f_1, \cdots, f_d]$. It follows that if S is Cohen-Macaulay and $f_1, \cdots, f_k$ is an R-sequence of forms in

M, then S is free over $K[f_1, \cdots, f_k]$, for $f_1, \cdots, f_k$ can be extended to an R-sequence of length d consisting of forms, and $K[f_1, \cdots, f_d]$ is free over $K[f_1, \cdots, f_k]$.

We next observe:

(2.2) If S is free R-algebra (or merely a free R-module), then for any family of ideals $\{I_\lambda : \lambda \in \Lambda\}$ in R, $(\bigcap_\lambda I_\lambda)S = \bigcap_\lambda (I_\lambda S)$.

(2.3) If S is a graded Noetherian K-algebra, where K is a field, and R is a subring of S generated over K by a finite number of forms $f_1, \cdots, f_m$ of positive degree such that S is R-flat, then S is free R-module. In fact, if $\{s_j : j\}$ is a family of forms in S such that $\{s_j'\}$ is a K-vector space basis for S/PS, where $P = (f_1, \cdots, f_m)$ and $'$ denotes reduction modulo P, then $\{s_j : j\}$ is a free basis for S over R. (We can always choose one of the s_j to be 1, if we wish.)

Proof. We use Noetherian induction on the homogeneous ideals of R. If $P = (0)$ then $R = K$ and there is nothing to prove. Otherwise there is a nonzero form f in P such that either (1) f is not a zerodivisor in R or (2) $fP = (0)$, according as grade $P > 0$ or grade $P = 0$. In either case, if $*$ denotes reduction modulo (f), we have that $\{s_j^* : j\}$ is a free basis for S/fS ever R/fR, by the induction hypothesis. (S/fS is flat over R/fR by (18.10) of [7].) The $\{s_j : j\}$ span S as an R-module, for if g were a form of least degree in $S - \sum_j Rs_j$, we would have $g = \sum_j r_j s_j + \sum_i f_i t_i$, where each $r_j \in R$, for each j, $\deg r_j + \deg s_j = \deg g$, and for each i, $\deg f_i + \deg t_i = \deg g$. Since the f_i have positive degree, each $t_i \in \sum_j Rs_j$, and since each $f_i \in R$, $g \in \sum_j Rs_j$.

Now suppose that there is a nonzero relation $\sum_j r_j s_j = 0$, where the $r_j \in R$. We can choose such a relation in which the r_j are forms and $\max_j \deg r_j$ is a small as possible. Since $\sum_j r_j^* s_j^* = 0$, each $r_j \in (f)$, and we have $r_j = f t_j$, where $t_j \in R$, for each j. In case (1), since f is not a zerodivisor in R and S is R-flat, f is not a zerodivisor in S, and $\sum_j t_j s_j = 0$, contradicting the minimality of $\max_j \deg r_j$. In case (2) $\sum_j t_j s_j \in \mathrm{Ann}_s f = (\mathrm{Ann}_R f)S$ (since S is R-flat) $= PS$, so that $\sum_j t_j' s_j' = 0$. Hence, each $t_j \in P$, and each $r_j = f t_j = 0$.

(2.4) With the same hypothesis and notation as (2.3), if S/P is Cohen-Macaulay and $g_1, \cdots, g_t$ are forms of S such that $g_1', \cdots, g_t'$ is a system of parameters for S/P, then $g_1, \cdots, g_t$ are algebraically independent over R, and for each q, $0 \leq q \leq t$, S is free over $R[g_1, \cdots, g_q]$.

Proof. Since S/P is Cohen-Macaulay, it is a finitely generated graded free module over $K[g_1', \cdots, g_t']$. Let $h_1, \cdots, h_k$ be forms of S such that $h_1', \cdots, h_k'$ is a free basis for S/P over $K[g_1', \cdots, g_t']$. We

can assume that $h_1 = 1$. Then it is clear that the products $h_i'M'$, where M is monomial in $g_1, \cdots, g_t$, form a K-vector space basis for S/P over K. By (2.3), the products h_iM are a free basis for S over R, and the result is then clear.

(2.5) If T is a Cohen-Macaulay graded Noetherian K-algebra and I is a homogeneous ideal of T such that T/I is Cohen-Macaulay, then for every ideal J of T containing I, grade $J = $ gade $J/I + $ grade I.

Proof. Since T and T/I are graded and Cohen-Macaulay, all minimal primes of I have the same grade $g = $ grade I. It suffices to consider the case where J is a prime which contains I, and localizing at J will not change the grades of J, I or J/I. Hence, we may assume instead that T is a Cohen-Macaulay local ring with maximal ideal J, and that T/I is Cohen-Macaulay. Let P be any minimal prime of I. Then grade $I = $ rank P, grade $J/I = $ rank $J/I = $ rank J/P, grade $J = $ rank J, and the result follows.

