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$\mathscr{H}$-COMMUTATIVE SEMIGROUPS IN WHICH EACH HOMOMORPHISM IS UNIQUELY DETERMINED BY ITS KERNEL

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An eqivalence relation σ on a semigroup S is called a *congruence* if whenever $a\sigma b$ and $c\sigma d$ it follows that $ac\sigma bd$. There is a natural correspondence between congruences on S and homomorphic images of S. In this paper semigroups satisfying the following two conditions are considered.

(1) There exists g in S such that, if σ and ρ are congruences on S and $\{x \in S: x\sigma g\} = \{x \in S: x\rho g\}$, then σ and ρ coincide.

(2) For all $a, b \in S$, either $ab = ba$ or, for some x and y in S, $abx = ba$ and $yab = ba$.

In § 1, four examples of such semigroups are given. For example, in Type 1 (which is the most complicated of the four) the method of construction may be roughly described as follows. Start with an arbitrary group G, having g as its identity element. Adjoin to G any number of extra pieces. Each of these must be an interval which has been cut out from some subgroup of the additive real number. Each interval must have 0 as one endpoint and (with possibly one exception) must actually contain 0. The resulting set is turned into a semigroup by the convention that the extra elements act, wherever the operation is not already defined, in the same manner as the group identity g. Finally, one can, optionally, adjoin a zero element.

In § 3 the following result is obtained.

MAIN THEOREM. *Every semigroup satisfying* (1) *and* (2) *is isomorphic to a semigroup of one of the four types constructed in* § 1.

The effect of (1) is to assert that each congruence is uniquely determined by its *kernel relative to* g (that is, by the equivalence class to which g belongs). Thus if S is a group with identity element g, an elementary theorem of group theory states that (1) is satisfied. Indeed, it is easy to see that g could be taken to be any element of the group, and (1) would still be true. On the other hand, (1) does not hold for arbitary semigroups. For example, we shall see (as a consequence of Lemma 1) that (1) does not hold for the positive integers, if we take either multiplication or addition as the semigroup operation.

(1) can be better understood if we recall that Ljapin [4, 5] gave a fairly simple condition for a subset of a semigroup to be a possible congruence class, and called such subsets *normal.* Thus (1) states that there is a natural one-to-one correspondence between homomorphisms of S and those normal subsets which contain g. Moreover, given a normal subset N of any semigroup, one can construct two particular congruences (defined by Ljapin [4, 5] and Teissier [9]) which are respectively the smallest and the largest congruence having N as a class. Thus (1) states that these two congruences are identical for every normal set N containing g, so that, in the lattice of congruences on S, the "closed interval" consisting of all those congruences having N as a class reduces to a single "point."

Our main purpose is to determine all commutative semigroups satisfying (1). However, it seemed more natural to replace commutativity by (2), which is a slighty weaker condition. Recall that Green [3] defined an equivalence relation $\mathscr{H}$ on an arbitrary semigroup by: $a\mathscr{H}b$ if and only if either $a = b$ or, for some x, y, u, v in S, $ax = ya = b$ and $bu = vb = a$. Since (2) is simply commutativity modulo $\mathscr{H}$, we might call a semigroup satisfying (2) $\mathscr{H}$*-commutative.* It is easy to see that (2) implies that $\mathscr{H}$ is itself a congruence. In fact, (2) is equivalent to the assertion that $\mathscr{H}$ is a congruence and the homomorphic image determined by $\mathscr{H}$ is commutative. Some examples of $\mathscr{H}$-commutative semigroups are furnished by those semigroups (studied by Clifford [1]) which are unions of groups and in which the idempotents commute. In fact, if S is a union of groups, then (2) holds if and only if the idempotents commute.

