FUNDAMENTAL GROUPS OF COMPACT COMPLETE LOCALLY AFFINE COMPLEX SURFACES

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The fundamental group of a compact complete locally affine complex manifold of two complex dimensions is a solvable group which is a finite cyclic extension of a nilpotent or abelian group. Such a manifold has vanishing Euler characteristic and is finitely covered by a nilmanifold. A description of these manifolds and their fundamental groups is obtained in the course of the proofs of these facts.

1. **Introduction.** A locally affine manifold is a manifold with an affine connection having zero curvature and torsion. A complete locally affine real manifold is of the form R^n/Γ ([3]) and a complete locally affine complex manifold is of the form C^n/Γ([7]); Γ denotes a freely-acting properly discontinuous group of real or complex affine transformations, and the connection is induced from the usual one on R^n or C^n. This representation allows a group-theoretic study of complete locally affine spaces, the most difficult aspect of which is determining which abstract groups can be embedded in the group of affine transformations of R^n or C^n to give a Γ as described above. Such groups are of course the fundamental groups of complete locally affine spaces.

Kuiper ([5]) has studied compact locally affine real surfaces, benefiting from the knowledge of fundamental groups of compact real surfaces in general. Auslander ([1]) has studied compact locally Hermitian complex surfaces, benefiting from the fact that these are finitely covered by tori, a fact which is a consequence of Bieberbach's theorems on crystallographic groups. Vitter ([8]) has studied arbitrary compact locally affine complex surfaces using the results of Kodaira on general complex surfaces.

In this paper, we prove several results about the fundamental group Γ of a compact complete locally affine complex surface C^2/Γ which are necessary for a detailed study of such structures. We show: Γ is a finite cyclic extension of a subgroup Γ_0; Γ_0 is either abelian or nilpotent and its structure can be described precisely. Furthermore: C^2/Γ_0 is a nilmanifold, and the Euler characteristic of C^2/Γ vanishes. The methods used here are in the spirit of Auslander and Kuiper and are quite different from those of Kodaira and Vitter.

2. **Algebraic preliminaries.** In this section, we derive several facts about subgroups Γ of the group $A(2, C)$ of complex affine trans-

formations of C^2, using only the assumption that Γ acts freely on C^2.

A transformation $A \in A(2, C)$ may be identified with a nonsingular complex matrix $\begin{pmatrix} a & b & r \\ c & d & s \\ 0 & 0 & 1 \end{pmatrix}$. The action of A, on the left of C^2, sends (x, y) to (x', y'), where

$$x' = ax + by + r$$

$$y' = cx + dy + s .$$

If $A = \begin{pmatrix} a & b & r \\ c & d & s \\ 0 & 0 & 1 \end{pmatrix} \in A(2, C)$, we denote by $h(A)$ the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, C)$, the "holonomy part" of A.

LEMMA 2.1. *If* $\Gamma \subset A(2, C)$ *acts freely on* C^2, *then each element of* $h(\Gamma)$ *has* 1 *as an eigenvalue.*

Proof. The point (x, y) is a fixed point of $\begin{pmatrix} a & b & r \\ c & d & s \\ 0 & 0 & 1 \end{pmatrix} \in A(2, C)$ exactly if

$$(a - 1)x + by = -r$$

$$cx + (d - 1)y = -s .$$

These equations have a solution unless 1 is an eigenvalue of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let G_1 denote the group of all complex matrices of the form $\begin{pmatrix} a & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$ with $a \neq 0$; let G_2 denote the group of all complex matrices of the form $\begin{pmatrix} 1 & b & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix}$ with $d \neq 0$.

PROPOSITION 2.2. (Cf. [5].) *If* $\Gamma \subset A(2, C)$ *acts freely on* C^2, *then* Γ *is conjugate in* $A(2, C)$ *to a subgroup of* G_1 *or a subgroup of* G_2.