We are now ready to give the proof of Theorem 1. By replacing R, S by $L \otimes_K R, L \otimes_K S$, where L is an algebraic closure of K, we may assume without loss of generality that K is algebraically closed. By (2.3) S is faithfully flat (even free) over R, and since S is Cohen-Macaulay we can conclude that R is Cohen-Macaulay and that for each maximal ideal Q of R that S/QS is Cohen-Macaulay.

We know that every radical ideal of R is an intersection of maximal ideals of R, and it follows from (2.2) and (2.3) that it is enough to show that for each maximal ideal Q of R, QS is radical in S. To show that QS is radical it is necessary and sufficient to prove that there is an element f in a defining ideal for the singular locus of the variety defined by QS such that f is not a zerodivisor on QS.

Let us represent S as a quotient of a polynomial ring $T = K[x_1, \cdots, x_t]$ by a homogeneous ideal I. (The x_i need not have degree one.) Let $F_1, \cdots, F_m$ be forms in T whose residues modulo I are $f_1, \cdots, f_m$, and let us denote a set of forms which generate I by $F_{m+1}, \cdots, F_r$. Thus, $T/(F_1, \cdots, F_r) \cong S/P$, where $P = (f_1, \cdots, f_m)$.

Any maximal ideal Q of R has the form $(f_1 - c_1, \cdots, f_m - c_m)$ for suitable choices of $c_1, \cdots, c_m$ in K. When $Q = P$, $c_1 = \cdots = c_m = 0$.

We claim that for any associated prime Q_1 of QS in S, rank $Q_1 = d$, where $d = \dim R$. In fact, since S/QS is Cohen-Macaulay, every associated prime Q_1 of QS is minimal, and since S is flat over R, $Q_1 \cap R = Q$. Then $R_Q \to S_{Q_1}$ is a faithfully flat local extension, and the fiber S_{Q_1}/PS_{Q_1} has dimension zero, since Q_1 is minimal for PS. Hence, rank $Q_1 = \dim S_{Q_1} = \dim R_Q + \dim S_{Q_1}/PS_{Q_1}$ (see [2], Corollaire

6.1.2, p. 135) $= \dim R_Q = \operatorname{rank} Q = \dim R$, for R is homogeneous and Cohen-Macaulay and all maximal ideals therefore have the same rank.

The associated primes of $(F_1 - c_1, \cdots, F_m - c_m, F_{m+1}, \cdots, F_r) = Q^*$ in T are in one-to-one correspondence with those of QS in S, in an obvious way, so that if g denotes the grade of I, then by (2.5) each associated prime H of Q^* has grade $(= \operatorname{rank})$ $g + d$. It follows that a defining ideal for the singular locus of the variety defined by Q^* is $Q^* + I_{g+d}(\partial F_i/\partial x_j)$, where $I_{g+d}(\partial F_i/\partial x_j)$ is the ideal generated by the $g + d$ by $g + d$ minors of the r by t Jacobian matrix obtained from $F_1 - c_1, \cdots, F_m - c_m, F_{m+1}, \cdots, F_r$ (the constants drop out after differentiation). See [7], (46.3) and (39.11).

In the case where $Q = P$, we know that P^* is radical, and it follows from (2.1) that there is a form F in $P^* + I_{g+d}(\partial F_i/\partial x_j)$ which is not a zerodivisor on P^*. Clearly, we can choose such an F in $I_{g+d}(\partial F_i/\partial x_j)$, and we then have that the image f of F in S is not a zerodivisor on PS. By (2.4), S is a free module over $R[f]$, and f is an indeterminate over R.

Now let Q be any maximal ideal of R. Then $F \in I_{g+d}(\partial F_i/\partial x_j) \subset Q^* + I_{g+d}(\partial F_i/\partial x_j)$, and to complete the proof it suffices to show that f is not a zerodivisor on QS. Since S is free over $R[f]$, it is enough to show that f is not a zerodivisor on $QR[f]$. Since Q is an ideal of R and f is an indeterminate over R, this is clear.

3. *Proof of Theorem 2.* We first restate Propositions 21 and 22 of [5].

(3.1) Let R be a Noetherian ring and let $x_1, \cdots, x_n$ be elements of R which generate an ideal of grade g. Let $t_{ij}, 1 \leq i \leq g, 1 \leq j \leq n$, be gn new indeterminates. Then the elements $\sum_j t_{ij} x_i$ form an R-sequence in $R[t_{ij}: i, j]$ and if R is a domain (respectively, reduced) and $i \leq g - 1$ then the ideal generated by the first i terms is prime (respectively, radical).

