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ARCHIMEDEAN EXTENSIONS OF DIRECTED INTERPOLATION GROUPS

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P. F. Conrad has obtained some properties of archimedean extensions (a-extensions) of lattice ordered groups (l-groups). In particular, Conrad proved that every abelian l-group has an $\mathscr{A}$-closure (an abelian a-extension which has no proper abelian a-extension). D. Khuon proved that every l-group has an a-closure (an a-extension which has no proper a-extension). Using a slightly different definition, Conrad and Bleier defined an a^*-extension of an l-group and proved that every abelian l-group has an a^*-closure and every archimedean l-group has a unique a^*-closure. These results have been extended to another class of l-groups by Glass and Holland (unpublished).

The purpose of this paper is to extend the l-group results to the class of directed interpolation groups. The obvious definitions give rise to some negative results; the situation for abelian $\mathscr{P}$-groups is more propitious and it is proved that any such group has an $\mathscr{A}$-closure in this class. However, taking less direct definitions of a-extensions and a^*-extensions gives $\mathscr{A}$-closures and $\mathscr{A}^*$-closures in restricted classes of abelian directed interpolation groups.

It is assumed that the reader is familiar with [1], [2], [3], [4], and [5]

1. Definitions and notation. Throughout this paper, additive notation will be used for all groups, abelian or not. $\subset$ will denote strict containment and $\subseteqq$ will denote strict containment or equality. *On* will denote the collection of all ordinals.

If A is a p.o. set and $\alpha, \beta \in A$, then $\alpha \| \beta$ will stand for $\alpha \nleqq \beta$ and $\beta \nleqq \alpha$.

If G is a group and $X \subseteqq G$, $\langle X \rangle$ will denote the subgroup of G generated by X. If G and H are groups $G \oplus H$ will denote the cartesian sum of G and H. If G and H are p.o. groups, then $G \overrightarrow{\oplus} H$, the *lexicographic sum of G over H*, is the *group* $G \oplus H$ ordered by: $(g, h) > 0$ if and only if $g > 0$ (in G) or $g = 0$ and $h > 0$ (in H). If $\{G_\alpha: \alpha \in A\}$ is a family of p.o. groups, then $\Pi\{G_\alpha: \alpha \in A\}(\Sigma\{G_\alpha: \alpha \in A\})$ will denote the cartesian product (sum) of the family of groups $\{G_\alpha: \alpha \in A\}$ ordered by: $g \geqq 0$ if $g_\alpha \geqq 0$ (in G_α) for all $\alpha \in A$; $\Pi^*\{G_\alpha: \alpha \in A\}$ is the same group as above but ordered by: $g > 0$ if and only if $g_\alpha > 0$ (in G_α) for all $\alpha \in A$.

If G is a p.o. group, G^+ will denote the *positive cone of* $G = \{g \in G: g \geqq 0\}$ and G^* will denote the *strictly positive cone of* $G = \{g \in G: g > 0\}$. $\boldsymbol{R}(\boldsymbol{Z})$ will denote the additive o-group of reals (integers) but $\boldsymbol{R}^+(\boldsymbol{Z}^+)$ will denote the strictly positive reals (integers).

Let G be a p.o. group. The partial order is said to be *dense* if and only if for all $g, h \in G$, if $g < h$ there exists $f \in G$ such that $g < f < h$. The partial order satisfies the *interpolation property* if whenever $g, h, f, k \in G$ and $g, h \leqq f, k$, there exists $x \in G$ such that $g, h \leqq x \leqq f, k$. A p.o. group satisfying the interpolation property is called an *interpolation group*. A directed interpolation group G in which $f \vee g$ (or $f \wedge g$) exists only when $f \leqq g$ or $g \leqq f$ is said to be an *antilattice*. A directed group G such that for all $f, g, h, k \in G$ whenever $g, h < f, k$, there exists $x \in G$ such that $g, h < x < f, k$ is called a *tight Riesz group*. Let $\bar{P} = \{g \in G: g \in G^+$ or g is pseudo-positive$\}$ and let the cone of $\leqq$ be $\bar{P}$. If G has no pseudo-identities, then $\leqq$ is said to be a *compatible tight Riesz order* for $(G, \leqq)$.

Let A be a partially ordered set. For each $\alpha \in A$. Let R_α be a partially ordered abelian group. Let K be the cartesian product of $\{R_\alpha: \alpha \in A\}$. The set of all $k \in K$ such that $\{\alpha \in A: k_\alpha \neq 0\}$ satisfies the ascending chain condition (in A) forms abelian group which is denoted by $V(A, R_\alpha)$. For each $v \in V(A, R_\alpha)$, let $M(v) = \{\alpha \in A: v_\alpha \neq 0$ and $v_\beta = 0$ for all $\beta \in A$ such that $\beta > \alpha\}$. $V(A, R_\alpha)$ is a partially ordered abelian group under the ordering: $v > 0$ if and only if $v_\alpha > 0$ for all $\alpha \in M(v)$.