Proof. Suppose first that Γ contains an element A such that $h(A)$ has an eigenvalue $\lambda \neq 1$. Put $h(A)$ in diagonal form $\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$ by conjugating by $P \in GL(2, C)$. Suppose $B \in \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \Gamma \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}^{-1}$. Write $h(B) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $h(A)h(B) = \begin{pmatrix} \lambda a & \lambda b \\ c & d \end{pmatrix}$. Since both $h(A)$ and $h(AB)$ have 1 as an eigenvalue, we get $(a-1)(d-1) - bc = 0$ and $(\lambda a - 1)(d - 1) -$

$\lambda bc = 0$. Multiply the first equation by λ and subtract it from the second to obtain $(\lambda - 1)(d - 1) = 0$. Hence $d = 1$, and then $bc = 0$. That is, $h(B) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ or $h(B) = \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix}$. We cannot have both kinds of $h(B)$ occurring; for if both $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} a' & 0 \\ c' & 1 \end{pmatrix}$ were in $Ph(\Gamma)P^{-1}$, with $b \neq 0$ and $c' \neq 0$, we would have $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a' & 0 \\ c' & 1 \end{pmatrix} = \begin{pmatrix} aa' + bc' & b \\ c' & 1 \end{pmatrix} \in Ph(\Gamma)P^{-1}$, but this matrix does not have 1 as an eigenvalue. Hence we have, in this case, that $\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}\Gamma\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}^{-1}$ is contained in G_1 or in the group of all complex matrices of the form $\begin{pmatrix} a & 0 & r \\ c & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$; the latter is conjugate to G_2 via $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and we are done.

Now suppose every element of $h(\Gamma)$ has both eigenvalues 1. If $h(\Gamma)$ consists only of the identity, we are done. Otherwise, some conjugate of Γ contains an element of the form $\begin{pmatrix} 1 & 1 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}$. Let $\begin{pmatrix} a & b & r \\ c & d & s \\ 0 & 0 & 1 \end{pmatrix}$ be an arbitrary element of this conjugate of Γ. Then both $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + c & b + d \\ c & d \end{pmatrix}$ have both their eigenvalues 1. Thus $(a-1)(d-1) - bc = 0$ and $(a + c - 1)(d - 1) - (b + d)c = 0$. Subtracting these equations gives $c = 0$. Hence $a = d = 1$, and we are done.

COROLLARY 2.3. *If $\Gamma \subset A(2, C)$ acts freely on C^2, then Γ is solvable.*

Proof. The third derived group of Γ is trivial since this is true of G_1 and G_2.

LEMMA 2.4. *If $\Gamma \subset A(2, C)$ acts freely on C^2 and $h(\Gamma)$ is abelian, then Γ is conjugate in $A(2, C)$ to a subgroup of the group of all matrices of the form $\begin{pmatrix} 1 & 0 & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix}(d \neq 0)$ or to a subgroup of the group of all matrices of the form $\begin{pmatrix} 1 & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}.$*

Proof. If $h(\Gamma)$ consists only of the identity, we are done. Suppose $h(\Gamma)$ contains a non-identity element A which is diagonalizable. Conjugate A in $GL(2, C)$ to $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$, $\lambda \neq 1$. If $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the corresponding conjugate of $h(\Gamma)$, the fact that $AB = BA$ implies $b = c = 0$, and the fact that Γ acts freely implies $a = 1$ or $d = 1$.

If a were $\neq 1$ we would have $AB = \begin{pmatrix} a & 0 \\ 0 & \lambda \end{pmatrix}$, in contradiction to 2.1. Hence every element of $h(\Gamma)$ is simultaneously conjugate to a matrix of the form $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$, and we are done in this case.

Suppose now that no element of $h(\Gamma)$ is diagonalizable. Let A be a non-identity element of $h(\Gamma)$. Conjugate A in $GL(2, C)$ to $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, $\lambda \neq 0$. Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be in the corresponding conjugate of $h(\Gamma)$. The fact that $AB = BA$ implies $c = 0$ and $a = d$, so necessarily $a = d = 1$. As before, we are done.

