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# EXTREME POINTS AND UNICITY OF EXTREMUM PROBLEMS IN $H^1$ ON POLYDISCS

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**In his recent work, K. Yabuta tries to extend the classical results of deLeeuw and Rudin on extreme points and extremum problems in the Hardy class $H^1$ on the unit disc to the $n$-dimensional case. In this paper, it is shown that simple induction arguments provide some extension of the results as well as simplification of the arguments in Yabuta's work.**

Let $U$ be the open unit disc and $T$ the unit circumference in the complex plane. Let $U^n$ and $T^n$ be the $n$-dimensional open unit polydisc and the $n$-dimensional torus, respectively, i.e., the subsets of the $n$-dimensional complex Euclidean space $C^n$ which are cartesian products of $n$ copies of $U$ and $T$, respectively. We shall denote by $H^1(U^n)$ the class of all holomorphic functions $f$ in $U^n$ for which

$$||f||_1 = \sup\left\{\int_{T^n} |f(rw)|\, dm_n(w) \colon 0 \leqq r < 1\right\} < +\infty ,$$

where $m_n$ is the normalized Lebesgue measure on the torus $T^n$. It is well known (cf. [5]) that each $f$ in $H^1(U^n)$ has nontangential boundary values on $T^n$, which determine a well-defined element $f^*$ of the space $L^1(m_n)$ on $T^n$ with respect to the measure $m_n$ and that the mapping $f \to f^*$ is an isometry of $H^1(U^n)$ onto a subspace of $L^1(m_n)$. For simplicity of notations, we use $f$ instead of $f^*$ and denote by the same symbol $H^1(U^n)$ the corresponding subspace of $L^1(m_n)$.

In the one-dimensional case, deLeeuw and Rudin [1] showed that $f \in H^1(U)$ is an extreme point of the unit ball of $H^1(U)$ if and only if $||f||_1 = 1$ and $f$ is outer, i.e.,

$$\log |f(0)| = \int_T \log |f(w)|\, dm_1(w) > -\infty .$$

For higher dimensions, Rudin [5] called a function $f \in H^1(U^n)$ outer if

$$\log |f(0)| = \int_{T^n} \log |f(w)|\, dm_n(w) > -\infty .$$

A simple modification of deLeeuw-Rudin's arguments shows that every outer function of norm one in $H^1(U^n)$ is an extreme point of the unit ball of $H^1(U^n)$. But the converse is false for $n \geqq 2$, as was shown recently by Yabuta [6]. He proved that $\pi(z_1 + z_2)/4$ is not an outer function but an extreme point of the unit ball of $H^1(U^n)$ for $n \geqq 2$.

A proof to this fact was given in our previous paper [2]. On the other hand, extremum problems for Hardy classes on the unit disc have a long history (cf. [4] and [1]). Yabuta [7; 8; 9; 10] studies, among others, unicity of extremum problems in the $n$-dimensional Hardy classes and tries to extend results due to deLeeuw-Rudin [1] and others. We should note that the unicity problem has not yet attained to its final solution even in the one-dimensional case.

In this paper, we shall first give a sufficient condition for extreme points of the unit ball of $H^1(U^n)$, which covers the case of outer functions as well as Yabuta's example. Our method is to simply extend the one-dimensional result due to deLeeuw and Rudin by use of the mathematical induction. The key theorem in our approach is Theorem 2.1, which is an abstract formulation of our induction steps. In this way we shall arrive at a class of functions which we call separately outer functions. Separately outer functions defined in §2 may be one of natural generalizations of outer functions in one dimension. In §3 we discuss the unicity problem by the same method, where the induction procedure is furnished by Theorem 3.1. In our approach, the main problem is to find better one-dimensional results, from which the corresponding higher dimensional results follow almost automatically. In this way we can reproduce most $n$-dimensional results obtained so far and even extend them, as we shall see below.

A question naturally arises as to whether there exist results which cannot be obtained by mere combination of the one-dimensional results and the induction arguments. For example, we may ask if the separate outerness can characterize the extreme points of the unit ball of $H^1(U^n)$, $n \geqq 2$. The answer is negative, as Professor Rudin has suggested to us recently. In fact, he has given a one-parameter family of extreme points of the unit ball of $H^1(U^2)$ which are not separately outer. His example will be described in §4. We thank Professor Rudin for allowing us to include this example in the present paper.

