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ON THE FITTING LENGTH OF A SOLUBLE LINEAR GROUP

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Let G be a finite soluble completely reducible linear group of degree n over a perfect field. It is shown that the Fitting length $l(G)$ of G satisfies the inequality

$$l(G) \leqq 3 + 2\log_3(n/2),$$

and that this bound is best possible for infinitely many values of n.

Let G be a soluble completely reducible linear group of degree $n > 1$ over a perfect field k. Huppert shows in [3], Satz 10, that the derived length of G is at most $6\log_2 n$. This is therefore an upper bound for the Fitting length $l(G)$ of G as well. In this note we assume in addition that G is finite and prove that

$$l(G) \leqq 3 + 2\log_3(n/2).$$

We show further that this bound is actually achieved for infinitely many values of n.

LEMMA 1. *Let K be a normal subgroup and M a maximal subgroup of a finite soluble group G. Then $l(M \cap K) \geqq l(K) - 2$.*

Proof. Let $l(K) = l$. The result is clearly true when $l \leqq 2$; therefore assume $l > 2$ and proceed by induction on $|G|$. Set $F = F(K)$, the Fitting subgroup of K. Since F is the direct product of its Sylow subgroups, each of which is normal in G, we may assume by the R_0-closure of the class $L(l)$ of groups of Fitting length at most l that F has a Sylow p-complement S such that $l(K/S) = l$. Suppose $S \neq 1$. If $S \leqq M$, then M/S is maximal subgroup of G/S, and so by induction $l(M \cap K/S) \geqq l(K/S) - 2 = l - 2$. But then $l(M \cap K) \geqq l - 2$, as required. On the other hand, if $S \nleqq M$, then $M \cap K/M \cap S \cong S(M \cap K)/S = K/S$ has Fitting length l, whence $l(M \cap K) = l$. Hence we may assume that $S = 1$ and that F is a p-group. Set $L/F = F(K/F)$. Then $l(K/L) = l - 2$ and L/F is a p'-group. There are two possibilities to consider:

(a) $L \nleqq M$. In this case $L(M \cap K) = K$ and therefore $M \cap K/M \cap L \cong K/L$ has Fitting length $l - 2$. Hence $l(M \cap K) \geqq l - 2$.

(b) $L \leqq M$. In this case, denoting the Fitting subgroup of $M \cap K$ by $\bar{F}$, since $F \leqq \bar{F}$ and $C_K(F) \leqq F$, we see that $\bar{F}$ is a p-group. But then $\bar{F}/F$ is a normal p-subgroup of $M \cap K/F$, and so $\bar{F}/F \leqq$

$C_{K/F}(L/F) \leqq L/F$, a p'-group. Thus $F = \bar{F}$ and $l(M \cap K) = 1 + l(M \cap K/F)$. By induction, $l(M \cap K/F) \leqq l(K/F) - 2 = l - 3$, and so in this case too the conclusion of the lemma holds.

LEMMA 2. *Let T be an extra-special group of order 2^7, and let A be a soluble subgroup of* Aut(T) *acting irreducibly on $T/\Phi(T)$. Then $l(A) \leqq 4$.*

Proof. By Huppert, [4], III.13.9 (b), A is a subgroup of an orthogonal group of dimension 6 over the field of 2 elements; hence by Dieudonné [2], p. 68, $|A|$ divides $2^7 \cdot 3^2 \cdot 5 \cdot 7$ or $2^7 \cdot 3^4 \cdot 5$. Since $T/\Phi(T)$ is an irreducible $Z_2[A]$-module, $2 \nmid |F(A)|$ and hence $|F(A)|$ is a divisor of $3^2 \cdot 5 \cdot 7$ or $3^4 \cdot 5$. Let $F(A)/K$ be a chief factor of A and set $\bar{A} = A/C_A(F(A)/K)$. If $|F(A)/K| = 3, 3^2, 5$ or 7, examination of the corresponding linear groups shows that $l(\bar{A}) \leqq 3$. If $|F(A)/K| = 3^3$, $\bar{A}$ is isomorphic with a subgroup of $GL(3, 3)$. Its order therefore divides $|GL(3, 3)| = 2^5 \cdot 3^3 \cdot 13$. But its order also divides $2^7 \cdot 3^4 \cdot 5/3^3$ and therefore divides $2^5 \cdot 3$. But then $0_{2,3,2}(\bar{A}) = \bar{A}$ and so we have $l(\bar{A}) \leqq 3$. Since a Sylow 3-subgroup of $GL(6, 2)$ has order 3^4 and is non-Abelian, there are no other possibilities for the order of $F(A)/K$. Hence $A/F(A)$, which is a subdirect product of the groups $\bar{A}$, has Fitting length at most 3. Thus $l(A) \leqq 4$, as claimed.

