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SEMI-PRIMARY SPLIT RINGS

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This paper is concerned with Hochschild's "maximal algebra" which has also been discussed by Jans, Jans and Nakayama, and Zaks.

A category of rings for which the maximal algebra construction is valid is first defined. These are the "split rings" of the title. These rings include the split rings introduced by Jans and Nakayama but they need not be semi-primary. A second category consisting of a kind of sheaf over a directed graph is introduced. Using this second category the maximal algebra construction is exhibited as a composition of adjoint functors, and hence gives the universal mapping property of the maximal algebra. The properties of the sheaf category are then used to show that a semi-primary split ring is the image of a semi-primary ring of a special type.

In [6] the spectrum of a right perfect ring was modified to a rected graph. Although the resulting functor $(\text{rings})^{op} \to \text{graphs}$ preserves and reflects coproducts it does not possess an adjoint. We modify the graph to a kind of sheaf by putting stalks of rings over the vertices and stalks of bimodules over the arrows. The resulting ring $\to (\text{sheaves})^{op}$ has a right adjoint $(\text{sheaves})^{op} \to \text{rings}$ and the composition functor $\text{rings} \to \text{rings}$ is essentially Hochschid's "maximal algebra" ([7], [8] and [9]).

Coupling this with known results (Theorem 2) gives the universal mapping property of the maximal ring of a split semi-primary ring [9].

Lastly we use the geometry of the sheaf category to show that that every semi-primary split ring is a homomorphic image of a certain type of subring of a split, hereditary generalized triangular matrix ring. This result is stated twice below. First, as Theorem 3, it is stated in the special terminology of the sheaf category. Following Theorem 3 we make such definitions as are necessary to restate the result in ring theoretic terms: Theorem 3′.

The formality of the proofs somewhat obscures the construction of the special ring so the construction is indicated for a special case following Theorem 3′.

1. Definitions.

In this section we define the categories and functors to be used in subsequent sections. The symbols introduced will retain their meaning throughout the paper.

By a "ring homomorphism" we understand an additive and multiplicative mapping of rings with unit which carries unit to unit. Unidentified maps are the obvious ones.

DEFINITION. A *split ring* R is a quadruple (a, b, c, d) such that (R1) and (R2) below hold.

(R1) a: $R \to \Gamma$, b: $\Gamma \to R$ are ring homomorphisms with $ab = 1_\Gamma$ and Γ a semi-simple ring (with minimum condition).

If we set $I = \ker a$ then I and I^2 are two-sided ideals of R and hence $R - R$ bimodules. The map b then induces a $\Gamma - \Gamma$ bimodule structure on I and I^2.

(R2) c: $I \to I/I^2$, d: $I/I^2 \to I$ are $\Gamma - \Gamma$ bimodule homomorphisms such that $cd = 1_I$.

We set $A = \operatorname{Im} d$ and then we have $R = \Gamma \oplus A \oplus I^2$, $I = A \oplus I^2$ as $\Gamma - \Gamma$ bimodules. This is the Jans-Nakayama definition (see [9]) when I is the Jacobson radical of R and R is semi-primary.

DEFINITION. A *morphism of split rings* $R \to R'$ is a quadruple $(\alpha, \beta, \gamma, \delta)$ satisfying (R3), (R4), and (R5) below.

(R3) α: $R \to R'$ and β: $\Gamma \to \Gamma'$ are ring homomorphisms, γ and δ are induced by α, and the diagrams

$$\begin{array}{ccc} R & \underset{b}{\overset{a}{\rightleftarrows}} & \Gamma \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ R' & \underset{b'}{\overset{a'}{\rightleftarrows}} & \Gamma' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} I & \underset{d}{\overset{c}{\rightleftarrows}} & I/I^2 \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ I' & \underset{d'}{\overset{c'}{\rightleftarrows}} & I'/I'^2 \end{array}$$

commute.

(R4) If A' is a maximal two-sided ideal of Γ' then

$$\ker [\Gamma \longrightarrow \Gamma' \longrightarrow \Gamma'/A']$$

is a maximal two-sided ideal of Γ.