1. Quasi-analytic subspaces. Let $(X, \mu)$ be a finite measure space, where $X$ is a set and $\mu$ is a completely additive positive measure defined on a given Borel field in $X$, and let $L^1(\mu)$ be the complex $L^1$-space on $X$ with respect to $\mu$. We denote by $\|\cdot\|_\mu$ or $\|\cdot\|_1$ the usual Banach space norm of $L^1(\mu)$. Let $E$ be a linear subspace of $L^1(\mu)$. $E$ is said to be *quasi-analytic with respect to* $(X, \mu)$, if a function $f$ in $E$ vanishes $\mu$-almost everywhere on $X$ whenever it vanishes on a set of positive $\mu$-measure. It is well known that the space $H^1(U^n)$ is quasi-analytic with respect to $(T^n, m_n)$. Another example of quasi-analytic spaces is given by the space $H^1$ over the Bohr compactification of the real line. For this space, quasi-analyt-

icity was shown by Helson, Lowdenslager and Malliavin (cf. [3]). First we prove the following

LEMMA 1.1. *Let $(X, \mu)$ and $(Y, \nu)$ be two finite measure spaces, and let $E$, $F$ and $G$ be subspaces of $L^1(\mu)$, $L^1(\nu)$ and $L^1(\mu \times \nu)$, respectively. Suppose that*

(1.1) *$E$ and $F$ are quasi-analytic with respect to $(X, \mu)$ and $(Y, \nu)$, respectively; and*

(1.2) *For each $h \in G$, $h(x, \cdot)$ belongs to $F$ for almost every $x \in X$ and $h(\cdot, y)$ belongs to $E$ for almost every $y \in Y$.*

*Then, $G$ is quasi-analytic with respect to $(X \times Y, \mu \times \nu)$.*

*Proof.* We may suppose without loss of generality that both $\mu$ and $\nu$ are probability measures, i.e., $\mu(X) = \nu(Y) = 1$. Given $h \in G$, let $X_h$ (resp. $Y_h$) be the set of points $x \in X$ (resp. $y \in Y$) for which $h(x, \cdot)$ (resp. $h(\cdot, y)$) belongs to $F$ (resp. $E$). Then, (1.2) implies that $\mu(X_h) = \nu(Y_h) = 1$ for each $h \in G$. Now suppose that $h \in G$ vanishes on a set $D$ of positive $(\mu \times \nu)$-measure. Then there exists a measurable subset $D_1$ of $X_h$ with $\mu(D_1) > 0$ such that, for each $x \in D_1$, the set $\{y \in Y: (x, y) \in D\}$ is a measurable subset of $Y$ of positive $\nu$-measure. Since $F$ is quasi-analytic with respect to $(Y, \nu)$, we see that $h(x, \cdot) = 0$ a.e. on $Y$ for each $x \in D_1$. We put $Z(D_1, h) = \{(x, y) \in D_1 \times Y: h(x, y) = 0\}$. Then, $Z(D_1, h)$ is measurable and $(\mu \times \nu)(Z(D_1, h)) = \mu(D_1)$. Thus there exists a subset $D_2$ of $Y_h$ with $\nu(D_2) = 1$ such that, for each $y \in D_2$, the intersection $Z(D_1, h) \cap (X \times \{y\})$ is a measurable set of measure equal to $\mu(D_1) > 0$. The quasi-analyticity for $E$ now implies that $h(\cdot, y) = 0$ a.e. on $X$ for each $y \in D_2$. It follows easily that $h = 0$ a.e. on $X \times Y$, as was to be proved.

By use of the mathematical induction, we can prove the following

COROLLARY 1.2. *Let $E_i$ $(1 \leqq i \leqq n)$ be a subspace of $L^1(\mu_i)$ which is quasi-analytic with respect to $(X_i, \mu_i)$ and let $J_n$ be a subspace of $L^1(\mu_1 \times \cdots \times \mu_n)$ which satisfies the following:*

(1.3) *For each $h \in J_n$ and for each $i$ with $1 \leqq i \leqq n$, the function $x_i \to h(x_i, y_{(i)})$ belongs to $E_i$ for almost every $y_{(i)} \in X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_n$, where the circumflex indicates factors which are omitted.*

*Then, $J_n$ is quasi-analytic with respect to $(X_1 \times \cdots \times X_n, \mu_1 \times \cdots \mu_n)$.*

## 2. Extremal functions.

We again consider a linear subspace $E$ of $L^1(\mu)$. A function $f$ in $E$ is called *extremal in $E$* if $f = 0$ or if $f \neq 0$ and $f/\|f\|_\mu$ is an extreme point of the unit ball of $E$.

THEOREM 2.1. *Let $E$, $F$ and $G$ be as in Lemma 1.1. Let $f$ be* *a function in G such that, for almost every $y \in Y_f$, $f(\cdot, y)$ is extremal in E, and, for almost every $x \in X_f$, $f(x, \cdot)$ is extremal in F. Then, f is extremal in G.*

*Proof.* We may assume as before that $\mu$ and $\nu$ are probability measures. Clearly we have only to consider the case $\|f\|_{\mu\times\nu} = 1$. Let $X_f'$ (resp. $Y_f'$) be the set of $x \in X_f$ (resp. $y \in Y_f$) for which $f(x, \cdot)$ (resp. $f(\cdot, y)$) is extremal in $F$ (resp. $E$). Then our assumption imply that $\mu(X_f') = \nu(Y_f') = 1$. Now we take any $h \in G$ such that

$$\|f + h\|_{\mu\times\nu} = \|f - h\|_{\mu\times\nu} = \|f\|_{\mu\times\nu} = 1 . \tag{1}$$

We claim $h = 0$, which will show the theorem.