We state without proof the following elementary arithmetical facts.

LEMMA 3. (a) *If $d \geqq 3$, $3^d \geqq d\sqrt{12}$;*
(b) *If $d \geqq 4$, $2^d \geqq d\sqrt{12}$;*

We now come to our main result.

THEOREM. *Let G be a finite soluble completely reducible linear group of degree n over a perfect field k. Let $l(G) = l > 1$. Then*

$$n \geqq 2 \cdot 3^{\eta(l-3)/2} ,$$

where $\eta = 0$ for $l = 2, 3$ and $\eta = 1$ for $l \geqq 4$.

Proof. Since a linear group of degree one is Abelian, the theorem is clearly true for $l = 2, 3$. Therefore assume $l \geqq 4$. We may suppose there is an n-dimensional k-space V on which G acts (faithfully and completely reducibly). We proceed by induction on the integer $m = |G| + \dim_k(V)$, assuming the theorem has already been proved for all groups G and all fields k giving smaller values of m. Let $V = \mathscr{U}_1 \oplus \cdots \oplus \mathscr{U}_r$ be a decomposition of V into irreducible components $\mathscr{U}_i$. Set $K_i = \ker(G \text{ on } \mathscr{U}_i)$. If $G/K_i \in L(l-1)$ for every i, we have, $G \in R_0L(l-1) = L(l-1)$ since $\bigcap_{i=1}^{r} K_i = 1$. Since this is not the case, we have $l(G/K_i) = l$ for some i, and therefore when $r > 1$ we may apply induction to the triple $(G/K_i, \mathscr{U}_i, k)$ to give the result. Therefore assume V is irreducible as a $k[G]$-module. Since G is finite and k is perfect, we can find a finite extension $\bar{k}$ of k which is a splitting field for G and its subgroups such that $\bar{V} = \bar{k} \otimes_k V$ is completely reducible; in fact $\bar{V} = V_1 \oplus \cdots \oplus V_s$ is the direct sum of algebraically conjugate irreducible $\bar{k}[G]$-modules. If $s > 1$, $\dim_{\bar{k}}(V_i) < \dim_{\bar{k}}(\bar{V}) = \dim_k(V)$. Since $\ker(G \text{ on } V_i) = \ker(G \text{ on } V) = 1$, we can apply induction to the triple $(G, V_i, \bar{k})$ to give the result. Therefore we may assume that $s = 1$ and without loss of generality that $k = \bar{k}$ is a splitting field for G and its subgroups.

Let H be a subgroup of G critical for the class $L(l-1)$; thus $H \in L(l) \backslash L(l-1)$ and all proper subgroups of H belong to $L(l-1)$. By Lemma 5.2 and Theorem 5.3 of [1] there is a prime q dividing $|F(G)|$ such that H has a special normal q-subgroup Q such that $Q/\Phi(Q)$ is a chief factor of H on which H induces a group of automorphisms of Fitting length exactly $l-1$. If k has finite characteristic p, by the irreducibility of V we have $O_p(G) = 1$; thus $q \neq \operatorname{char} k$. Hence there exists a composition factor V^* of $V|_H$ not centralized by Q. The subgroup $Q^* = C_Q(V^*)$ is proper and normal in H, and therefore $Q^*\Phi(Q) = Q$ or $\Phi(Q)$. But $Q^*\Phi(Q) = Q$ implies Q^* is not proper. Therefore $Q^* \leqq \Phi(Q)$. But then $l(H/Q^*) = l$. If $H < G$, induction applied to the triple $(H/Q^*, V^*, k)$ gives the result. Therefore we suppose $H = G$ is critical for $L(l)$.