In § 2 we develop some preliminary results. Some of the methods and results of this section (for instance, the partition in Lemma 6 of a portion of S into certain equivalence classes each of which is totally ordered by the divisibility relation) are very similar to those used by Ljapin [6] in his study of a more restricted class of semigroups, viz., commutative semigroups in which the only normal sets are ideals and single elements. Lemma 8 is a slightly altered version of an imbedding theorem due to Clifford [2].

After proving the main theorem, we derive some corollaries in § 4. For example, in the finite case the construction becomes particularly simple (Corollary 1). Corollary 2 specializes to the class of semigroups studied by Ljapin [6]. Our results are similar to his but more explicit because of our use of subgroups of real numbers. Finally, if (2) is strengthened by requiring g to be an identity element, S must be either a group or a group with a zero element adjoined (Corollary 3).

1. Examples.

Type 1. Let G be an arbitrary group, and g its identity element. Let $G_0, G_1, \cdots$ be any (finite or finite) number of multiplicative subgroup of the positive real numbers. (Since the multiplicative positive reals are isomorphic to the additive reals, we could justas well take $G_0, G_1, \cdots$ to be additive groups, and make a few small changes in what follows. We choose multiplication mainly to agree with our use elsewhere in this paper of multiplicative notation.) Let I_0 be optionally either the interval $(1/2, 1)$ or the interval $[1/2, 1)$. For each $i \neq 0$, let I_i be optionally either $(1/2, 1]$ or $[1/2, 1]$. For each i, let $T_i = G_i \cap I_i$. Let $S = (\bigcup_i T_i) \cup G$. (In the formation of S, we regard all the T_i and G as mutually disjoint. Formally, this involves replacing each T_i by another set in one-to-one correspondence with T_i.) We define multiplication $\circ$ in S by: $a \circ b = ab$, the group product, if $a, b \in G$; $a \circ b = b \circ a = a$, if $a \in G$, $b \in T_i$; $a \circ b = ab$, the numerical product, if $a, b \in T_i$ and $ab \in T_i$; $a \circ b = g$, the group identity, in all other cases. Finally, we adjoin to S an optional zero element.

Type 2. Let G be an arbitrary group, and g any element of G. Adjoin to G an optional zero element and an optional element a. Define $a^2 = g^2$, $ah = gh$, $ha = hg$, for all $h \in G$.

Type 3. Let G be a multiplicative subgroup, containing 1/2, of the positive real numbers. Let S be the intersection of G with either the interval $[1/2, 1)$ or the interval $[1/2, 1]$, with an extra element 0 adjoined. Let g be the number 1/2. Define: $a \circ b = ab$, the numerical product, if $a \neq 0 \neq b$ and $ab \geqq 1/2$; $a \circ b = 0$, in all other cases.

Type 4. Let G be as in Type 3. Let S be $G \cap [1/2, 1]$, with two extra elements, 0 and h, adjoined. Define the operation as in Type 3, with the additional provision that $h \circ 1 = 1 \circ h = 1/2$, and $h \circ a = a \circ h = 0$, if $a \neq 1$.

It is fairly straightforward to check that semigroups of these four types satisfy (1) and (2). For (2), one needs only note that Types 3 and 4 are actually commutative, while in Types 1 and 2 the only non-commutativity which exists is that arising from the group G. Thus in Types 1 and 2, ab and ba are always either equal or both in G, so that (2) holds.

For (1), the best procedure is to determine all the congruences. For example, suppose σ is any congruence on a semigroup of Type 1 (with the zero element adjoined). The σ-class containing 0 must be an ideal, I. If $I \neq \{0\}$, then I contains G, and every other σ-class must be a single element. On the other hand, suppose $I = \{0\}$. Then σ restricted to G gives a congruence on G, which can be expressed as the partition of G into the cosets of some normal subgroup H. Finally, each element of the T_i must be either alone in its σ-class, or identified with H. The situation with the other types is similar but easier.