We can now establish the lemma. It is clear that if I is semi-regular, then the ideals in question are radical, for the ideal M/I will have rank and hence grade d in the reduced Cohen-Macaulay ring R/I, and hence the power associated ideal generated by the images of $x_1, \cdots, x_n$ will also have grade d.

To prove the converse, suppose that the ideals described are radical. To show that R/I is Cohen-Macaulay, it suffices to show that $(R/I)_{M/I}$ is Cohen-Macaulay (see, for example, Proposition 19 of [5]). It is then sufficient to prove the following local result:

(3.2)   Let (R, M) be a reduced local ring of dimension d, and let $x_1, \cdots, x_n$ be elements of M which generate an ideal whose radical is M. Let t_{ij} be $(d-1)n$ new indeterminates over R, as before. Then R is Cohen-Macaulay if and only if each of the ideals $(\sum_j t_{1j}x_j, \cdots, \sum_j t_{ij}x_j)$, $1 \leqq i \leqq d - 1$, is radical in $R[t_{ij}: i, j]$.

Proof.   The necessity of the condition is clear from (3.1). Now suppose that the ideals are radical. If $d = 0$ there is nothing to prove. If $d > 0$, then since (0) is radical M cannot be a minimal prime of (0) and hence cannot be an associated prime of (0), and the grade of M is $\geqq 1$.

We use induction on d. If $d = 1$, we are done. If $d > 1$, then we know that $\sum_j t_{1j}x_j$ is not a zerodivisor in $R[t_{1j}: j]$, and hence also not a zerodivisor in $R(t_{1j}: j) = R[t_{1j}: j]_{M'}$, where M' is the extension of M to $R[t_{1j}: j]$. $R(t_{1i}: j)$ is a faithfully flat local extension of R, and $R(t_{1j}: j)/M = K(t_{1j}: j)$. It suffices to show that $R(t_{1j}: j)$ is Cohen-Macaulay. But we may now apply the induction hypothesis to $R(t_{1j}: j)/(\sum_j t_{1j}x_j)$.

We are now ready to prove Theorem 2.   Let $d = \dim K[x]/I$. If $d = n, I = (0)$ and we are done. Assume that $d \leqq n - 1$, so that $d - 1 \leqq n - 2$. Then it suffices to show that if $1 \leqq p \leqq n - 2$, and $T = (t_{ij})$ is a p by n matrix of new indeterminates, then $I + (\sum_j t_{1j}x_j, \cdots, \sum_j t_{pj}x_j)$ is a radical ideal in $K[x, t]$. Hence, it suffices to show that the image of I in $R_1 = K[x, t]/(\sum_j t_{1j}x_j, \cdots, \sum_j t_{pj}x_j)$ is radical. Let $Y = (y_{ij})$ be the matrix of indeterminates described in Theorem 2, and let $Z = (z_{ij})$ be a p by $n - 1$ matrix of new indeterminates.

We have a K-homomorphism ϕ from $K[x, t]$ to $K[y, z]$ which takes each x_i to Δ_i, and which takes each entry of T to the corresponding entry of the product matrix ZY. Let $P = \ker \phi$. We shall show that $P = P'$, where $P' = (\sum_j t_{1j}x_j, \cdots, \sum_j t_{pj}x_j)$. It is clear that $P' \subset P$, and P' is prime by (3.1). It is then easy to see that $P' = P$: the point is that given $n - 2$ or fewer vectors in L^n, where L is any field, and a nonzero (respectively, a zero) vector v orthogonal to all of them, then we can choose $n - 1$ independent (respectively, dependent) vectors in the orthogonal complement of v such that (1) the original $n - 2$ or fewer vectors are in their span, and (2) if they form the rows of a matrix, the $n - 1$ by $n - 1$ minors of that matrix are the components of v.

Hence, we may regard $K[x, t]/P$ as a subring of $K[y, z]$. Let I^* be the image of I in $K[x, t]/P$. Our hypothesis gives us that $I^*K[y, z]$ is radical: the presence of the z's is irrelevant, since they do not occur in the generators for I^*.

Let R_2 be the subring of $K[y, z]$ generated by the Δ_i and the entries of the product matrix ZY, so that $R_2 = \operatorname{Im} \phi$ and $R_1 \cong R_2$. It suffices to show that $I^*K[y, z] \cap R_2 = I^*R_2$. It is actually the case that if J is any ideal of R_2 then $JK[y, z] \cap R_2 = J$.