(R5) If A', B' are maximal two-sided ideals of Γ' and

$$P' = \ker [R' \longrightarrow \Gamma' \longrightarrow \Gamma'/A'], \qquad Q' = \ker [R' \longrightarrow \Gamma' \longrightarrow \Gamma'/B']$$
$$P = \ker [R \longrightarrow R' \longrightarrow R'/P'], \qquad Q = \ker [R \longrightarrow R' \longrightarrow R'/Q']$$

then the induced monomorphism $R/PQ \to R'/P'Q'$ carries the Jacobson radical of R/PQ onto the Jacobson radical of $R'/P'Q'$.

By the *split spectrum*, $\operatorname{SSpec}(R)$, of R we shall mean the set of ideals of R of the form $\ker [R \to \Gamma \to \Gamma/A]$ as A runs through the set of maximal two-sided ideals of Γ. The axiom (R4) then assures

that a morphism $R \to R'$ induces a map $\mathrm{SSpec}(R') \to \mathrm{SSpec}(R)$.

In [6] we introduced in a somewhat different context the skeleton, $Sk_r R$, of a right perfect ring. Axiom (R5) assures that a morphism $R \to R'$ induces a morphism $Sk_r R' \to Sk_r R$. We shall denote the category of split rings by *SRng*.

DEFINITION. A *sheaf of simple rings over a finite graph, F,* consists of

(S1) An oriented graph G. That is, a set of arrows A_G, a set of vertices V_G and two functions $h, t: A_G \to V_G$ called *head* and *tail.* We assume that A_G and V_G are finite.

(S2) For each $x \in V_G$ a ring $R(x)$ which is simple (with d.c.c.).

(S3) For each $a \in A_G$ a left $R(ta)$ and right $R(ha)$ bimodule $M(ta, ha)$.

We further assume

(S4) The map $(t, h): A_G \to V_G \times V_G$ is one-to-one.

From now on we shall abbreviate "sheaf of simple rings over a finite graph" to "sheaf."

DEFINITION. A morphism of sheaves $\eta: F \to F'$ consists of

(S5) A morphism $\eta_G: G \to G'$ of graphs. That is, a pair of functions $A_G \to A_{G'}$, $V_G \to V_{G'}$ which commute with t and h.

(S6) For each $x \in V_G$ a ring homomorphism $\eta(x): R'(\eta x) \to R(x)$.

(S7) For each $a \in A_G$ a homomorphism

$$\eta(ta, ha): M(t'\eta a, h'\eta a) \longrightarrow M(ta, ha) \quad \text{of} \quad R'(t'\eta a) - R'(h'\eta a)$$

bimodules. Here the bimodule structure on $M(ta, ha)$ is induced by $\eta(ta)$ and $\eta(ha)$.

We further assume

(S8) $\eta(ta, ha)$ is an isomorphism for each a in A_G.

We denote this category of sheaves by *grass.*

Next we define a functor S: $(SRng)^{op} \to grass$.

Let $R \in SRng$. For the graph of SR we adopt the skeleton of [6] to split rings. The vertices of the graph, G, are the elements of the set $\mathrm{SSpec}(R)$. If $P \in \mathrm{SSpec}(R)$ we set $R(P) = R/P$. We have an arrow $P \to Q \in A_G$ whenever the Jacobson radical of R/PQ is non-zero. This radical is $P \cap Q/PQ$ which is a left R/P and right R/Q bimodule. If $P \to Q \in A_G$ we set $M(P, Q) = P \cap Q/PQ$. The axioms (R4) and (R5) now assure that the canonical definition of S on morphisms defines a functor to *grass.*

PROPOSITION 1. *Let $R \in SRng$ and give R/I^2 the canonical split ring structure. Then we may identify SR and $S(R/I^2)$ using the*

natural map $R \to R/I^2$.

Proof. The correspondence $P \leftrightarrow P/I^2$ is a one-to-one correspondence between SSpec(R) and SSpec(R/I^2). We use the Nöther isomorphisms to identify R/P with $(R/I^2)/(P/I^2)$ and R/PQ with $(R/I^2)/(P/I^2)(Q/I^2)$.