2. Preliminary results. We now assume that S is a semigroup satisfying (1) and (2). By (2), left and right divisibility coincide, that is, if $ax = b$ then $ya = b$ for some y. Thus we shall say simply that a *divides* b, and write $a \mid b$, if either $a = b$ or $ax = b$ for some $x \in S$. The divisibility relation thus defined is reflexive and transitive, but it may happen that each of two distinct elements divides the other.

LEMMA 1. *At most one element of S (which is necessarily a zero element) fails to divide g.*

Proof. Let $A = \{x \in S: x \nmid g\}$. If $A \neq \emptyset$, A is an ideal. Hence we can consider the *Rees congruence modulo A* (introduced by Rees [8]), that is, the congruence having A as one class, every other class being a single element. This congruence and the *identity congruence* (in which each class consists of a single element) both have $\{g\}$ as their kernel. Hence they coincide, so that A consists of a single element, which must be a zero element.

Now let $I = \{x \in S: g \mid x\}$. Let G be the $\mathscr{H}$-class containing g, that is $G = \{x \in S: g \mid x \text{ and } x \mid g\}$. Then I is G together with a possible zero element.

LEMMA 2. *If $x, y \notin G$ and $x \mathscr{H} y$, then $x = y$.*

Proof. Let σ be the congruence having G as one class, every other class being a single element. Then σ and $\mathscr{H}$ are congruences with the same kernel, G. Hence σ and $\mathscr{H}$ coincide.

LEMMA 3. *Suppose J is an ideal containing g, and a and b are distinct elements of S not in J. Then for some $x \in S$ either $ax \in J$, $bx \notin J$ or $ax \notin J$, $bx \in J$.*

Proof. Define (following Teissier [9] and Pierce [7]) a congruence σ by: $x \sigma y$ if and only if every multiple of the pair x, y (including the pair x, y itself) consists of elements $x(a)$, $y(a)$ either both belonging to J, or neither belonging to J. Then σ has the same kernel as the Rees congruence modulo J. Hence σ coincides with the Rees congruence.

LEMMA 4. *Let $x \in S$. Then either $x^n \in I$, for some positive integer n, or else $x^2 = x$ and x is maximal relative to $|$ (i.e., has no divisors except itself).*

Proof. Suppose that, for all n, $x^n \neq I$. Let $J = \{y\colon \text{for all } n,\ y \nmid x^n\}$. If x^2 had a proper divisor, a, we could apply Lemma 3 to the pair a, x^2, to obtain z such that exactly one of the pair az, x^2z is in J. But this is impossible. For $a \mid x^2$ implies $az \mid x^2z$, so that if $az \in J$ then so is x^2z. On the other hand, if $az \notin J$, then $az \mid x^n$ for some n, and hence $z \mid x^n$, so that $x^2z \mid x^{n+2}$, and hence $x^2z \notin J$. Thus we have shown that x^2 is maximal relative to $|$, and in particular that $x^2 = x$.

Now let T be the complement of I in S.

LEMMA 5. *Suppose $a, b \in T$, and a and b are incomparable relative to $|$ (that is $a \nmid b$ and $b \nmid a$). Then there exists an idempotent e which is maximal relative to $|$ and which divides exactly one of a, b.*

Proof. Let $J = \{x \in S\colon x \nmid a \text{ and } x \nmid b\}$. Then $J \supseteqq I$. Apply Lemma 3, to obtain $x \in S$ with exactly one of ax, bx (say bx) in J. Then $ax \notin J$. Hence $ax = a$. Hence, for all n, $ax^n = a$. Hence, for all n, $x^n \notin I$. By Lemma 4, x is a maximal idempotent. $x \mid a$ since $ax = a$. Finally, $x \nmid b$, for if $x \mid b$ then $b = xy$ for some $y \in S$, and hence $xb = x(xy) = x^2y = xy = b \notin J$, so that $xb \notin J$ and by $\mathscr{H}$-commutativity $bx \notin J$, contradicting our earlier assumption.

Now we define a relation π on T by: $a\pi b$ if and only if either $a \mid b$ or $b \mid a$.