LEMMA 2.5. If $\begin{pmatrix} a & b & r \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has no fixed points in C^2, then $b = 0$.

Proof. If $b \neq 0$, $(0, -r/b)$ is a fixed point.

LEMMA 2.6. If $\Gamma \subset G_1$ acts freely on C^2, then $h(\Gamma)$ is abelian.

Proof. Let $A = \begin{pmatrix} a & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} a' & b' & r' \\ 0 & 1 & s' \\ 0 & 0 & 1 \end{pmatrix}$ be elements of Γ. Then $ABA^{-1}B^{-1} = \begin{pmatrix} 1 & f & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. By 2.5, $f = 0$.

COROLLARY 2.7. If $\Gamma \subset A(2, C)$ acts freely on C^2, then Γ is conjugate in $A(2, C)$ to a subgroup of G_2.

Proof. By 2.2, Γ is conjugate to a subgroup of G_1 or G_2. If Γ is conjugate to a subgroup of G_1, then $h(\Gamma)$ is abelian by 2.6. Then, by 2.4, Γ is conjugate to a subgroup of G_2.

LEMMA 2.8. If $\Gamma \subset A(2, C)$ is abelian and acts freely on C^2, then Γ is conjugate in $A(2, C)$ to a subgroup of the group of all matrices of the form $\begin{pmatrix} 1 & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$ or to a subgroup of the group of all matrices of the form $\begin{pmatrix} 1 & 0 & r \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $(d \neq 0)$.

Proof. $h(\Gamma)$ is abelian, so by 2.4 we can conjugate Γ into the group of all $\begin{pmatrix} 1 & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$, in which case we are done, or into the group

of all $\begin{pmatrix} 1 & 0 & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix}$. In the latter case: If all entries d which occur are 1, we are done. If some element has $d \neq 1$, further conjugation by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & s/(d-1) \\ 0 & 0 & 1 \end{pmatrix}$ takes this element to $\begin{pmatrix} 1 & 0 & r \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and other elements to $\begin{pmatrix} 1 & 0 & r' \\ 0 & d' & s' \\ 0 & 0 & 1 \end{pmatrix}$. Since Γ is abelian, we must have all $s' = 0$.

3. **Topological preliminaries.** The hypotheses that Γ acts properly discontinuously on C^2 and that C^2/Γ is compact are brought into play in this section.

We note the following important fact ([4], p. 357): The dimension of a real Euclidean space on which a group Γ acts freely, properly discontinuously, and with compact orbit space is determined by Γ itself, namely as the projective dimension of the integer group ring of Γ.

As a first application of this remark, we prove the following from Auslander ([2]).

LEMMA 3.1. *Suppose that $\Gamma \subset A(2, C)$ acts freely and properly discontinuously, and that C^2/Γ is compact. Then the set of translational parts (r, s) of elements $\begin{pmatrix} a & b & r \\ c & d & s \\ 0 & 0 & 1 \end{pmatrix}$ of Γ contains a basis for C^2 as a real vector space.*

Proof. Let V be the real subspaces of C^2 spanned by the translational parts of elements of Γ. Then the action of Γ on C^2 sends V to itself. Further, Γ acts freely and properly discontinuously on V, and V/Γ is compact. By the remark above, V and C^2 have the same dimension, so $V = C^2$.

COROLLARY 3.2. *If $\Gamma \subset A(2, C)$ is abelian, acts freely and properly discontinuously, and C^2/Γ is compact, then Γ is conjugate in $A(2, C)$ to a subgroup of the group of all matrices of the form $\begin{pmatrix} 1 & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$.*

Proof. By 2.8, the only alternative is that Γ can be conjugated to a subgroup of the group of all matrices of the form $\begin{pmatrix} 1 & 0 & r \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$. By 3.1, this cannot happen.