Finally, we define T: $(grass)^{op} \to SRng$. Let $F \in grass$ with associated graph G. We first define a sequence of abelian groups T_iF, $i = 0, 1, 2, \cdots$

Let $A_G^0 = V_G$ and A_G^n be the set of all sequences of arrows

$$x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_n$$

in G. Given such a sequence of arrows we may form the left $R(x_0)$ and right $R(x_n)$ bimodule

$$M(x_0, x_1) \otimes_{R(x_1)} M(x_1, x_2) \otimes \cdots \otimes_{R(x_{n-1})} M(x_{n-1}, x_n) .$$

We define T_0F to be the ring direct product $\Pi R(x)$, $x \in V_G$ and T_nF, $n > 0$, to be the abelian group direct product over A_G^n of the above bimodules.

Now set $TF = \bigoplus_0^\infty T_iF$, the direct sum of abelian groups. To define a ring structure on TF it is sufficient to define pq for

$$p \in M(x, \cdot) \otimes \cdots \otimes M(\cdot, y) \quad \text{and} \quad q \in M(z, \cdot) \otimes \cdots \otimes M(\cdot, w) .$$

We define $pq = p \otimes q$ if $y = z$ and $pq = 0$ otherwise. If p or q is in T_0F we omit the tensor symbol. With this definition T_0F is a semi-simple ring with d.c.c., T_1F is a T_0F bimodule and TF is the tensor ring of T_1F over T_0F.

For a split ring structure on TF we take $\Gamma = T_0F$, $I = \bigoplus_1^\infty T_iF$ and $A = T_1F$. Then $TF = \Gamma \oplus A \oplus I^2$ as $\Gamma - \Gamma$ bimodules and we use the natural projections and injections for (a, b, c, d).

Next assume that η: $F \to F' \in grass$. To define a ring homomorphism $T\eta$: $TF' \to TF$ it is sufficient to construct $T_0\eta$: $T_0F' \to T_0F$ a ring homomorphism and $T_1\eta$: $T_1F' \to T_1F$ compatible with $T_0\eta$.

We define $T_0\eta$ to be the unique map such that all diagrams

$$\begin{array}{ccc} \Pi_{V_{G'}} R'(x) & \xrightarrow{T_0\eta} & \Pi_{V_G} R(y) \\ \Big\downarrow{\scriptstyle \pi'_x} & & \Big\downarrow{\scriptstyle \pi_y} \\ R'(x) & \xrightarrow{\alpha_{xy}} & R(y) \end{array}$$

commute where the vertical maps are direct product projections and α_{xy} is $\eta(x)$ if $y = \eta_G x$ and $\alpha_{xy} = 0$ otherwise. $T_1\eta$ is defined similarly after replacing V_G by A_G.

2. Properties of the functors.

PROPOSITION 2. *(grass)$^{op} \to SRng$ is a functor and ST is naturally equivalent to the identity functor on grass.*

Proof. Let $\eta: F \to F' \in grass$. We must first verify that $T\eta$ is a morphism of split rings. Let A be a maximal two-sided ideal of T_0F. Then there is an $x \in V_G$ such that A is the kernel of the projection $\pi_x: T_0F \to R(x)$. Then

$$T_0F' \xrightarrow{T_0\eta} T_0F \xrightarrow{\pi'_x} R(x) = T_0F' \xrightarrow{\pi_y} R'(y) \xrightarrow{\eta(x)} R(x)$$

where $y = \eta_G x$. Since $R(x)$ and $R'(y)$ are simple and $\eta(x)$ is a homomorphism of rings with unit, $\eta(x)$ is one-to-one and

$$\ker \eta(x)\pi'_y = \ker \pi'_y$$

which is a maximal two-sided ideal of T_0F'. Thus (R4) is satisfied.