LEMMA 6. *π is an equivalence relation on T.*

Proof. Clearly, π is reflexive and symmetric. We must show that π is transitive. Thus suppose that $a\pi b$ and $b\pi c$. There are four cases:

Case 1: $a \mid b$ and $b \mid c$. Here, clearly $a \mid c$, so that $a\pi c$.

Case 2: $b \mid a$ and $c \mid b$. Here, clearly $c \mid a$, so that $a\pi c$.

Case 3: $a \mid b$ and $c \mid b$. Suppose $a\pi c$ does not hold. By Lemma 5, obtain a maximal idempotent e with, say, $e \mid c$, $e \nmid a$. Since $a \mid b$, $ae \mid bc$. But since e is an idempotent dividing b, we have $be = b$. Hence aeb, so that $ae \in T$. Now let $J = \{x\colon x \nmid ae$. Then $J \supseteqq I$. Moreover $ae \neq a$, for otherwise $e \mid a$. Apply Lemma 3 to obtain x with exactly one of ax, aex in J. If $ax \in J$, then, since $ax \mid aex$, we have $aex \in J$. Hence we must have $aex \in J$, $ax \notin J$. Hence $ax \mid ae$, and so $axe \mid ae^2 = ae$. But $axe = aex$ by $\mathscr{H}$-commutativity and Lemma 2. Hence $aex \notin J$, a contradiction.

Case 4: $b \mid a$ and $b \mid c$. Suppose $a\pi c$ does not hold. By Lemma 5 let e be a maximal idempotent with, say, $e \mid c$, $e \nmid c$. Then, since $b \mid c$ and $e \mid c$, we have $b\pi e$ by Case 3. But since e is maximal this implies $e \mid b$. Hence $e \mid a$.

Let the π-classes be called $S_0, S_1, \cdots$. Clearly each $S_i \cup I$ is an ideal. Let R_i be the homomorphic image of the semigroup $S_i \cup I$ determined by the Rees congruence modulo I. Let T_i' be R_i with the maximal idempotent of R_i (if there is one) removed.

LEMMA 7. *Each T_i' is a commutative semigroup with zero element* 0, *satisfying*:

(3) *T_i' is naturally totally ordered (in the sense of Clifford* [2]), *that is for all distinct $a, b \in T_i'$ either $a \mid b$ or $b \mid a$ but not both.*

(4) *Every $a \in T_i'$ is nilpotent, that is $a^n = 0$ for some positive integer n.*

Proof. T_i' is commutative because it is essentially part of an $\mathscr{H}$-commutative semigroup, considered modulo the relation $\mathscr{H}$. The zero element is simply I collapsed to a single point. To prove (3), let distinct a and b be given. If one is 0, the other divides it, but not conversely. If neither is 0, then $a, b \in S_i$. But S_i is a π-class. Hence one divides the other. By Lemma 2, it is impossible for each to divide the other. To prove (4), let $a \neq 0$ be given. Since $a \in S_i \subseteqq S$, and a is not a maximal idempotent, we conclude by Lemma 4, that, in S, $a^n \in I$ for some n. But this means that, in T_i', $a^n = 0$.

LEMMA 8. *Suppose T is any commutative semigroup with zero satisfying* (3) *and* (4). *Then T is isomorphic to the intersection with the interval* $(0, 1)$ *of some subgroup of the positive real numbers, with either the interval* $(0, 1/2)$ *or the interval* $(0, 1/2]$ *collapsed to a point.*