LEMMA 3.3. *Suppose* $\Gamma \subset A(2, C)$ *acts properly discontinuously on* C^2 *and contains elements* $A = \begin{pmatrix} 1 & b & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & f & u \\ 0 & h & v \\ 0 & 0 & 1 \end{pmatrix}$ *such that* $d \neq 1$ *and* $AB \neq BA$. *Then* d *is a root of unity.*

Proof. By direct computation we verify that

$$A^n = \begin{pmatrix} 1 & \dfrac{d^n - 1}{d - 1}b & \dfrac{d^n - 1}{d - 1}\dfrac{bs}{d - 1} + n\left(r - \dfrac{bs}{d - 1}\right) \\ 0 & d^n & \dfrac{d^n - 1}{d - 1}s \\ 0 & 0 & 1 \end{pmatrix}$$

for all integers n, and that the matrix $C_n = A^{-n}BA^nB^{-1} = \begin{pmatrix} 1 & f_n & u_n \\ 0 & 1 & v_n \\ 0 & 0 & 1 \end{pmatrix}$ has entries given by

$$f_n = (d^n - 1)\left(\frac{f}{h} - \frac{b}{d - 1} + \frac{d}{(d - 1)h}\right)$$

$$u_n = (d^n - 1)\left(-\frac{fv}{h} - \frac{bv}{(d - 1)h} + \frac{bv}{d - 1} + \frac{bs}{(d - 1)^2} + \frac{sf}{d - 1} - \frac{bsh}{(d - 1)^2}\right)$$

$$\quad + (d^{-n} - 1)\left(-\frac{bsh}{(d - 1)^2} + \frac{bv}{d - 1} + \frac{bs}{(d - 1)^2}\right)$$

$$v_n = (d^{-n} - 1)\left(v + \frac{s}{d - 1} - \frac{sh}{d - 1}\right).$$

We claim that if d is not a root of unity, then the matrices C_n are distinct. For suppose that $C_m = C_n$ with $m \neq n$. This would give $f/h - b/(d - 1) + b/((d - 1)h) = 0$ and $v + s/(d - 1) - sh/(d - 1) = 0$; that is, $(d - 1)f + b(1 - h) = 0$ and $(d - 1)v + s(1 - h) = 0$. Multiply the first of the latter two equations by s and the second by b; subtract to obtain $(d - 1)(sf - bv) = 0$. The equations $(d - 1)f = b(h - 1)$, $(d - 1)v = s(h - 1)$, and $sf = bv$ imply $AB = BA$; a contradiction.

Assume that d is not a root of unity and consider the points (x_n, y_n) of C^2 obtained by applying the distinct transformations C_n of Γ to the point $(0, v - sh/(d - 1))$. We have

$$x_n = f_n\left(v - \frac{sh}{d - 1}\right) + u_n = (d^{-n} - 1)\left(-\frac{bsh}{(d - 1)^2} + \frac{bv}{d - 1} + \frac{bs}{(d - 1)^2}\right)$$

$$y_n = v - \frac{sh}{d - 1} + v_n = d^{-n}\left(v + \frac{s}{d - 1} - \frac{sh}{d - 1}\right) - \frac{s}{d - 1}.$$

If $|d| = 1$, we may find a sequence n_i tending to infinity such that

$\lim d^{n_i} = 1$. Then the sequence of points (x_{n_i}, y_{n_i}) has a limit in C^2, contradicting the assumption that Γ acts properly discontinuously. If $|d| \neq 1$, we may assume $|d| > 1$. Again this sequence of points has a limit point and we obtain a contradiction.