LEMMA. *Let $R \in SRng$ with $I^2 = 0$, $\mathrm{SSpec}(R) = \{P_1, \cdots, P_n\}$ and and $\beta: \Pi_1^n R/P_i \to \Gamma$ the natural isomorphism. Let e_i be the image of the identity of R/P_i under $b\beta$. Then $e_ie_j = \delta_{ij}$, $1 = \sum e_i$ and the natural map $R \to R/P_iP_j$ carries e_iIe_j monomorphically onto the Jacobson radical of R/P_iP_j.*

Proof. Clearly $\sum e_i = 1$ and $e_ie_j = \delta_{ij}$. Since $I = \bigcap_1^n P_i$ we see that $P_i = \sum e_kX_k$ where $X_k = R$ if $k \neq i$ and $X_i = I$. Similarly $P_j = \sum X_ke_k$ where $X_k = R$ if $k \neq j$ and $X_j = I$. Thus we have $P_iP_j = \sum e_kX_{kl}e_l$ where $X_{kl} = R$ if $k \neq i$ or $l \neq j$, $X_{il} = I$ of $l \neq j$, $X_{kj} = I$ if $k \neq i$ and $X_{ij} = I^2 = 0$.

Now as Γ-bimodules, $R = I \oplus b\Gamma$ and hence $e_kIe_l = e_kRe_l$ if $k \neq l$. Thus writing $R = \sum e_kRe_l$ we see that if $i \neq j$ then R/P_iP_j is isomorphic to the matrix ring

$$\begin{bmatrix} e_iRe_i/e_iIe_i & e_iIe_j \\ 0 & e_jRe_j/e_jIe_j \end{bmatrix}.$$

Since β^{-1} a carries e_kRe_k onto R/P_k with kernel e_kIe_k we see that the lemma holds for $i \neq j$. For $i = j$ we have that R/P_iP_j is isomorphic to e_iRe_i which has radical e_iIe_i, since $I^2 = 0$ implies that I is the radical of R.

Now let $R' \in SRng$ and set $R = TSR'$. Identify SR' and $S(R'/I^2)$ (Proposition 1). We define a map $R \to R'$ as follows. Since

$$\Gamma = T_0SR' = \Pi_1^n R'/P_i'$$

we take $\beta\colon \Gamma \to \Gamma'$ as in the lemma. An arrow $P_i' \to P_j'$ is in the graph of SR' iff the radical of $R'/P_i'P_j'$ is nonzero and then the associated stalk is the radical. The group T_1SR' is the product of these radicals. On the othe hand, I'/I'^2 is the product of the groups $e_i(I'/I^2)e_j$. Thus we have a mapping $\delta\colon T_1SR' \to R'$ and this mapping is a T_0SR'-bimodule isomorphism. Thus, by the remarks preceding the lemma we have a ring homomorphism $\Phi(R)\colon TSR' \to R'$ which satisfies (R3). Since β and δ are isomorphisms (R4) and (R5) are satisfied as well. It is not difficult to see that $\Phi(R)$ is natural in R.

THEOREM 1. *The functors* $S\colon (SRng)^{op} \to grass$ *and* $T\colon (grass)^{op} \to SRng$ *are adjoints of the type*

$$\mathrm{Hom}\,(TF, R) \cong \mathrm{Hom}\,(SR, F)\ .$$

Proof. If we consider S to be a functor $SRng \to (grass)^{op}$ then the theorem states that S is a right adjoint for T. Let

$$\Psi(F)\colon F \longrightarrow STF \in (grass)^{op}$$

be the natural equivalence of Proposition 2. To prove the theorem it is sufficient (see [4] or [10]) to show that the compositions

$$S[\Phi(R)] \circ \Psi(SR)\colon SR \longrightarrow STSR \longrightarrow SR$$

and

$$\Phi(TF) \circ T(\Psi(F))\colon TF \longrightarrow TSTF \longrightarrow TF$$

are identities. This is almost clear.

3. Split semi-primary rings.

DEFINITION. A split ring $R = \Gamma + I$ is *semi-primary* if $I^n = 0$ for some $n > 0$.

A semi-primary split ring is semi-primary in the usual sense with Jacobson radical I.

In what follows we shall often confuse the quadruple $\Phi(R)$ with its first component.

PROPOSITION 3. *Let* R *be a semi-primary split ring. Then* $\Phi(R)$: $TSR \to R$ *is onto with kernel contained in* $(\bigoplus_1^\infty T_iSR)^2$.

Proof. From the discussion preceding Theorem 1 we see that $\Phi(R)$ carries T_0SR isomorphically onto $b\Gamma$ and T_1SR isomorphically onto $A = \operatorname{Im} d$. Thus $I = I^2 + \phi(R)[\bigoplus_1^\infty T_iSR]$. Since I is nilpotent it follows from [8, Lemma 1.2] that $\Phi(R)$ is onto. It follows from (R3) that the kernel is in $(\bigoplus_1^\infty T_iSR)^2$.