Proof. Clifford [2, especially Theorem 4, page 642] showed essentially that every such semigroup can be imbedded in the additive positive reals with either the interval $(1, \infty)$ or the interval $[1, \infty)$ collapsed to a point. Thus, in our multiplicative notation, we have T imbedded in $(0, 1)$ with either $(0, 1/2)$ or $(0, 1/2]$ collapsed to a point. Let J be that portion of $(0, 1)$ which is not collapsed, so that J is either $[1/2, 1)$ or $(1/2, 1)$. By virtue of the imbedding, we can regard the set U of nonzero elements of T as a subset of J. Let G be the multiplicative group of real numbers generated by U, that is let $G = \{u_1 \cdots u_m v_1^{-1} \cdots v_n^{-1} : u_i, v_i \in U\}$. We shall show that $G \cap J = U$. Clearly $G \cap J \supseteqq U$. Suppose $x \in G \cap J$. Then $x = u_1 \cdots u_m v_1^{-1} \cdots v_n^{-1}$, for some $u_i, v_i \in U$. If $m + n = 1$, we are finished, because then either $x = u_1 \in U$ or $x = v_1^{-1}$, and the second case cannot arise since $x \in J$. Thus we can proceed by induction on $m + n$. If $u_1 = v_1$, we can write $x = u_2 \cdots u_m v_2^{-1} \cdots v_n^{-1}$, so that by induction $x \in U$. On the other hand, suppose $u_1 \neq v_1$. By (3) we have, for some $y \in T$, either $v_1 y = u_1$ or $u_1 y = v_1$. Since y cannot be the zero element of T, we have $y \in U$, where either $y = u_1 v_1^{-1}$ or $y = v_1 u_1^{-1}$. Thus we can write either $x = y u_2 \cdots u_m v_2^{-1} \cdots v_n^{-1}$ or $x = u_2 \cdots u_m y^{-1} v_2^{-1} \cdots v_n^{-1}$. Thus by induction $x \in U$. It is now clear, that, if we start with G and perform the construction stated in the lemma, we obtain U together with the collapsed interval, that is, we essentially obtain T.

3. Proof of the main theorem. We divide the proof into four cases.

Case I: $g^2 \in G$, and there exists $a \in T$ such that $a^2 \in T$.

Case II: $g^2 \in G$, and, for all $a \in T$, $a^2 \notin T$.

Case III: $g^2 \notin G$, and there is none or one π-class.

Case IV: $g^2 \notin G$, and there is more than one π-class.

Note first that, in Cases I and II, G is a group. This follows from the known fact (Green [3]) that an $\mathscr{H}$-class containing two elements and their product must be a group.

LEMMA 9. *In Cases I and II, if e is the identity of the group G, and a is any element of T, we have $ae = ea = g$.*

Proof. If $0 \neq s \in S$, then es and se are in G. Thus $es = ese = se$. Thus, e commutes with every element of S. Define congruences σ and ρ on S by: $x\sigma y$ if and only if either $x = y$, or $xe = ye$ and a properly divides x and y; $x\rho y$ if and only if either $x = y$, or $xe = ye$, $a \mid x$, $a \mid y$. The kernel of σ is $\{x \in S\colon xe = g$, a properly divides $x\}$, and the kernel of ρ is $\{x \in S\colon xe = g, a \mid x\}$. But σ and ρ are distinct (since a and ae are in the same ρ-class but not in the same σ-class). Hence their kernels are distinct. Hence $ae = g$.

Now in Case I, we show that S must be isomorphic to a semigroup of Type 1. By Lemmas 7 and 8, we have a one-to-one correspondence between each T_i' and the intersection with either (1/2, 1) or [1/2, 1) of a suitable subgroup G_i of the positive reals. By Lemma 5, each of the T_i, with one possible exception, consists of the corresponding T_i' with a maximal idempotent e_i adjoined. We can handle this by numbering in such a way that G_0 gives rise to the exception. (If there is no exception, let $G_0 = \{1\}$.) Then for $i \neq 0$, intersecting G_i with (1/2, 1] or [1/2, 1] gives T_i, since the number 1 can correspond to e_i. Finally we saw that G is a group. Thus we have a one-to-one correspondence between S and a Type 1 semigroup.