2. A naïve approach to a-extensions of directed interpolation groups. Let H be a directed interpolation group and let G be a subgroup of H. G is said to be an interpolation subgroup of H if and only if for all $x, y \in G$ and $z, t \in H$, $x, y \leqq z, t$ implies there exists $g \in G$ such that $x, y \leqq g \leqq z, t$. Note that this condition is equivalent to: $x, y \in G$ and $z, t \in H$ and $x, y \geqq z, t$ imply there exists $g \in G$ such that $x, y \geqq g \geqq z, t$.

It should be observed that if H is an l-group and G an interpolation subgroup of H, then G is an l-subgroup of H. However, let $H = V(A, R_\alpha)$ and $G = V(B, R_\beta)$ where $A = \{\bar{1}, \bar{2}, \bar{3}\}$ ordered by: $\bar{1}, \bar{2} > \bar{3}$ and $\bar{1} \| \bar{2}$, $B = \{\bar{1}, \bar{2}\}$ and $R_\alpha = \boldsymbol{R}$ for all $\alpha \in A$. H is a directed interpolation group and G is an l-group under the induced ordering. However, G is not an interpolation subgroup of H as $(0, 0, 1) \leqq (1, 0, 0), (0, 1, 0)$ but there is no $g \in G$ such that $(0, 0, 1) \leqq g \leqq (1, 0, 0), (0, 1, 0)$.

$h_1, h_2 \in H^+$ are said to be *a-equivalent* if and only if there exist $n_1, n_2 \in \boldsymbol{Z}^+$ such that $h_1 \leqq n_2 h_2$ and $h_2 \leqq n_1 h_1$. H is an *a-extension of* G if and only if every $h \in H^+$ is a-equivalent to some $g \in G^+$ and G is an interpolation subgroup of H. This coincides with the definition given in [2] for l-groups and is a natural extension to directed interpolation groups. Notice that if K is an a-extension of H and H is an a-extension of G, then K is an a-extension of G. Using the method of [2] and the results of [4] and [5] it is easy to see that H is an a-extension of G if and only if there is a $(1:1)$ map of the convex d-subgroups of G onto those of H which preserves inclusion and maps prime subgroups of G (polars of G) to prime subgroups of H (polars of H).

G is *a-closed* if and only if G has no proper a-extension. In view of [2] and [6], we cannot hope for there to be a unique a-closure of a directed interpolation group but we can try to prove that *any* path of proper a-extensions eventually terminates. The next two theorems shatter this dream—the second gives a path of proper a-extensions of a certain class of abelian directed interpolation groups which cannot be closed up; the first, more dramatically, proves that *no* path of a-extensions of $\boldsymbol{R}$ can ever be closed up.

THEOREM A. *Any a-extension (in the class of directed interpolation groups) of a dense antilattice is a dense antilattice. Consequently, no dense antilattice other than* $\{0\}$ *has an a-closure and no abelian dense antilattice other than* $\{0\}$ *has an a-closure in the class of abelian directed interpolation groups.*

Proof. It is easy to see that any a-extension of an antilattice is an antilattice. Suppose G is a dense directed interpolation group and that H is an a-extension of G. We prove that H is dense.

Assume that $0 < h \in H$ and that there is no $k \in H$ such that $0 < k < h$. Since G is dense, $h \notin G$. We first show that, under these hypotheses, some multiple of h belongs to G. Let g be a-equivalent to h and assume that $m \in \boldsymbol{Z}^+$-is the least such that $mg \geqq h$. Then $g, h \geqq 0, h-(m-1)g$. Since H is an interpolation group, there exists $k \in H$ such that $g, h \geqq k \geqq 0, h-(m-1)g$. Thus $h \geqq k \geqq 0$ and so $h = k$ by the hypothesis and the choice of m. Consequently, $g \geqq h$. Let $n \in \boldsymbol{Z}^+$ be least such that $nh \geqq g$ and let $p \in \boldsymbol{Z}^+$ be greatest such that $g \geqq ph$. Now $p \leqq n$ and, by hypothesis, $p \neq n$. Hence $0, g-(n-1) \leqq g-gh, h$ and, as before, $h \leqq g-ph$, a contradiction. Thus $nh = g$ for some $n \in \boldsymbol{Z}^+$ and $g \in G^+$. Choose $g \in G^+$ so that n is minimal. Since G is dense, there exists $g' \in G$ such that $0 < g' < g = nh$. Let $p \in \boldsymbol{Z}^+$ be least such that $g' \leqq ph$. Then $p \leqq n$ and $g'-(p-1)h, 0 \leqq g', h$. Hence there exists $k \in H$ such that $g'-(p-1)h, 0 \leqq k \leqq g', h$. By hypothesis, $k = h$ and so $h \leqq g'$. Let $q \in \boldsymbol{Z}^+$ be greatest such that $qh \leqq g'$. Thus $q \leqq p$. If $q < p$, then $q'-(p-1)h, 0 \leqq g'-qh, h$ and, as before, $h \leqq g'-qh$. It follows that $(q+1)h \leqq g'$, a contradiction. Consequently, $q = p$ and $g' = ph$. By the choice of n, $p = n$. Hence $g' = ph = nh = g$, a contradiction.