DEFINITION. Let $F \in grass$ with graph G. We say that F has *no closed paths* if for each path $x \to y \to z \cdots \to z$ in G, $x \neq z$.

Suppose that F has no closed paths, then TF is isomorphic to a generalized triangular matrix ring in Harada's sense [5]. To see this first index the vertices of the graph of F as follows. First assign the integers $1, 2, \cdots, n_{k_1}$ to those vertices which are not the tails of any arrows. Erase these vertices and all arrows leading to them obtaining a graph G'. Assign the integers $n_{k_1} + 1, \cdots, n_{k_2}$ to these vertices of G' which are not the tails of any arrows in G'. Continue in this manner until V_G is exhausted. Now set

$$M_{ii} = R_i = R(x_i)$$

and let M_{ij} be the direct sum of all left R_i and right R_j modules $M(x_i, \cdot) \otimes \cdots \otimes M(\cdot, x_j)$ for which $x_i \to \cdots \to x_j$ is a path in G. The multiplication in TF then induces functions $M_{ij} \times M_{jk} \to M_{ik}$ and the resulting generalized triangular matrix ring $T(R_i, M_{jk})$ is isomorphic to TF.

Since $T_0F = \Gamma$ is semi-simple, $M \otimes_\Gamma N = 0 \Rightarrow M = 0$ or $N = 0$. Thus we have

PROPOSITION 4. *Let $F \in grass$ then TF is a semi-primary split ring iff F has no closed paths.*

The next statement rephrases known results in our language. By results in [1] we need not distinguish between left and right homological dimensions.

THEOREM 2. *Let R be a semi-primary split ring. There is an $F \in grass \ni R \cong TF \in SRng$ if and only if the ring R is hereditary.*

Proof. Assume that R is hereditary. Since I is the radical of R [3, Theorem 8] shows that the global dimension of R/I^2 is finite. Thus by [6, Theorem 3.1] SR has no closed paths and hence TSR is a semi-primary split ring. Since $\Phi(R)$: $TSR \to R$ is onto with kernel in $(\bigoplus_1^\infty T_iSR)^2$ [2, Theorem I] shows that $\Phi(R)$ is an isomorphism.

Conversely, assume $R \cong TF$. Then TF is semi-primary and by [9, Proposition 4] TF is an hereditary ring.

An example due to Zaks ([11] or [12]) shows that not every hereditary generalized triangular matrix ring is split and hence not every hereditary generalized triangular matrix ring is of the form TF for some $F \in grass$.

DEFINITION. A morphism $\eta: C \to F \in grass$ is a *covering* if the corresponding map on graphs, $H \to G$, is onto and for each $x \in V_H$ or $\alpha \in A_H$ the stalkwise maps $\eta(x)$ and $\eta(ta, ha)$ are identities.

LEMMA. *If $F \in grass$ then there is a covering $\eta: C \to F$ in grass where C has no closed paths. Further, one may assume that C has at most one path between any two vertices of its graph.*

Proof. Let G be the graph of F and let $x \in V_G$. We define a graph G' as follows. $V_{G'} = V_G \amalg \{x''\}$ (disjoint union) and $A_{G'} = A_G$. Let $h, t: A_G \to V_G$ be the head and tail functions for G,

$$H_x = \{a \in V_G \mid h(a) = x\} \quad \text{and} \quad T_x = \{a \in V_G \mid t(a) = x\} .$$

Choose decompositions $H_x = H_1 \amalg H_2$ and $T_x = T_1 \amalg T_2$.

We next define $h', t': A_{G'} \to V_{G'}$. If $a \in H_x$ we set $h'(a) = x$ if $a \in H_1$ and $h'(a) = x'$ if $a \in H_2$. If $a \in A_{G'} \setminus H_x$ we set $h'(a) = h(a)$. If $a \in T_x$ we set $t'(a) = x$ if $a \in T_1$ and $t'(a) = x'$ if $a \in T_2$. If $a \in A_{G'} \setminus T_x$ we set $t'(a) = t(a)$. This defines G'. We shall say that G' is obtained from G by *making a cut at x*.