To see this, let $SL(n - 1, K)$ act on the 1-forms of $K[y, z]$ thus: if $C \in SL(n - 1, K)$, let C act by taking the entries of Y (respectively, Z) into the corresponding ones of CY (respectively, ZC^{-1}). This extends to a rational action of $SL(n - 1, K)$ on $K[y, z]$. The ring of invariants is known classically to be precisely R_2. (See [8], Theorem (2.6.A), p. 45.) Note that if we had $p \geqq n - 1$, the reasoning would fail, for the $n - 1$ by $n - 1$ minors of Z would also be invariant.) Since $SL(n - 1, K)$ is reductive, we have a Reynolds retraction $K[y, z] \to R_2$ (which is a map of R_2-modules), and it follows that every ideal of R_2 when expanded to $K[y, z]$ and then contracted back to R_2 contracts to itself. (See [1], §V-2, [5], Propositions 9 and 10.)

4. **Linearly radical spaces.** Throughout this section, K is a field, R is a K-algebra, and V is a K-vector subspace of R. We first note two trivial facts:

(4.1) If V is linearly radical in R, and $I = (f_1, \cdots, f_m)$, where $f_1, \cdots, f_m \in V$, then $V/V \cap I$ is linearly radical in R/I.

(4.2) If V is linearly radical in R and $x_1, \cdots, x_n$ are indeterminates over R, then $V + \sum_i Kx_i$ is linearly radical in $R[x_1, \cdots, x_n]$.

Note that if $V \subset R$ is linearly radical, then R is reduced, for $0 \in V$, and (0) must be radical.

The next result is the crucial point in the proof of Theorem 3b).

(4.3) If V is linearly radical in R, x is an indeterminate over R, and $f \in V$, then the subspace of $R[x]$ spanned by V and xf is linearly radical.

Proof. Each finite-dimensional subspace of $V + Kxf$ has a basis consisting of at most one element of the form $g + cxf, g \in V, c \in K$, and several elements $v_1, \cdots, v_k$ of V. We may assume that we have already divided out by $(v_1, \cdots, v_k)$. We may also assume that K is algebraically closed. It then suffices to show that if $f, g \in V$, then $(fx + g)$ is radical in $R[x]$.

Let $F \in R[x]$ be an element of the radical of $(fx + g)$ not in $(fx + g)$. If we localize at $K[x] - \{0\}$, the resulting ring is $K(x) \otimes_K R$, and then $fx + g \in K(x) \otimes_K V$, and so generates a radical ideal. It follows that there is a nonconstant polynomial p in $K[x] - \{0\}$ such

that $pF \in (fx + g)$. We can write $p = (x - c_1) \cdots (x - c_m)$, where $c_1, \cdots, c_m \in K$. Let $p_t = (x - c_1) \cdots (x - c_t)$, $0 \leq t \leq m$ ($p_0 = 1$). Then for some t, $0 \leq t \leq m - 1$. $p_t F \notin (fx + g)$ while $(x - c_{t+1})p_t F \in (fx + g)$. Replacing F by $p_t F$, we can assume that $F \notin (fx + g)$ but $(x - c)F \in (fx + g)$ for a certain $c \in K$. After a change of variable $x - c \mapsto x$, we can assume that $xF \in (fx + g)$, but $F \notin (fx + g)$ (of course, F is altered by the change of variable, and we also have a new g: $fx + g = f(x - c) + (g + cf)$). After replacing F by a suitable power, we can also assume that $F \notin (fx + g)$ but $F^2 \in (fx + g)$.

Let us write $F = xF_1 + r_0$, where $r_0 \in R$ is the constant term of F. Since $xF, F^2 \in (fx + g)$, we have that $r_0 F \in (fx + g)$. We can also write:

$$(*) \qquad xF = (Gx^2 + s_1 x + s_0)(fx + g) ,$$

where $s_0, s_1 \in R$. Equating like coefficients, we have that $s_0 g = 0$, and that $r_0 = s_1 g + s_0 f$. We can also write $r_0 F = (Hx + t_0)(fx + g)$, where $t_0 \in R$, and equating constant terms gives that $r_0^2 = t_0 g$. Then:

$$
\begin{aligned}
(s_0 f)^3 &= (r_0 - s_1 g)^2(s_0 f) = r_2^0(s_0 f) - (s_0 g)f(2r_0 s_1 + s_1^2 g) \\
&= (t_0 g)(s_0 f) - 0 \ (\text{since } r_0^2 = t_0 g \ \text{ and } \ s_0 g = 0) \\
&= 0 \ .
\end{aligned}
$$

Hence, $s_0 f = 0$, since R is reduced. But then (*) yields $xF = (Gx^2 + s_1 x)(fx + g)$, or $F = (Gx + s_1)(fx + g) \in (fx + g)$, as required.