Finally suppose K is an a-cloure of a dense antilattice $G \neq \{0\}$. Then K is a dense antilattice. Let L be any abelian trivially ordered group. It is immediate that $K \overrightarrow{\oplus} L$ is a dense antilattice which is a proper a-extension of K, a contradiction.

COROLLARY A.1. *$\boldsymbol{R}$ has no a-closure in either the class of directed interpolation groups or the class of abelian directed interpolation groups.*

THEOREM B. *Suppose A is a* p.o. *set and for each $\alpha \in A$, $R_\alpha \neq \{0\}$ is a subgroup of either $\boldsymbol{R}$ or the trivially ordered additive group of reals. If $V(A, R_\alpha)$ is a directed interpolation group, then there exist $\{G_\beta: \beta \in On\}$ such that $G_0 = V(A, R_\alpha)$ and if $\lambda, \mu \in On$ and $\lambda < \mu$, then G_μ is a proper a-extension of G_λ in the class of abelian directed interpolation groups.*

Proof. Let B be a maximal totally ordered subset of A. If α_0 is a minimal element of B and $R_{\alpha_0} \cong \boldsymbol{Z}$, let $S_\alpha = R_\alpha$ if $\alpha \neq \alpha_0$ and $S_{\alpha_0} = \boldsymbol{R}$. Then $W = V(A, S_\alpha)$ is a directed interpolation group by Teller's conditions (see [8]) and is an a-extension of $V = V(A, R_\alpha)$ so we may assume that if B has a minimal element α_0, R_{α_0} is a dense o-group or a subgroup of the trivially ordered additive group of reals. Let $\Gamma = A \cup \{\gamma\}$ where $\gamma \notin A$. Γ is a p.o. set under the ordering: $\gamma_1 < \gamma_2$ if and only if $\gamma_1 < \gamma_2$ in A or $\gamma_1 = \gamma$ and $\gamma_2 \in B$. Let $R_\delta = R_\alpha$ if $\delta \in A$ and R_γ be the trivially ordered additive group of reals. Then $U = V(\Gamma, R_\delta)$ is a directed interpolation group (by Teller's conditions) and an a-extension of V. Continuing in this fashion, the theorem is proved.

Even removing pseudo-identities does not help since $\boldsymbol{R} \boxplus^* \boldsymbol{R}$ is an a-extension of $\{(a, a): a \in \boldsymbol{R}\} \cong \boldsymbol{R}$ in the class of directed interpolation groups without pseudo-identities.

3. a-extensions of abelian $\mathscr{P}$-groups.

Using the results of [3] and the methods of [2], the following generalizations of theorems of [2] are obtained:

THEOREM C. 1. *If G is an abelian $\mathscr{P}$-group, then G has an a-closure in the class of abelian $\mathscr{P}$-groups. If H is any such a-closure of G and Δ is a plenary subset of $C_1(G)$, then there exists an "l"-isomorphism of H into $V = V(\Delta, R_\delta)$.*

THEOREM C. 2. *If A is a* p.o. *set and $R_\alpha = \boldsymbol{R}$ for all $\alpha \in A$, then*

$V(A, R_\alpha)$ and $F(A, R_\alpha)$ are a-closed in the class of abelian $\mathscr{P}$-groups. Moreover, $F(A, R_\alpha)$ is an a-closure of $\Sigma(A, R_\alpha)$ in this class.

A p.o. *group* G is said to be *archimedean* if and only if for all $f, g \in G$, $nf \leqq g$ for all $n \in \boldsymbol{Z}$ implies $f = 0$.