We start from the following obvious inequality

$$|f + h| + |f - h| - 2|f| \geqq 0 \qquad \text{on } X \times Y .$$

Integrating this over $X \times Y$, we get

$$\int_{X\times Y} \{|f + h| + |f - h| - 2|f|\} d\mu d\nu = \|f + h\|_{\mu\times\nu} + \|f - h\|_{\mu\times\nu} - 2\|f\|_{\mu\times\nu} = 0 .$$

By Fubini's theorem, there exists a subset $Y_1$ of $Y_f' \cap Y_h$ with $\nu(Y_1)=1$ such that, for each $y \in Y_1$,

$$\int_X \{|f(\xi, y) + h(\xi, y)| + |f(\xi, y) - h(\xi, y)| - 2|f(\xi, y)|\} d\mu(\xi) = 0 . \tag{2}$$

For the sake of simplicity, we use the notations $A_y$, $B_y$ and $C_y$ for the quantities $\|f(\cdot, y) + h(\cdot, y)\|_\mu$, $\|f(\cdot, y) - h(\cdot, y)\|_\mu$ and $\|f(\cdot, y)\|_\mu$, respectively. So we have $A_y + B_y - 2C_y = 0$ for each $y \in Y_1$. Fubini's theorem also shows that the functions $y \to A_y$, $y \to B_y$ and $y \to C_y$ are measurable on $Y$.

We fix $y \in Y_1$ for a moment. If $C_y = 0$, then $h(\cdot, y) = 0 = f(\cdot, y)$ a.e. on $X$. If $C_y \neq 0$ and $A_y = 0$ (resp. $B_y = 0$), then $h(\cdot, y) = -f(\cdot, y)$ a.e. on $X$ (resp. $h(\cdot, y) = f(\cdot, y)$ a.e. on $X$). Suppose finally that $A_y \neq 0$, $B_y \neq 0$ and $C_y \neq 0$. In this case, we put

$$\phi = \frac{f(\cdot, y) + h(\cdot, y)}{A_y}$$

and

$$\psi = \frac{f(\cdot, y) + h(\cdot, y)}{B_y} .$$

Then, both $\phi$ and $\psi$ belong to the unit ball of $E$, and $(A_y/2C_y)\phi + (B_y/2C_y)\psi = f(\cdot, y)/C_y$. Since $f(\cdot, y)/C_y$ is an extreme point of the unit ball of $E$, we have $\phi = \psi = f(\cdot, y)/C_y$ a.e. on $X$. It follows that $h(\cdot, y) = ((A_y - C_y)/C_y) \times f(\cdot, y)$ a.e. on $X$. Summing up, we have shown that there exists a bounded measurable function $K(y)$ on $Y$ such that, for each $y \in Y_1$, $h(\cdot, y) = K(y)f(\cdot, y)$ a.e. on $X$ and moreover $K(y) = 0$ if $C_y = 0$. Similarly, there exist a subset $X_1$ of $X_f' \cap X_h$ with $\mu(X_1) = 1$ and a bounded measurable function $K'(x)$ on $X$ having the following properties: for each $x \in X_1$,

$$(3) \quad \int_Y \{|f(x, \eta) + h(x, \eta)| + |f(x, \eta) - h(x, \eta)| - 2|f(x, \eta)|\} d\nu(\eta) = 0 ,$$

$h(x, \cdot) = K'(x)f(x, \cdot)$ a.e. on $Y$, and $K'(x) = 0$ if $\|f(x, \cdot)\|_\nu = 0$.

Now we define $k(x, y)$ by putting $k(x, y) = 0$ if $f(x, y) = 0$ and $= h(x, y)/f(x, y)$ if $f(x, y) \neq 0$. Then it follows from (2) and (3) that $-1 \leqq k(\cdot, y) \leqq 1$ a.e. on $X$ for each $y \in Y_1$ and $-1 \leqq k(x, \cdot) \leqq 1$ a.e. on $Y$ for each $x \in X_1$. Clearly $k(x, y)$ is measurable. Let $y \in Y_1$. Then, the quasi-analyticity of $E$ implies that $\|f(\cdot, y)\|_\mu = 0$ if $f(\cdot, y)$ vanishes on a set of positive $\mu$-measure. So we have, for each fixed $y \in Y_1$, $k(\cdot, y) = K(y)$ a.e. on $X$ and, for each fixed $x \in X_1$, $k(x, \cdot) = K'(x)$ a.e. on $Y$. We put $k_1(x, y) = K(y)$ for all $x \in X$ and all $y \in Y_1$ and $k_2(x, y) = K'(x)$ for all $x \in X_1$ and all $y \in Y$. Then, both $k_1$ and $k_2$ are bounded and measurable on $X \times Y$. We have