From (4.3) we can deduce at once:

(4.4)   If V is a linearly radical subspace of R, $f_1, \cdots, f_m \in V$, and $U = (u_{ij})$ is an m by n matrix of inderminates over R, then the subspace W of $R[u_{ij}: i, j]$ spanned by V and the entries of the product matrix $(f_1 \cdots f_m) U$ is linearly radical. (Note that (4.3) is the case $m = n = 1$.)

Proof. By repeated applications of (4.3), the space V' spanned by V and the elements $f_i u_{ij}$ is linearly radical, and W is a subspace of V'.

The following result contains Theorem 3b).

(4.5)   Let Λ be a partially ordered set as in Theorem 3b) and suppose that for each minimal element $\lambda \in \Lambda$, U_λ is a one-rowed matrix with entries in a linearly radical subspace V or R, while for the set of nonminimal λ the U_λ are matrices of indeterminates with all entries distinct. Then the space spanned by V and the entries of the product matrices $U_{\lambda_1} \cdots U_{\lambda_n}$ of the same type described in

Theorem 3b) (let us call these the *admissible* products) is linearly radical.

Proof. Clearly, we may reduce to the case where $\varLambda$ is finite. Let $\varLambda_1$ be the set of minimal elements of $\varLambda$, let $\varLambda' = \varLambda - \varLambda_1$, and let $\varLambda_2$ be the set of minimal elements of $\varLambda'$. Let V' be the space spanned by V and the entries of the admissible products $U_{\lambda_1} U_{\lambda_2}$, where, necessarily, $\lambda_i \in \varLambda_i$, $i = 1, 2$, and $\lambda_1 < \lambda_2$. By repeated applications of (4.4), V' is linearly radical.

If we replace $V, \varLambda, \{U_\lambda : \lambda \in \varLambda\}$ by $V', \varLambda', \{U'_\lambda : \lambda \in \varLambda'\}$, where $U'_\lambda = U_\lambda$ if $\lambda \in \varLambda' - \varLambda_2$, while $U'_\lambda = U_{\lambda_1} U_{\lambda_2}$ if $\lambda = \lambda_2 \in \varLambda_2$ (and λ_1 is the unique predecessor of λ_2), then $V', \varLambda'$, and $\{U'_\lambda : \lambda \in \varLambda'\}$ again satisfy our hypotheses, and the result follows by induction on the length of the longest chain in $\varLambda$.

Consider now the situation of Theorem 3b) with $\varLambda$ finite. The entries of the admissible products $U_{\lambda_1} \cdots U_{\lambda_n}$ such that λ_n is maximal can be shown to generate the ring of invariants of an action of a product of general linear groups on R, at least if K has characteristic zero. It was facts of this kind which led the author to prove Theorem 3b).

Suppose that K has characteristic zero and consider the class of ideals $I_{H,n}$ discussed in [4] and [5]. Working from Propositions 25, 26, 27, and 28 of [5], we have an embedding $K[X]/I_H \to K[u, v]$, such that the image $K[U^*V]$ is the ring of invariants of a reductive linear algebraic group acting on $K[u, v]$. Just as in the last paragraph of the proof of Theorem 2, there is a Reynolds operator from $K[u, v]$ to $K[U^*V]$, and to show that $I_{H,n}$ is radical it suffices to show that $I_{H,n}/I_H$ expands to a radical ideal in $K[u, v]$. But $I_{H,n}/I_H$, expanded to $K[u, v]$, is generated by a subset of the first row of the matrix U^*V, and this first row spans a linearly radical space, by Theorem 3b).

As was mentioned before, proving that the ideals $I_{H,n}$ are radical is critical in proving that they are also semiregular.

The situation with respect to the result of Theorem 3a) is quite similar. Suppose we knew that the space spanned by the minors is linearly radical for all $r \leqq s$, or even the weaker result that any subset of the minors generates a radical ideal (which we prove below for $r = 2$). A Reynolds operator argument then shows that within the homogeneous coordinate ring of the Grassmannian of r-dimensional subspaces of K^r, the ideal generated by a subset of the r by r minors is radical (this homogeneous coordinate ring is generated as a K-algebra by the r by r minors of an r by s matrix of indeterminates over K) when K has characteristic zero. The point is that the r by r

minors generate the ring of invariants of an action of $SL(r, K)$ on the polynomial ring generated over K by the entries of the r by s matrix of indeterminates. In particular, the ideals $\mathfrak{A}(B)$, $B \in \mathscr{B}$, and in particular, the ideals $\mathfrak{A}(B_o)$, discussed in [3], are radical, and this is a critical point in showing that homogeneous coordinate rings of Grassmann varieties and their Schubert subvarieties are Cohen-Macaulay.