PROPOSITION 3.4. *If $\Gamma \subset A(2, C)$ acts freely and properly discontinuously on C^2, and C^2/Γ is compact, then Γ contains a unipotent normal subgroup Γ_0 of finite index with Γ/Γ_0 cyclic.*

Proof. By 2.7, we may assume $\Gamma \subset G_2$. Assume that Γ contains a central element $\begin{pmatrix} 1 & b & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix}$ with $d \neq 1$. Conjugate Γ by

$$\begin{pmatrix} 1 & \dfrac{-b}{d-1} & 0 \\ 0 & 1 & \dfrac{s}{d-1} \\ 0 & 0 & 1 \end{pmatrix} \in G_2$$

to obtains a subgroup of G_2 containing the central element $\begin{pmatrix} 1 & 0 & r' \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Since this element is central, all the elements of this new subgroup have the form $\begin{pmatrix} 1 & 0 & u \\ 0 & h & 0 \\ 0 & 0 & 1 \end{pmatrix}$. But, according to 3.1, this cannot happen.

Now in general, Γ is the fundamental group of the compact manifold C^2/Γ, so it is finitely generated. Let $A_i = \begin{pmatrix} 1 & b_i & r_i \\ 0 & d_i & s_i \\ 0 & 0 & 0 \end{pmatrix} (1 \leq i \leq k)$ be a set of generators of Γ. If A_i is central, $d_i = 1$ by the preceding paragraph. If A_i is not central, d_i is a root of unity by 3.3. Hence the image of the homomorphism $\begin{pmatrix} 1 & b & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix} \to d$ of Γ into the unit circle group in C is a finite group. This finite group, being a subgroup of the multiplicative group of a field is cyclic. The kernel of this homomorphism is the desired subgroup Γ_0.

4. **The main theorem.** In this section, we sharpen the statement of Proposition 3.4 and interpret our results terms of compact complete locally affine complex surfaces.

Let D_k ($k \geq 1$) denote the torsion free nilpotent group with generators: A, B, C, and D; and relations: $ABA^{-1}B^{-1} = C^k$, C and D central.

THEOREM 4.1. *Let $\Gamma \subset A(2, C)$ act freely and properly discon-*

tinuously on C^2, and let C^2/Γ be compact. Then Γ contains a unipotent normal subgroup Γ_0 of finite index such that Γ_0 is isomorphic to Z^4 or D_k (for some $k \geq 1$) and Γ/Γ_0 is cyclic of order $1, 2, 3, 4, 5, 6, 8, 10,$ or 12.

Proof. Let Γ_0 be the subgroup of Γ obtained in 3.4. By results of Malcev ([6]) on nilmanifolds, Γ_0 may be considered as a discrete subgroup of a unique connected simply-connected nilpotent Lie group N such that N/Γ_0 is compact. Then Γ_0 acts freely and properly discontinuously on the space N and the orbit space is compact. N is topologically Euclidean, so by the remark preceding 3.1, the real dimension of N is four. Now, there are only two four-dimensional connected simply-connected Lie groups, namely R^4 and $\left\{\begin{array}{l}\text{all real} \\ \text{matrices} \\ \text{of the form}\end{array}\right.$

$\left. \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \right\} \times R$. For these two groups, the discrete subgroups with compact quotients are known to be isomorphic to Z^4 in the first case and the D_k in the second case.

Let $G \subset C$ be the additive group of complex numbers b such that $\begin{pmatrix} 1 & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma_0$. Let $S \in \Gamma$ be an element whose image in Γ/Γ_0 generates this cyclic group. Then $S = \begin{pmatrix} 1 & f & u \\ 0 & \lambda & v \\ 0 & 0 & 1 \end{pmatrix}$ with $\lambda^n = 1$, where n is the index of Γ_0 in Γ. $S^{-1}\begin{pmatrix} 1 & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}S = \begin{pmatrix} 1 & \lambda b & r' \\ 0 & 0 & \lambda^{-1}s \\ 0 & 0 & 1 \end{pmatrix}$ shows that $\lambda b \in G$ for $b \in G$. If G is the trivial subgroup (0) of C, then the "holonomy group" $h(\Gamma)$ is finite cyclic of order n and we are in the case studied by Auslander in [1]. In this case, $n = 1, 2, 3, 4,$ or 6. Assume now that G is not trivial. Then G is a free abelian group of rank r, with $1 \leq r \leq 4$, since Γ_0 can be generated by four elements. Let $b_1 \cdots, b_r$ be a basis of G. Expressing λb_i in terms of this basis and taking a determinant, we obtain a polynomial of degree r with integer coefficients which is satisfied by λ. Hence the field generated by λ over the rationals is of degree at most r. This field is the field generated by a primitive n^{th} root of unity, so it has degree $\varphi(n)$, where φ is Euler's totient. Thus $\varphi(n) \leq r$. The only solutions of $\varphi(n) \leq 4$ are those listed in the statement of the theorem.