$$\iint_{X\times Y} |k - k_1| d\mu d\nu = \int_Y d\nu(y) \int_X |k(x, y) - k_1(x, y)| d\mu(x)$$
$$= \int_{Y_1} d\nu(y) \int_X |k(x, y) - K(y)| d\mu(x) = 0 ,$$

so that $k = k_1$ a.e. on $X \times Y$. Similarly, we have $k = k_2$ a.e. on $X \times Y$. Consequently, $k_1 = k_2$ a.e. on $X \times Y$. From this follows immediately that $k_1$, $k_2$ and also $k$ are equal to a constant $c$ with $-1 \leqq c \leqq 1$ almost everywhere on $X \times Y$. Thus, we have $h = cf$ a.e. on $X \times Y$ and, in view of (1), $c = 0$. Hence $h = 0$, as was to be proved.

Let us consider a finite number of finite measure spaces $(X_i, \mu_i)$, $1 \leqq i \leqq n$. Then we have the following

COROLLARY 2.2. *Let $E_i$ $(1 \leqq i \leqq n)$ be a subspace of $L^1(\mu_i)$ which is quasi-analytic with respect to $(X_i, \mu_i)$, and let $J_n$ be a subspace of $L^1(\mu_1 \times \cdots \times \mu_n)$ which satisfies the condition* (1.3). *Let $f$ be a function in $J_n$ which satisfies the following conditions for all $i$ with $1 \leqq i \leqq n$:*

(2.1)$_i$ *For almost every $y_{(i)} \in X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_n$, the function $x_i \rightarrow f(x_i, y_{(i)})$ belongs to $E_i$ and is extremal in $E_i$.*

*Then $f$ is extremal in $J_n$.*

*Proof.* We use induction to prove the assertion. We may assume as before that all $\mu_i$ are probability measures. We know that the assertion is true for $n = 2$. We now suppose that it is true for $n = k$ ($\geqq 2$).

Let $g$ be a generic element in $J_{k+1}$ and put

$$D_i[g] = \{y_{(i)} \in X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_{k+1}: g(\cdot, y_{(i)}) \in E_i\}$$

for $1 \leqq i \leqq k$. The hypothesis (1.3) for $n = k + 1$ then says that

$$(\mu_1 \times \cdots \times \hat{\mu}_i \times \cdots \times \mu_{k+1})(D_i[g]) = 1 \qquad \text{for } 1 \leqq i \leqq k .$$

We denote by $\chi_i[g]$ the characteristic function of the set $D_i[g]$, and by $D_{i,k+1}[g]$ the set of points $x_{k+1} \in X_{k+1}$ such that

$$g(\cdot, x_{k+1}) \in L^1(\mu_1 \times \cdots \times \mu_k) ,$$

$\chi_i[g](\cdot, x_{k+1})$ is measurable, and

$$\int \chi_i[g](w, x_{k+1}) d(\mu_1 \times \cdots \times \hat{\mu}_i \times \cdots \times \mu_k)(w) = 1 .$$

Putting $D[g] = \cap \{D_{i,k+1}[g]: 1 \leqq i \leqq k\}$, we have $\mu_{k+1}(D[g]) = 1$.

Let $J'$ be the (not necessarily closed) subspace of $L^1(\mu_1 \times \cdots \times \mu_k)$ which is generated by all $g(\cdot, x_{k+1})$ with $g \in J_{k+1}$ and $x_{k+1} \in D[g]$. We claim that $J'$ satisfies the condition (1.3) for $n = k$. To see this, take any function $h$ from $J'$. Then, there exist a finite number of elements $g_1, \cdots, g_s$ in $J_{k+1}$ such that $h = \sum_{j=1}^{s} g_j(\cdot, x_{k+1}^{(j)})$ where $x_{k+1}^{(j)} \in D[g_j]$. We take any $i$ with $1 \leqq i \leqq k$. Since $x_{k+1}^{(j)}$ is in the set $D_{i,k+1}[g_j]$ for each $j$, we have

$$\int \chi_i[g_j](w, x_{k+1}^{(j)}) d(\mu_1 \times \cdots \times \hat{\mu}_i \times \cdots \times \mu_k)(w) = 1 .$$

So, putting $W_{(i,j)} = \{w \in X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_k: \chi_i[g_j](w, x_{k+1}) = 1\}$, we see that $(\mu_1 \times \cdots \times \hat{\mu}_i \times \cdots \times \mu_k)(W_{(i,j)}) = 1$ and $g_j(\cdot, w, x_{k+1}^{(j)})$ belongs to $E_i$ for each $w \in W_{(i,j)}$. We set $W_{(i)} = \cap \{W_{(i,j)}: 1 \leqq j \leqq s\}$. Then $(\mu_1 \times \cdots \times \hat{\mu}_i \times \cdots \times \mu_k)(W_{(i)}) = 1$ and

$$h(\cdot, w) = \sum_{j=1}^{s} g_j(\cdot, w, x_{k+1}^{(j)}) \in E_i$$

for each $w \in W_{(i)}$. As $i$ is arbitrary, this proves that $J'$ satisfies (1.3) for $n = k$.