We now prove Theorem 3a) for the case $s = r + 1$. It is easy to see that the linear changes of coordinates of the y_{ij} which are uniform on the colums of $Y = (y_{ij})$ are in one-to-one correspondence with the linear changes of coordinates in the vectors space spanned by the Δ_i. From this it follows that we need only prove:

(4.6) Let $Y = (y_{ij})$ be an r by $r + 1$ matrix of indeterminates over a field K. Let J_t be the ideal $(\Delta_i: 1 \leq i \leq r + 1 - t)$, where Δ_i is the r by r minor obtained by deleting the ith column. Then J_t is radical for each t.

Proof. If $t = 0$, J_t is known to be prime (see, for example, Proposition 26 of [5]) while the result is clear is $t \geq r$. Hence, we assume that $r > t > 0$, and we use induction on t, i.e. we assume the result for $t' < t$. It is easy to see that $\sqrt{J_t} = J_0 \cap I_t$, where I_t is the ideal generated by the t by t minors of the matrix formed from the last t columns of Y, and is therefore prime. In fact, since we are working with radical ideals, we need only see a corresponding fact about unions of varieties, and the result reduces to the observation that given an r by $r + 1$ matrix with entries in a field, then the r by r minors which contain the last t columns all vanish if and only if either all the r by r minors vanish (i.e. the matrix has rank $\leq r - 1$) or else the t by t minors of the last t columns vanish (the last t columns are dependent). For if the last t columns are independent and the matrix has rank r, we can find r independent columns containing the last t, and they give rise to a nonvanishing minor. Now $J_{t-1} = J_t + (\Delta)$ is radical, where $\Delta = \Delta_{r-t+2}$, by the induction hypothesis, so that $\sqrt{J_t} \subset J_t + (\Delta)$, and it follows that $\sqrt{J_t} \subset J_t + \Delta(\sqrt{J_t}: \Delta)$. But $\sqrt{J_t}: \Delta = (J_0: \Delta) \cap (I_t: \Delta)$, $J_0: \Delta$ is the unit ideal, and I_t is a prime not containing Δ, so that $I_t: \Delta = I_t$. Thus, $\sqrt{J_t} = J_t + \Delta I_t$. It suffices to show that if A is any t by t minor of the last t columns of Y, then $A\Delta \in J_t$. We might as well consider the case where A is formed from the first t rows. Now Y can be decomposed into submatrices thus:

$$Y = \begin{pmatrix} Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \end{pmatrix},$$

where the Y_i have t rows, the Z_i have $r - t$ rows, and Y_i and Z_i both have either $r + 1 - t$, 1, or $t - 1$ columns, according as $i = 1$, 2, or 3. Thus

$$\Delta = \det\begin{pmatrix} Y_1 & Y_3 \\ Z_1 & Z_3 \end{pmatrix}$$

and $A = \det(Y_2 \ Y_3)$. Form an auxiliary $r + t$ by $r + t$ matrix

$$Y^* = \begin{pmatrix} Y_1 & Y_2 & Y_3 & 0 \\ Z_1 & Z_2 & Z_3 & 0 \\ 0 & Y_2 & Y_3 & Y_3 \end{pmatrix}.$$

Then $\det Y^* = \pm A\Delta$. To see this, subtract each of the last $t - 1$ columns from the corresponding one of the next to last $t - 1$ columns. After permuting the columns, we have that $\det Y^*$ is the same, up to sign, as the determinant of

$$\begin{pmatrix} Y_1 & Y_3 & Y_2 & 0 \\ Z_1 & Z_3 & Z_2 & 0 \\ 0 & 0 & Y_2 & Y_3 \end{pmatrix}.$$

This matrix has block form, and the determinant is clearly $A\Delta$. On the other hand, if we subtract each of the first t rows of Y^* from the corresponding one of the last t, we find that Y^* has the same determinant as

$$\begin{pmatrix} Y_1 & Y_2 & Y_3 & 0 \\ Z_1 & Z_2 & Z_3 & 0 \\ -Y_1 & 0 & 0 & Y_3 \end{pmatrix}.$$

Any nonzero r by r minor of the first r rows comes from Y, and if that minor does not contain the last t columns of Y, then its cofactor has a column which is 0. Hence, the determinant of this last matrix is in J_t, and $A\Delta \in J_t$ as required.

We need the following result before we can prove the rest of Theorem 3a).