THEOREM D. *If G is an archimedean abelian $\mathscr{P}$-group, then so is any a-extension of G in the class of abelian $\mathscr{P}$-groups.*

Proof. Suppose that an abelian $\mathscr{P}$-group H is an a-extension of the archimedean abelian $\mathscr{P}$-group G and $nh \leqq k$ for all $n \in \boldsymbol{Z}(h, k \in H)$. If $h, k \geqq 0$, then there exist $f, g \in G$ which are a-equivalent to h and k respectively. It is easy to see that $ng' \leqq f'$ for all $n \in \boldsymbol{Z}$ where f', g' are fixed multiples of f, g respectively. Hence $g' = 0$ and so $g = 0$. Thus $h = 0$. Since H is directed, it may be assumed that $k \geqq 0$. If $h \leqq 0$, the proof is the same as above so assume $h \| 0$. There exist $h_1, h_2 \in H$ such that $h = h_1 - h_2$ and h_1, h_2 are pseudo-disjoint. There exist $g_1, g_2 \in G$ such that g_i is a-equivalent to $h_i (i = 1, 2)$; say $m_i g_i \geqq h_i$ and $n_i h_i \geqq g_i (i = 1, 2)$. Now $ng_1 \leqq nn_1 h_1 = nn_1(h + h_2) \leqq k + nn_1 m_2 g_2$ for all $n \in \boldsymbol{Z}$ and there exists $f \in G$ a-equivalent to k; say, $k \leqq pf$, $p \in \boldsymbol{Z}^+$. Now $n(g_1 - n_1 m_2 g_2) \leqq pf$ for all $n \in \boldsymbol{Z}$ and hence $g_1 = n_1 m_2 g_2$. It follows that $h_1 \leqq m_1 g_1 = m_1 n_1 m_2 g_2 \leqq m_1 m_2 n_1 n_2 h_2$ which is impossible since h_1 and h_2 are pseudo-disjoint. Thus H is archimedean.

In [2], Conrad proved that every archimedean abelian $\mathscr{P}$-group (and hence every integrally closed abelian $\mathscr{P}$-group) is an l-group. Hence we have shown:

COROLLARY D. 1. *Every a-extension of an archimedean l-group in the class of abelian $\mathscr{P}$-groups is an archimedean l-group.*

In [2], the result was proved in the class of all l-groups not only abelian l-groups. Consequently, that result cannot be captured by the above proof. In view of Example 6.4 of [2], a-closures of archimedean abelian $\mathscr{P}$-groups are not unique.

K. M. van Meter has proved that every archimedean $\mathscr{P}$-group is an archimedean l-group and has proved that any a-extension of an archimedean $\mathscr{P}$-group in the class of $\mathscr{P}$-groups is an archimedean $\mathscr{P}$-group (see [9]) and so has proved a stronger theorem then Theorem D. The proof given here is more direct and was discovered independently and at the same time.

In view of Theorem D and Theorem 3.1 of [1]:

COROLLARY D. 2. *Every archimedean (abelian) $\mathscr{P}$-group has a unique a^*-closure in the class of (abelian) $\mathscr{P}$-groups.*

4. Archimedean extensions of compatible tight Riesz groups. It is easy to see that G is a tight Riesz group if and only if G is a dense antilattice. By Theorem A, any a-extension of a tight Riesz group is a tight Riesz group. We will restrict our attention to the class of directed interpolation groups without pseudo-identities and confine ourselves to those tight Riesz groups compatible with an l-group. Such partially ordered groups will be called *acceptable tight Riesz groups*. If $(G, \leqq)$ is an acceptable tight Riesz group, then $(G, \leqq)$ will be the l-group whose positive cone is the positive cone of $(G, \leqq)$ together with its pseudo-positive elements. Let $T = \{g \in G\colon g > 0\}$. Then there exists $\{M_\alpha\colon \alpha \in A\}$ a collection of prime l-subgroups of $(G, \leqq)$ such that $T = G^+ \setminus \cup \{M_\alpha\colon \alpha \in A\}$ (see Theorem 2.6 of [7]). Recall that if $(H, \leqq)$ is an l-group which is an a-extension of the l-group $(G, \leqq)$, then there is a (1: 1) map ϕ of the prime l-subgroups of $(G, \leqq)$ onto the prime l-subgroups of $(H, \leqq)$ which preserves containment. Let G be an abelian group and $X \subseteqq G$. Let $\bar{G}$ be the divisible closure of G and $\bar{X} = \{y \in G\colon ny \in X \text{ for some } n \in \boldsymbol{Z}^+\}$. If $(G, \leqq)$ is an l-group, then $(\bar{G}, \leqq)$ is an l-group where $h \in \bar{G}^+$ if and only if $nh \in G^+$ for some $n \in \boldsymbol{Z}^+$ (i.e., $\bar{G} = \bar{G}^+$). Then $\bar{T} = \bar{G}^+ \setminus \bigcup \{\bar{M}_\alpha\colon \alpha \in A\}$ is the strict positive cone of a compatible tight Riesz group $(\bar{G}, \leqq)$. Moreover, $(\bar{G}, \leqq)$ is an a-extension of $(G, \leqq)$ and there is a (1: 1) map of the prime subgroups of $(G, \leqq)$ onto those of $(\bar{G}, \leqq)$ which preserves containment. Using these facts, we define what is meant by an $\mathscr{A}$-extension of an abelian acceptable tight Riesz group. In view of the above remarks, we need only concern ourselves with divisible abelian acceptable tight Riesz groups. If $(K, \leqq)$ is an abelian acceptable tight Riesz group and $(K, \leqq)$ the corresponding l-group, let $K^+ = \{k \in K\colon k \geqq 0\}$ and $T_K = \{k \in K\colon k > 0\}$.