Let A be an Abelian normal subgroup of G. Let $V|_A = W_1 \oplus \cdots \oplus W_t$ be the decomposition into homogeneous components W_i. Suppose $t > 1$, and let M be a maximal subgroup of G containing the stabilizer S of W_1. By Clifford theory W_1 is an irreducible S-module and $V = W_1^G = (W_1^M)^G$. Furthermore, $Y = W_1^M$ is an irreducible $k[M]$-module. Applying induction to the triple $(M/\ker(M \text{ on } Y), Y, k)$ gives $\dim_k(Y) \geqq 2.3^{\eta'(l'-3)/2}$, where $l' = l(M)$. If $|G:M| = 2$, then $M \lhd G$ and clearly $l' = l-1$. But then $n = 2\dim_k(Y) \geqq 2\cdot 2\cdot 3^{\eta'(l-4)/2} > 2\cdot 3^{\eta(l-3)/2}$. Therefore suppose $|G:M| \geqq 3$. By Lemma 1 $l(M) \geqq l-2$, and so again by induction we have $n \geqq 3\dim_k(Y) \geqq 3\cdot 2\cdot 3^{\eta'(l-5)/2} \geqq 2\cdot 3^{\eta(l-3)/2}$. Therefore we may assume that $t = 1$, and, since k is a splitting field for the subgroups of G, that every Abelian normal subgroup of G is cyclic and contained in $Z(G)$.

Thus Q is an extra-special group, say of order q^{2d+1}. By Huppert [4], V.16.14, the faithful irreducible $k[Q]$-modules have dimension q^d. Since V is faithful for Q, we have $n = \dim_k(V) \geqq q^d$. G induces on $U = Q/\Phi(Q)$ a soluble irreducible group S of symplectic linear transformations over Z_q, and $l(S) = l-1$. If $l = 4$ or 5, $q^d \geqq 6$; for the irreducible soluble subgroups of $\mathrm{Sp}(2,2)$, $\mathrm{Sp}(2,3)$, $\mathrm{Sp}(2,5)$ and $\mathrm{Sp}(4,2) \cong S_6$ all have Fitting length at most 2. In these cases we have $n \geqq q^d \geqq 6 \geqq 2\cdot 3^{(l-3)/2}$. If $l \geqq 6$, induction applied to $(G/\ker(G \text{ on } U), U, Z_q)$ shows that $d \geqq 3^{(l-4)/2} \geqq 3$. Thus, if $q \neq 2$, by induction and Lemma 3(a), we have

$$n \geqq q^d \geqq 3^d \geqq d\sqrt{12} \geqq 2\cdot\sqrt{3}\cdot 3^{(l-4)/2} = 2\cdot 3^{(l-3)/2} .$$

And if $l \geqq 6$ and $q = 2$, by Lemma 2 and induction we have $d \geqq \max\{4, 3^{(l-4)/2}\}$. Hence using Lemma 3(b) we have

$$n \geqq 2^d \geqq d\sqrt{12} \geqq 2\cdot 3^{(l-3)/2} .$$

This completes the proof.

The bound for this theorem can actually be achieved whenever l is odd and $k = Z_3$. For let $l = 2l' + 1$ and let H be the holomorph of an elementary Abelian group A of order 9. $H/A \cong GL(2, 3)$ has Fitting length 3. Let $W = (\cdots(H \wr S_3) \wr \cdots \wr S_3)$, the successive wreath product of H with $l' - 1$ copies of the symmetric group of degree 3 according to its natural representation. It is easy to check that W has a self-centralizing elementary Abelian normal 3-subgroup N such that $l(W/N) = 2(l' - 1) + 3 = l$. N is a faithful irreducible $Z_3[W/N]$-module of Z_3-dimension $2\cdot 3^{l'-1} = 2\cdot 3^{(l-3)/2}$.

We conclude by remarking that the above methods give better bounds for $l(G)$ in terms of n if the smallest prime divisor of $|G|$ is greater than 2 or, more generally, if the 2-length of G is restricted.

References

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