The groups Z^4 and D_k of 4.1 do occur as subgroups of $A(2, C)$ which act freely and properly discontinuously on C^2 with compact orpit space. Which cyclic extensions of these groups can occur is a more delicate question.

EXAMPLE 4.2.   Let $A, B, C,$ and D be the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1\sqrt{-1} & \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & \sqrt{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

respectively.   These matrices generate a subgroup Γ_0 of $A(2, C)$ which is isomorphic to D_1. Γ_0 acts freely and properly discontinuously on C^2 and C^2/Γ_0 is a compact complete locally affine complex surface which is even a nilmanifold.   Let S be the matrix $\begin{pmatrix} 1 & 0 & 1/2\sqrt{-1} \\ 0 & -1 & \sqrt{-1} \\ 0 & 0 & 1 \end{pmatrix}$.
Then $S^2 = D$ and conjugation by S sends A, B, C, D to $A^{-1}, B^{-1}, C,$ D respectively. S and the group Γ_0 generate a subgroup Γ of $A(2, C)$ containing Γ_0 as a normal subgroup of index two. Γ acts freely and properly dicontinuously on C^2 and C^2/Γ is a compact complete locally affine complex surface.   The fundamental group Γ of this surface is solvable but not nilpotent.   The commutator subgroup of Γ is generated by $A^2, B^2,$ and C, from which we obtain the first Betti number of C^2/Γ as $b_1 = 1$.   Using Poincaré duality and the vanishing of the Euler characteristic, proved below, we find all Betti numbers of C^2/Γ are given by 1, 1, 0, 1, 1. C^2/Γ is an example of a compact complete locally affine complex surface with non-abelian "holonomy group" $h(\Gamma)$.

A complete locally affine surface may be represented in two ways as C^2/Γ and C^2/Γ''.   This corresponds to Γ and Γ'' being conjugate in $A(2, C)$; the element of $A(2, C)$ effecting the conjugation amounts to a change of coordinates on the surface.   This allows us to interpret 4.1 as follows.

Consequences 4.3.   1.   A compact complete locally affine complex surface has a fundamental group which is solvable and is an extension of Z^4 or some D_k by a finite cyclic group of order 1, 2, 3, 4, 5, 6, 8, 10, or 12.

2.   Such a surface is finitely covered by another such surface which is a nilmanifold with fundamental group Z^4 or some D_k; the cover is normal with deck transformation group cyclic of one of the above orders.

We conclude with a proof of the following theorem.   This result was also obtained by Vitter ([8]) using the fact that the presence of a locally affine structure on a compact complex manifold implies that all its Chern classes, excepting the zeroth, are zero.

THEOREM 4.4.   *A compact complete locally affine complex surface has Euler characteristic zero.*

Proof.   It suffices to prove this for a finite cover of the surface

and this cover may be represented as C^2/Γ_0 with

$$\Gamma_0 \subset \left\{ \begin{matrix} \text{all complex} \\ \text{matrices of} \\ \text{the form} \end{matrix} \quad \begin{pmatrix} 1 & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

With coordinates (x, y) on C^2, the vector field $\partial/\partial x$ has non-vanishing real part on C^2 which is invariant under the action of Γ_0. This gives a non-vanishing vector field on C^2/Γ_0, and hence the Euler characteristic of this surface is zero.

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