(4.7) Let s be an integer such that if $x_1, \cdots, x_s$ and $y_1, \cdots, y_s$ are indeterminates over a field K, then any subset of the 2 by 2 minors of

$$\begin{pmatrix} x_1 & \cdots & x_s \\ y_1 & \cdots & y_s \end{pmatrix}$$

generates a radical ideal in $R = K[x_1, y_1, \cdots, x_s, y_s]$. Then for every field K, for every $c \in K$, and for every nonnegative integer n, if $x_1, \cdots, x_s, y_1, \cdots, y_s,$ and $u_1, \cdots, u_n$ are independent indeterminates over K, then any subset of the 2 by 2 minors of

$$\begin{pmatrix} u_1 \cdots u_n & x_1 \cdots x_s \\ cu_1 \cdots cu_n & y_1 \cdots y_s \end{pmatrix}$$

generates a radical ideal in $R[u] = R[u_1, \cdots, u_n]$.

Proof. We use induction on s. Notice that if s has the required property so does each $s' < s$, and we may assume that if a matrix has the required form with $s' < s$ and any n, then any subset of its 2 by 2 minors generates a radical ideal. We then use induction on n. First note that by subtracting c times the first row from the second we may assume that $c = 0$ (there is a linear change of coordinates on the x's and y's). Let J be any ideal generated by a subset of the 2 by 2 minors. Let I be the subideal of J generated by those of the minors which do not involve the column which contains u_n. By the induction hypothesis, I is radical, and J is generated over I by certain of the products $u_n y_i$. After renumbering, we can assume that $J = I + u_n(y_1, \cdots, y_k), 1 \leq k \leq s$ (if $J = I$ we are done). We claim that $J = (I + (u_n)) \cap (I + (y_1, \cdots, y_k))$. Call the intersection J', and suppose that $j = i + u_n t \in J'$, where $i \in I$. Then $u_n t \in I + (y_1, \cdots, y_k)$. u_n is not a zerodivisor on $I + (y_1, \cdots, y_k)$, since the generators of that ideal do not involve u_n. Hence, $t \in I + (y_1, \cdots, y_k)$, and our claim is established.

It suffices then to show that $I + (u_n)$ and $I + (y_1, \cdots, y_k)$ are radical. The first is clear, since I is radical and its generators do not involve u_n. On the other hand, $R[u]/(I + (y_1, \cdots, y_k))$ is isomorphic to a quotient of $K[x_{k+1}, y_{k+1}, \cdots, x_s, y_s][u_1, \cdots, u_n, x_1, \cdots, x_k]$ by an ideal generated by a subset of the 2 by 2 minors of

$$\begin{pmatrix} u_1 \cdots u_n & x_1 \cdots x_k & x_{k+1} \cdots x_s \\ 0 \cdots 0 & 0 \cdots 0 & y_{k+1} \cdots y_s \end{pmatrix},$$

and hence is reduced by a special case of the result for $s' = s - k < s$.

We can now prove the last statement of Theorem (3a). We use induction on s. We must show that an ideal J generated by a subset of the minors of

$$\begin{pmatrix} x & x_1 \cdots x_s \\ y & y_1 \cdots y_s \end{pmatrix}$$

is radical, given the result for smaller such matrices of indeterminates.

Let I be the subideal of J generated by those minors which do not involve the first column. I is radical by the induction hypothesis. After a renumbering we can assume that $J = I + J_1$, where J_1 is the ideal generated by those 2 by 2 minors which have first column equal to $\begin{matrix} x \\ y \end{matrix}$ and second column equal to $\begin{matrix} x_i \\ y_i \end{matrix}$ for some $i, 1 \leq i \leq k$, where k is a fixed integer such that $1 \leq k \leq s$ (if $J = I$ we are done). Let H be the ideal generated by the 2 by 2 minors of

$$\begin{pmatrix} x_1 \cdots x_k \\ y_1 \cdots y_k \end{pmatrix} .$$

We first show that $J = (I + (x, y)) \cap (J + H)$. Let J' denote the intersection. Suppose that $j' = j + h \in J'$, where $j \in J$ and $h \in H$. Then $h \in J'$ and it suffices to show that $h \in J$. But $h \in (H+I) \cap ((x, y)+I) = H(x, y) + I$, because the generators of H and I do not involve the indeterminates x and y. It suffices then to show that for $1 \leq i, j \leq k$, $z(x_i y_j - x_j y_i) \in J$, where z denotes x or y. But

$$x(x_i y_j - x_j y_i) = x_i(x y_j - x_j y) - x_j(x y_i - x_i y) \in J ,$$

and

$$y(x_i y_j - x_j y_i) = y_i(x y_j - x_j y) - y_j(x y_i - x_i y) \in J ,$$

as required.

$I + (x, y)$ is clearly radical, since I is. It suffices, then, to show that $J + H$ is radical. Thus, we have reduced to the case where $H \subset I$.