Now let $f$ be any function in $J_{k+1}$ satisfying the conditions $(2.1)_i$ with $1 \leqq i \leqq k + 1$. Let $D_i'[f]$ ($1 \leqq i \leqq k$) be the set of points $y_{(i)} \in X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_{k+1}$ for which $f(\cdot, y_{(i)})$ belongs to $E_i$ and is extremal in $E_i$. Then, $D_i'[f] \subseteqq D_i[f]$ and $(\mu_1 \times \cdots \times \hat{\mu}_i \times \cdots \times \mu_{k+1}) \times (D_i'[f]) = 1$. We denote by $\chi_i'[f]$ the characteristic function of $D_i'[f]$, and by $D'_{i,k+1}[f]$ the set of points $x_{k+1} \in D_{i,k+1}[f]$ such that $\chi'_i[f](\cdot, x_{k+1})$ is measurable and

$$\int \chi'_i[f](w, x_{k+1}) d(\mu_1 \times \cdots \times \hat{\mu}_i \times \cdots \times \mu_k)(w) = 1 .$$

Set $D'[f] = \cap \{D'_{i,k+1}[f]: 1 \leqq i \leqq k\}$. Then $\mu_{k+1}(D'[f]) = 1$.

Let $x_{k+1} \in D'[f]$ and take any $i$ with $1 \leqq i \leqq k$. Then, $x_{k+1} \in D'_{i,k+1}[f]$. Thus, putting $V_i(x_{k+1}) = \{w \in X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_k: \chi'_i[f](w, x_{k+1}) = 1\}$, we see that $(\mu_1 \times \cdots \times \hat{\mu}_i \times \cdots \times \mu_k)(V_i(x_{k+1})) = 1$ and, for each $w \in V_i(x_{k+1}), f(\cdot, w, x_{k+1})$ is extremal in $E_i$. As $x_{k+1} \in D[f]$, we see that $f(\cdot, x_{k+1}) \in J'$ and $f(\cdot, x_{k+1})$ satisfies the conditions $(2.1)_i$ for $1 \leqq i \leqq k$ with $n = k$. By the induction hypothesis, the assertion is true for $J'$ and therefore $f(\cdot, x_{k+1})$ is extremal in $J'$. This is true for almost every $x_{k+1} \in X_{k+1}$, because $\mu_{k+1}(D'[f]) = 1$. On the other hand, Corollary 1.2 shows that the space $J'$ is quasi-analytic with respect to $(X_1 \times \cdots \times X_k, \mu_1 \times \cdots \times \mu_k)$. Thus, we can apply Theorem 2.1 to the case $E = J', F = E_{k+1}, G = J_{k+1}$ and conclude that $f$ is extremal in $J_{k+1}$. This completes the proof.

For the space $H^1(U^n)$ we get the following

COROLLARY 2.3. *Let $f$ be a function in $H^1(U^n)$ and let $k, k'$ be natural numbers with $k + k' = n$. Suppose that, for almost every $w_1 \in T^k, f(w_1, \cdot)$ is extremal in $H^1(U^{k'})$ and, for almost every $w_2 \in T^{k'}$, $f(\cdot, w_2)$ is extremal in $H^1(U^k)$. Then, $f$ is extremal in $H^1(U^n)$.*

This is just a restatement of Theorem 2.1 for the space $H^1(U^n)$, because the hypotheses (1.1) and (1.2) are clearly satisfied with $X = T^k$, $Y = T^{k'}$, $\mu = m_k$, $\nu = m_{k'}$, $E = H^1(U^k)$, $F = H^1(U^{k'})$ and $G = H^1(U^n)$.

In order to state the next corollary, we use the following notations: $U_i$ (resp. $T_i$) $(1 \leqq i \leqq n)$ denote $n$ copies of $U$ (resp. $T$), so that $U^n = U_1 \times \cdots \times U_n$ (resp. $T^n = T_1 \times \cdots \times T_n$).

COROLLARY 2.4. *Suppose that a function $f$ in $H^1(U^n)$ satisfies the following conditions for all $i$ with $1 \leqq i \leqq n$:*

$(2.2)_i$ *For almost every $w_{(i)} \in T_1 \times \cdots \times \hat{T}_i \times \cdots \times T_n$, the function $z_i \rightarrow f(z_i, w_{(i)})$ belongs to $H^1(U_i)$ and is an outer function.*

*Then, $f$ is extremal in $H^1(U^n)$.*

This follows readily from Corollary 2.2, since deLeeuw and Rudin [1] showed that the outer functions of norm one are just the extreme points of the unit ball of $H^1(U)$. Using this corollary, we can construct some extremal functions in $H^1(U^n)$ $(n \geqq 2)$ which are not outer functions:

(i) Yabuta's function $z_1 + z_2$. For each fixed real $\theta$, $e^{i\theta} + z$ is easily seen to be an outer function. So the above corollary implies that $z_1 + z_2$ is extremal in $H^1(U^n)$ for $n \geqq 2$.