Let $(G, \leqq)$ and $(H, \leqq)$ be divisible abelian acceptable tight Riesz groups. $(H, \leqq)$ is an *$\mathscr{A}$-extension* of $(G, \leqq)$ if and only if $(H, \leqq)$ is an $\mathscr{A}$-extension of $(G, \leqq)$ and if $T_G = G^+ \setminus \bigcup \{M_\alpha\colon \alpha \in A\}$ where each M_α is a prime l-subgroup of $(G, \leqq)$, then $T_H = H^+ \setminus \bigcup \{N_\alpha\colon \alpha \in A\}$ where N_α is the prime l-subgroup of $(H, \leqq)$ corresponding to M_α. It follows at once from the facts concerning $\mathscr{A}$-extensions of l-groups that:

THEOREM E. *Every abelian acceptable tight Riesz group has an $\mathscr{A}$-closure (in the class of abelian acceptable tight Riesz groups) which is not necessarily unique. Moreover, any $\mathscr{A}$-extension of an archimedean abelian acceptable tight Riesz group is archimedean.*

The last fact follows because $(G, \leqq)$ is archimedean if and only if $(G, \leqq)$ is. The same is not true for integrally closed since if $(G, \leqq)$ is $\boldsymbol{R} \boxplus^* \boldsymbol{R}$, then $(G, \leqq) = \boldsymbol{R} \boxplus \boldsymbol{R}$ is integrally closed whereas $(G, \leqq)$ is not. It should be observed that if $(H, \leqq)$ is an $\mathscr{A}$-extension of

$(G, \leqq)$ where $(H, \leqq)$ and $(G, \leqq)$ are abelian divisible acceptable tight Riesz groups, then there is a (1:1) map of the convex d-subgroups of $(G, \leqq)$ onto the convex d-subgroups of $(H, \leqq)$ which preserves containment and for every $h \geqq 0$, there exists a $g \in G$ such that $g \geqq 0$ and $g \leqq nh$ and $h \leqq mg$ for some $m, n \in \boldsymbol{Z}^+$.

An abelian acceptable tight Riesz group $(G, \leqq)$ will be called a *good* tight Riesz group if and only if $T_G = G^+ \backslash \bigcup \{M_\alpha : \alpha \in A\}$ where each M_α is a closed prime subgroup of $(G, \leqq)$. As before, we need only consider divisible good tight Riesz groups.

Let $(H, \leqq)$ and $(G, \leqq)$ be divisible good tight Riesz groups. $(H, \leqq)$ is an a^*-extension of $(G, \leqq)$ if and only if $(H, \leqq)$ is an a^*-extension of $(G, \leqq)$ and if $T_G = G^+ \backslash \bigcup \{M_\alpha : \alpha \in A\}$, then $T_H = H^* \backslash \bigcup \{N_\alpha : \alpha \in A\}$ where N_α is the closed prime l-subgroup of $(H, \leqq)$ corresponding to the closed prime l-subgroup M_α of $(G, \leqq)$.

THEOREM F. 1. *Every good tight Riesz group has an a^*-closure (in the class of good tight Riesz groups).*

2. *Every $\mathscr{A}$-extension (in the class of divisible abelian acceptable tight Riesz groups) of a good tight Riesz group $(G, \leqq)$ is a good tight Riesz group which is an a^*-extension of $(G, \leqq)$.*

3. *Every archimedean good tight Riesz group has a unique a^*-closure (in the class of good tight Riesz groups).*

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