If we localize at $K[x, y] - \{0\}$, then the image of J is radical. In fact, in the quotient we have that y_j is identified with $c x_j$ for $1 \leq j \leq k$, where $c = y/x \in K(x, y)$, so that the residue class ring is isomorphic to a quotient of $K(x, y)[x_1, \cdots, x_s, y_{k+1}, \cdots, y_s]$ by an ideal generated by a subset of the 2 by 2 minors of the matrix

$$\begin{pmatrix} x_1 \cdots x_k & x_{k+1} \cdots x_s \\ c x_1 \cdots c x_k & y_{k+1} \cdots y_s \end{pmatrix} ,$$

and the claim follows from the induction hypothesis and (4.7).

It is also clear that we need only consider the case where K is algebraically closed. Hence, if we let R denote the reduced K-algebra $K[x_1, y_1, \cdots, x_s, y_s]/I$, it is evident that we need only establish the following lemma:

(4.8) Let R be a reduced K-algebra over an algebraically closed field K. Suppose that the 2 by 2 minors of the matrix

$$\begin{pmatrix} a_1 \cdots a_k \\ b_1 \cdots b_k \end{pmatrix}$$

vanish, where the a_i and b_i are in R. Let x and y be indeterminates over R, and let J be the ideal generated by the 2 by 2 minors of the matrix

$$\begin{pmatrix} x \ a_1 \cdots a_k \\ y \ b_1 \cdots b_k \end{pmatrix}$$

in $R[x, y]$. If the image of J in the localization $R[x, y]$ at $K[x, t] - \{0\}$ is radical, then J is radical in $R[x, y]$.

Proof. $R[x, y]$ is a graded R-algebra (x and y have degree 1), and J is generated by 1-forms. Hence, $\sqrt{J}$ is a homogeneous ideal. Suppose that $\sqrt{J} \neq J$ and let F be a form of $R[x, y]$ in $\sqrt{J} - J$. Let $P = J : (F)$. Then P is a homogeneous ideal, so that $P \cap K[x, y]$ is homogeneous. After localization at $K[x, y] - \{0\}$, J expands to a radical ideal, so that we can assume that P contains a nonzero form of positive degree in $K[x, y]$. Since K is algebraically closed, we can write this form as a product $L_1 \cdots L_m$, where the L_i are 1-forms in $K[x, y]$. By replacing F by $L_1 \cdots L_t F$ for a suitable t, $0 \leq t \leq m - 1$, we can assume that $F \notin J$ but $LF \in J$ for a suitable 1- form L. By a linear change of coordinates on x, y, we can suppose that $L = x$. By replacing F by a suitable power, we can also assume that $F \notin J$ but $F^2 \in J$. Let $\deg F = d$.

We can write $F = F_1 x + r_0 y^d$. Since $xF, F^2 \in J$, we also have that $r_0 y^d F \in J$. Since $xF \in J$, we have that

$$(*) \qquad xF = \sum_{i=1}^{k} (G_i x^2 + s_i x y^{d-1} + t_i y^d)(b_i x - a_i y)$$

where the G_i are forms of degree $d - 2$ and the s_i and t_i are in R. It follows that

$$\sum_{i=1}^{k} t_i a_i = 0$$

and that

$$r_0 = \sum_{i=1}^{k} t_i b_i - \sum_{i=1}^{k} s_i a_i \ .$$

Let $b = \sum_i t_i b_i$. Then $b \in (r_0) + (a_1, \cdots, a_k)$, so that

$$b^2 \in (r_0^2) + (a_1, \cdots, a_s) \ .$$

Since $r_0 y^d F \in J$, we have that

$$r_0 y^d F = \sum_{i=1}^{k} (H_i x + u_i y^{2d-1})(b_i x - a_i y) \ ,$$

where the H_i are forms of degree $2d - 2$ and the $u_i \in R$. If we equate coefficients of y^{2d}, we find that $r_0^2 \in (a_1, \cdots, a_k)$, and hence $b^2 \in (a_1, \cdots, a_k)$. Since R is reduced, if $b \neq 0$ we can find a prime Q not containing b. Let $'$ denote reduction modulo Q. Since $b'^2 \neq 0$, we have that the the row

$$(a_1' \cdots a_k')$$

is not 0. Since the 2 by 2 minors of

$$\begin{pmatrix} a_1' \cdots a_k' \\ b_1' \cdots b_k' \end{pmatrix}$$

vanish, there is a constant c in the fraction field of the domain R/Q such that $(b_1' \cdots b_k') = c(a_1' \cdots a_k')$. Then since $\sum_i t_i' a_i' = 0$, we have that $b' = \sum_i t_i' b_i' = c \sum_i t_i' a_i' = 0$, a contradiction.

Hence, we must have $b = 0$, i.e. $\sum_i t_i b_i = 0$. But then (*) yields:

$$F = \sum_i (G_i x + s_i y^{d-1})(b_i x - a_i y) \in J,$$

and we are done.

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