(ii) More generally, we see that a function of the form $f(z_1, \cdots, z_k) + g(z_{k+1}, \cdots, z_n)$ is extremal in $H^1(U^n)$ if $f$ and $g$ are inner functions in $H^1(U^k)$ and $H^1(U^{n-k})$, respectively. We may have more complicated examples, e.g., $(z_1 + z_2 z_3)^3$, etc.

Here we add a simple remark concerning functions that are described in Corollary 2.4. We say that a function $f$ in $H^1(U^n)$ is *separately outer* if it satisfies the conditions $(2.2)_i$ for all $i$ with $1 \leqq i \leqq n$. Thus we have shown that every separately outer function is extremal in $H^1(U^n)$. It may be natural to have the following

PROPOSITION 2.5. *Every outer function in $H^1(U^n)$ is separately outer.*

*Proof.* Suppose that $f \in H^1(U^n)$ is not separately outer. Then it does not satisfy $(2.2)_i$ for some $i$, say $i = 1$. So there exists a measurable subset $A$ of $T_2 \times \cdots \times T_n$ of positive measure such that

$$\log |f(0, \xi)| < \int_{T_1} \log |f(w_1, \xi)| \, dm_1((w_1)$$

for every $\xi \in A$. Integrating both sides over $T_2 \times \cdots \times T_n$, we have

$$\log |f(0)| < \int_{T^n} \log |f(w)| \, dm_n(w) .$$

Hence, $f$ is not outer, as was to be proved.

**3. Unicity of extremum problems.** Let $(X, \mu)$ be a finite measure space and $E$ a linear subspace of $L^1(\mu)$. Given a bounded linear functional $\Phi$ on $E$, we denote by $S^\Phi$ the set of functions $f \in E$ such that $\|f\|_1 = 1$ and $\Phi(f) = \|\Phi\|$. By Hahn-Banach's theorem, there exists a function $\phi \in L^\infty(\mu)$ such that $\|\Phi\| = \|\phi\|_\infty$ and

$$\Phi(f) = \int_X f \phi d\mu \qquad \text{for all} \quad f \in E .$$

Then $f \in S^\Phi$ implies

$$(4) \qquad f(x)\phi(x) \geqq 0 \qquad \text{a.e. on } X ,$$

and

$$(5) \qquad |\phi(x)| = \|\phi\|_\infty \qquad \text{for every} \quad x \in S(f) ,$$

where $S(f)$ denotes the support of $f$ and is determined up to a set of measure zero. Conversely, if $f \in E$ with $\|f\|_1 = 1$ satisfies both (4) and (5), then $f$ belongs to $S^\varphi$.

If the space $E$ is quasi-analytic with respect to $(X, \mu)$, then $S(f) = X$ for every nonzero element $f$ in $E$, so that the set $S^\varphi$ is determined by any of its elements. In fact, let $f \in S^\varphi$. Then, $g \in E$ belongs to $S^\varphi$ if and only if $\arg g(x) = \arg f(x)$ a.e. on $X$ and $\|g\|_1 = 1$. In this case, we may write $S^f$ instead of $S^\varphi$ whenever $f$ is in $S^\varphi$. We say that an element $f \in E$ is said to have *the unicity property in* $E$ if $f \neq 0$ and $S^g = \{g\}$ with $g = f/\|f\|_1$.

THEOREM 3.1. *Let $E$, $F$ and $G$ be as in Lemma* 1.1, *and let $f \in G$ be a nonzero function. Suppose that for almost every $x \in X_f$, $f(x, \cdot)$ has the unicity property in $F$ and, for almost every $y \in Y_f$, $f(\cdot, y)$ has the unicity property in $E$. Then, $f$ has the unicity property in $G$.*

*Proof*. We may suppose without loss of generality that $f(x, \cdot)$ (resp. $f(\cdot, y)$) has the unicity property for every $x \in X_f$ (resp. $y \in Y_f$). Let $g \in G$ be any element in $S^f$. Since $G$ is quasi-analytic by Lemma 1.1, we see that $\arg g = \arg f$ a.e. on $X \times Y$. So, for almost every $x \in X_f \cap X_g$, we have $g(x, \cdot) \neq 0$ a.e. on $Y$ and

$$\arg g(x, \cdot) = \arg f(x, \cdot) \qquad \text{a.e. on } Y. \tag{6}$$

We may assume that (6) is true for all $x \in X_f \cap X_g$. Since $f(x, \cdot)$ has the unicity property in $G$, we have for $x \in X_f \cap X_g$

$$\frac{f(x, \cdot)}{\|f(x, \cdot)\|} = \frac{g(x, \cdot)}{\|g(x, \cdot)\|} \qquad \text{a.e. on } Y.$$

Putting $K(x) = \|g(x, \cdot)\|/\|f(x, \cdot)\|$, we get a measurable function $K(x)$ which is finite almost everywhere on $X$ and $g(x, \cdot) = K(x)f(x, \cdot)$ a.e. on $Y$ for every $x \in X_f \cap X_g$. Similarly, we have a measurable function $K'(y)$ on $Y$ which is finite almost everywhere and $g(\cdot, y) = K'(y)f(\cdot, y)$ a.e. on $X$ for every $y \in Y_f \cap Y_g$.