Now define a morphism of graphs $\eta_G: G' \to G$ by taking η_G to be the identity on $V_{G'} \setminus \{x\}$ and $A_{G'}$ and setting $\eta_G(x') = x$. We extend η_G to a covering $\eta: C \to F$ by "pulling the stalks back along η_G." That is, if $x \in V_{G'}$ we set the stalk over x equal to $R(\eta_G x)$, the stalk of F over $\eta_G(x)$, and let $\eta(x): R(\eta_G x) \to R(\eta_G x)$ be the identity. Similarly for the stalks over arrows.

Since the composition of coverings is a covering we may make successively a finite number of cuts to obtain a covering $C \to F$ where C has no closed paths.

In proving the second statement we may assume that F has no closed paths. If $x, y \in V_G$ such that there are two paths from x to y whose final arrows α, β are such that $\alpha \neq \beta$ then make a cut at y such that α and β no longer have the same head. This gives a covering $F_1 \to F$. If there are two paths in the graph of F_1 with the same beginning and end, make another cut and get a covering $F_2 \to F_1$, etc. This process will end in a finite number of steps.

If $\eta: C \to F$ is a covering and C has no closed paths then $T\eta: TF \to TC$ is a morphism of TF to a split, hereditary generalized

triangular matrix ring. Further assume that there is at most one path between any two vertices of C and let R' be the set theoretic image of $T\eta$ in TC. The (x, y) entry of the matrix ring TC is zero if there is no path from x to y in the graph of C or varies in the $R(x)$, $R(y)$-bimodule associated to the unique path from x to y. If the path from x to y has the same image under η as the path from z to w then, as abelian groups, the (x, y) module for TC and the (z, w) module for TC are identical and if a matrix lies in R' then its (x, y) entry equals its (z, w) entry. The ring R' is obtained from TC by setting certain entries of TC equal to each other. For examples of rings R' see [13, Examples 5.7, 6.7, and 7.7].

Now suppose that $F = SR$ where R is a semi-primary split ring. We have $\Phi(R)$: $TF \to R$, an onto ring homomorphism. Thus R will be an image of R' if $\ker T\eta \subset \ker \Phi(R)$. If I is the radical of R and $I^{n-1} \neq 0$, $I^n = 0$ then $\ker \Phi(R) \supset \bigoplus_n^\infty T_iF$. Thus it is sufficient to have $\ker T\eta \subset \bigoplus_n^\infty T_iF$. This means that every path of length n in F should be the image of a path of length n in C under η. To accomplish this we merely construct C from F by making cuts as described in the proof of the lemma and then attach as many distinct paths of appropriate length as we need to the freshly cut ends. Thus we have

THEOREM 3. *Let R be a semi-primary split ring. Then there is a covering η: $C \to SR$ where C has no closed paths and at most one path between any two vertices and* $\ker T\eta \subset \ker \Phi(R)$.

The generalized triangular matrix rings TC are of a special type. One is given a set of simple rings S_i $1 \leqq i \leqq n$ (corresponding to the vertices of the graph C) and a set of left S_i and right S_j bimodules, ${}_iM_j$ (corresponding to the arrows of C).

Form a symbolic matrix M with S_i in the iith place and ${}_iM_j$ in the ijth place. Compute the formal products M^k using the tensor product to multiply bimodules and using the identities: $M \otimes_s S = M$, $S \otimes_s M = M$. The sequence $[M^k]$ stabilizes in less than n steps. The ring TC may be defined as the generalized matrix ring for which the ijth entry of a ring element varies in the ijth entry of M^n.

The condition that there be at most one path between any two vertices of C means that given i and j there is at most one sequence $i = i_1, i_2, \cdots, i_k = j$ such that ${}_{i_l}M_{i_{l+1}}$ is nonzero for each l. This means that any summations necessary to form the product M^k are automatically defined.