Since $f \neq 0$ a.e. on $X \times Y$, we can define $k(x, y) = g(x, y)/f(x, y)$, which is measurable and finite a.e. on $X \times Y$. It follows that, for every $x \in X_f \cap X_g$, $k(x, \cdot) = K(x)$ a.e. on $Y$ and for every $y \in Y_f \cap Y_g$, $k(\cdot, y) = K(y)$ a.e. on $X$. Thus, $k$ is equal to a constant $c$ almost everywhere on $X \times Y$; so $g = cf$ a.e. on $X \times Y$. Since both $f$ and $g$ are of norm one, we have $c = 1$. Hence, $S^f = \{f\}$, as was to be proved.

COROLLARY 3.2. *Let $E_i$ $(1 \leqq i \leqq n)$ and $J_n$ be as in Corollary* 2.2. *Let $f$ be a function in $J_n$ which satisfies the following conditions* $(3.1)_i$ *for all $i$ with $1 \leqq i \leqq n$:*

$(3.1)_i$ *For almost every* $y_{(i)} \in X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_n$, *the function* $x_i \to f(x_i, y_{(i)})$ *on* $X_i$ *belongs to* $E_i$ *and has the unicity property in* $E_i$.

*Then* $f$ *has the unicity property in* $J_n$.

The proof is similar to that of Corollary 2.2 and so omitted. Applying these facts to the spaces $H^1(U^n)$ $(n \geqq 2)$, we can formulate results similar to Corollaries 2.3 and 2.4. Here we state only the latter one.

COROLLARY 3.3. *Suppose that a nonzero function* $f$ *in* $H^1(U^n)$ *satisfies the following conditions for all* $i$ *with* $1 \leqq i \leqq n$:

$(3.2)_i$ *For almost every* $w_{(i)} \in T_1 \times \cdots \times \hat{T}_i \times \cdots \times T_n$, *the function* $z_i \to f(z_i, w_{(i)})$ $(z_i \in U_i)$ *belongs to* $H^1(U_i)$ *and has the unicity property in* $H^1(U_i)$.

*Then* $f$ *has the unicity property in* $H^1(U^n)$.

We thus obtain results for $H^1(U^n)$ by combining Corollary 3.3 with one-dimensional results. For example, we look at the following.

THEOREM 3.4 (Yabuta [7]). *Let* $f \in H^1(U^n)$. *Suppose that* $f$ *is outer and* $1/f \in L^1(m_n)$. *Then* $f$ *has the unicity property in* $H^1(U^n)$.

We can extend this by combining its one-dimensional form with Corollary 3.3. Namely we have

COROLLARY 3.5. *Let* $f \in H^1(U^n)$. *Suppose that* $f$ *is separately outer and satisfies the following conditions for all* $i$ *with* $1 \leqq i \leqq n$:

$(3.3)_i$ *For almost every* $w_{(i)} \in T_1 \times \cdots \times \hat{T}_i \times \cdots \times T_n$, *the function* $w_i \to 1/f(w_i, w_{(i)})$ $(w_i \in T_i)$ *is integrable on* $T_i$.

*Then* $f$ *has the unicity property in* $H^1(U^n)$.

Yabuta [9] also showed the following

THEOREM 3.6 (*Yabuta*). *Let* $f \in H^1(U^n)$ *and suppose that* $f$ *is not identically zero and* $\mathrm{Re}\, f \geqq 0$ *a.e. on* $T^n$. *Then* $f$ *has the unicity property in* $H^1(U^n)$.

Yabuta proved this by using certain properties of $n$-harmonic functions and others. But, in view of Corollary 3.3, it is enough for us to prove the case $n = 1$. Thus, let $f \in H^1(U)$ be not identically zero and satisfy $\mathrm{Re}\, f \geqq 0$ a.e. on $T$. Take any $g \in H^1(U)$ such that $\arg g = \arg f$ a.e. on $T$. If $\mathrm{Re}\, f$ vanishes at some point in $U$, then the minimum principle for harmonic functions shows that it vanishes identically in $U$. It follows at once that $f$ is a pure imaginary constant $b \neq 0$. Therefore $\arg g = \arg b$ $(=\pm\pi/2)$ a.e. on $T$, so that $g/b$ is an $H^1$ function with real boundary values a. e. on $T$. Hence $g/b$ is a constant, which is turned out to be positive. Next, suppose that $\mathrm{Re}\, f > 0$ in $U$. That $\arg g = \arg f$ a.e. on $T$ implies $\mathrm{Re}\, g > 0$ in $U$. Thus, by restricting the arguments of $f$ and $g$ to $[-\pi/2, +\pi/2]$, we see that $|\arg(g/f)| < \pi$ in $U$ and therefore that $\arg(g/f)$ is a well-defined harmonic function in $U$. Since $\arg(g/f)$ is bounded and equals zero a.e. on $T$, it vanishes identically on $U$. Thus $\log(g/f)$ is a real-valued holomorphic function on $U$, so that it is a constant. Hece $g/f$ is a positive constant, as was to be proved.

As before, we can improve Theorem 3.6 in the following way.

COROLLARY 3.7. *Let $f \in H^1(U^n)$ and suppose that it is not identically zero and satisfies the following conditions for all $i$ with $1 \leqq i \leqq n$:*

$(3.4)_i$ *For almost every $w_{(i)} \in T_1 \times \cdots \times \hat{T}_i \times \cdots \times T_n$, there exists a real number $\theta$, depending on $w_{(i)}$ and $f$, such that*

$$\theta - \frac{\pi}{2} \leqq \arg f(w_i, w_{(i)}) \leqq \theta + \frac{\pi}{2} \qquad (\mathrm{mod.}\ 2\pi)$$

*for almost all $w_i \in T_i$.*

*Then $f$ has the unicity property in $H^1(U^n)$.*

One of very special consequences of this is a theorem of Yabuta [8] which states that the function $z_1 + z_2$ has the unicity property. More generally, we have the following:

COROLLARY 3.8. *Every function of the form*

$$f(z_1, \cdots, z_k) + g(z_{k+1}, \cdots, z_n)$$

*has the unicity property if $f$ and $g$ are inner functions in $H^1(U^k)$ and $H^1(U^{n-k})$, respectively.*

We may have more complicated sufficient conditions. Anyway, as we already said in the introduction, the main problem in our approach is to find better one-dimensional results. As far as we are aware, the characterization of functions in the space $H^1$ having the unicity property is an open problem even in the one-dimensional case. Here we add a few more remarks.

PROPOSITION 3.9. *Let $g \in H^1(U)$ have the unicity property and let $f$ be an outer fuction in $H^1(U)$ such that $g/f \in L^\infty(m_1)$. Then $f$ has the unicity property.*

The proof is simple. Yabuta [8] proved this in the case $g(z) = \prod_{j=1}^{N}(a_j - z)$ where $\{a_j\}$ denote any $N$ distinct points on the unit circumference $T$. By using Corollary 3.3, the proposition can be extended to $H^1(U^n)$, which we do not state explicitly. In connection with this, we ask the following

*Problem*. Let $f$ and $g$ be outer functions in $H^1(U)$ such that $fg$ is also in $H^1(U)$. Under what conditions on $f$ and $g$, does $fg$ have the unicity property?

4. **Rudin's example.** Let $\alpha$ be a complex number with $|\alpha| > 1$ and put

$$f(z_1, z_2) = \alpha z_1 - z_2 .$$

We claim that the function $f$ is not separately outer but extremal in $H^1(U^2)$.

If $|w_2| = 1$, then $|w_2/\alpha| < 1$ and $f(w_2/\alpha, w_2) = 0$. Thus $f(\cdot, w_2)$ is not outer and so $f$ is not separately outer. To prove that $f$ is extremal in $H^1(U^2)$, it is enough to show that the constants are the only real functions $h$ on $T^2$ such that $fh \in H^1(U^2)$.

The Fourier coefficients of $g = fh$ are

$$\hat{g}(m, n) = \alpha\hat{h}(m-1, n) - \hat{h}(m, n-1) . \tag{7}$$

Since $g \in H^1(U^2)$, this implies that

$$\hat{h}(m+1, n-1) = \alpha\hat{h}(m, n) \qquad \text{if} \quad n < 0 .$$

By induction, we have

$$\hat{h}(m+k, n-k) = \alpha^k\hat{h}(m, n) \qquad \text{for} \quad k = 1, 2, \cdots; \ n < 0 .$$

Letting $k \to \infty$, we know $\hat{h}(m+k, n-k) \to 0$. Hence $\hat{h}(m, n) = 0$ if $n < 0$. Since $h$ is real, it follows that

$$\hat{h}(m, n) = 0 \qquad \text{if} \quad n \neq 0 . \tag{8}$$

Now (7) and (8) imply that

$$\hat{h}(m, 0) = \alpha\hat{h}(m-1, 1) = 0 \qquad \text{if} \quad m < 0 .$$

Since $h$ is real, we have $\hat{h}(m, 0) = 0$ if $m \neq 0$. Thus $\hat{h}(0, 0)$ is the only Fourier coefficient of $h$ that can be different from 0, so that $h$ is constant almost everywhere. This completes the proof.

The above example, together with the proof, has been suggested to us by Rudin. We should note that the same method was used once by us in order to prove the extremity of the function $z_1 + z_2$ (cf. [2]). Probably, Rudin's example will give a new insight into our problem and we will discuss it on another occasion.

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