Call the generalized triangular matrix rings defined in this way *pure tensorial triangular rings.*

The subrings R^1 of TC are easily defined in terms of M or M^n. If the elements on the main diagonal of TC in places $i_1, \cdots, i_k$ are to be identified, replace $S_{i_1}, \cdots, S_{i_k}$ in M by the same symbol, say S_{i_1}. To do this we must of course have $S_{i_1}, \cdots, S_{i_k}$ identical as abstract rings. Next replace ${}_iM_j$ and ${}_kM_l$ by the same symbol if the places i and k as well as the places j and l are to be identified. This process gives a new symbolic matrix M'. The subring R' of TC is defined by requiring that two entries of a matrix be equal whenever the corresponding entries of $(M')^n$ are equal. Call the rings R' defined in this way *special subrings* of pure tensorial triangular rings. For examples of such rings see [13, Examples 5.7, 6.7, and 7.7] and below.

With this notation we may restate Theorem 3.

THEOREM 3'. *Every semi-primary split ring is a homomorphic image of a special subring of a pure tensorial triangular ring. The kernel of the homomorphism is contained in the square of the radical.*

EXAMPLE. We will work through the construction of C and R' for a specific ring: a finite dimensional algebra R over an algebraically closed field F with the radical N such that R/N has two simple components and $N^4 = 0$.

First notice that such rings are split. By the Wedderburn principal theorem there is a semi-simple ring $\Gamma \cong R/N$ contained in R such that $R = \Gamma \oplus N$, a direct sum of Γ-bimodules. The ideal N^2 is a subbimodule of N and since F is the center of R this bimodule structure is equivalent to a left $\Gamma \otimes_F \Gamma^0$ structure. The semi-simple ring Γ is isomorphic to a product of full matrix algebras over F. The well-known isomorphism $M_n(F) \otimes_F M_m(F)^\circ \cong M_{nm}(F)$ shows that $\Gamma \otimes_F \Gamma^\circ$ is semi-simple and hence that N^2 is a bimodule summand of N. It follows that R is a split semi-primary ring.

The stalks over the vertices of SR are the simple components of Γ. Denote them by $S_i \cong M_{n_i}(F)$. The stalks ${}_iM_j$ over arrows are radicals of homomorphic images of R. Thus they are in fact left $M_{n_i}(F) \otimes_F M_{n_j}(F)^0$ modules and hence are semi-simple as bimodules. It follows that ${}_iM_j$ is isomorphic to a finite direct sum of copies of $M_{n_i,n_j}(F)$, the set of $n_i \times n_j$ matrices with entries in F.

The tensor product of $M_{i,j}(F)$ and $M_{j,k}(F)$ over $M_j(F)$ is isomorphic $M_{i,k}(F)$ as left $M_i(F)$ and right $M_j(F)$ bimodules. This isomorphism carries tensor products of individual matrices to the product of the matrices.

It follows from the last paragraph that entries of TC will be formal sums of matrices and the multiplication in TC will be given

by matrix multiplication plus the distributive laws. In particular if the bimodules ${}_iM_j$ are all simple TC will be isomorphic to a subring of $M_k(F)$ for some k.

Since Γ has two components there are ten possibilities for the graph of SR. These reduce to seven if R is assumed indecomposable [6, § 2]. Suppose the vertices and arrows of the graph G are given by

$$V = \{1, 2\}\,, \qquad A = \{12, 21, 11\}$$

where 12 is an arrow from 1 to 2 etc.

We wish to find a morphism of graphs η: $C \to G$ where C has at most one path between any two vertices. It may happen that $N^3 \neq 0$ so we also need the condition that any path of length three in G is the image of a path in C. One choice of C and η is given by

$$V_C = \{1, \cdots, 9\}\,,$$
$$A_C = \{41, 52, 61, 54, 76, 75, 87, 98\}$$

with η defined on V_C by

$$\eta(k) = 2 \qquad \text{if and only if } k = 8, 6, 2\,.$$

The matrix M' is 9×9. On the main diagonal M' has S_2 in the 8th, 6th and 2nd places and S_1 elsewhere. Off the main diagonal it has ${}_1M_1$ in places 41, 54, and 75; M_2 in places 52, 76, and 98; ${}_2M_1$ in places 63 and 87 and zeros elsewhere. If N' is the radical of R' then $R \cong R/I$ where $(N')^2 \supset I \supset (N')^4$.

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