Next, we show that g is the identity element of G. Since we are in Case I, let a be such that $a^2 \in T$. Then, using Lemma 9, we have $g = a^2e = a^2e^2 = a(ae)e = a(ea)e = (ae)(ae) = g^2$. Thus g is an idempotent in G, and hence is the group identity.

It remains to check that the one-to-one correspondence preserves the semigroup operation. If x, y, $xy \in T$, we must have x and y in the same S_i (for if $x \in S_i$, $y \in S_j$ with $i \neq j$, then xy would be in the ideal $S_j \cup I$ and also in the ideal $S_j \cup I$ and hence in I). Therefore x and y can be regarded as elements of some T_i' so that the operation is preserved by virtue of the isomorphism stated in Lemma 8. If $x, y \in T$ and $xy \in I$, we have $xy = xye = xey = xeye = e^2 = e$. If $x \in T$ and $y \in I$, we have $xy \in I$. Hence $xy = xye = xey = ey = y$. Similarly, if $x \in I$ and $y \in T$, we have $xy = x$. Finally, if x, $y \in I$, the operation is preserved, since we used G as the arbitrary group in the construction, and we saw by Lemma 1 that I is either G or G with a zero element adjointed.

LEMMA 10. *In Case II, T contains at most one element.*

Proof. Suppose a, $b \in T$ and $a \neq b$. If a and b are incomparable, then by Lemma 5 T contains an idempotent, contradicting Case II. Thus we can assume, say, $a \mid b$. Then $b = ac$ for some $c \in T$. If a and c were incomparable, we could apply Lemma 5 again, contradicting Case II. Let d be either a or c, whichever divides the other. Then $d \mid a$ and $d \mid c$. Hence $d^2 \mid ac$. Hence $d^2 \in T$, contradicting Case II.

Thus we can assume that either $T = \varnothing$ or $T = \{h\}$. In both cases, the isomorphism with a Type II semigroup is clear. For $h^2 \in I$, and hence $h^2 = h^2e = hehe = g^2$, by Lemma 9. On the other hand, $hx \in I$ for all $x \in I$. Hence $hx = hxe = hexe = gx$. Similarly, $xh = xg$ for $x \in I$.

LEMMA 11. *In Case III and IV, S contains a zero element* 0, $g^2 = 0$, *every element of S (except maximal idempotents) is nilpotent, and $G = \{g\}$.*

Proof. The first two statements are clear, since $g^2 \notin G$. Suppose x is not a maximal idempotent. By Lemma 4, $x^n \in I$ for some n, so that $g \mid x^n$. Hence $0 = g^2 \mid x^{2n}$, so that $x^{2n} = 0$. Finally, suppose $a \in G$. Then we have $ax = g$ and $gy = a$, for some $x, y \in S$. Hence $axy = a$, and, by induction on n, $a(xy)^n = a$ for all n. Thus xy is not nilpotent. Hence xy is a maximal idempotent, and so is x which divides it. Hence $ax = a$, so that $g = a$.

In Case III, there is at most one π-class, so that S itself (with the maximal idempotent, if one is present, removed) satisfies (3) and (4). Thus, by Lemma 8, there is an isomorphism between S and a Type 3 semigroup. (By Lemma 1, $g = 1/2$.)

LEMMA 12. *In Case IV there are exactly two π-classes, one of which consists of a single element.*

Proof. Suppose there were three π-classes, S_1, S_2 and S_3. By Lemma 5, at least two of these (say S_1 and S_2) contain maximal idempotents (e_1 and e_2). Then $e_2 \mid g$, and hence $e_1e_2e_1g = g$. Hence $e_1e_2 = g$. Similarly $e_2e_1 = g$. Hence $g^2 = e_1e_2e_1e_2 = e_1e_1e_2e_2 = e_1e_2 = g$, contradicting Case IV. Thus, there are exactly two π-classes, S_1 and S_2. Now suppose each of these contained more than one element, so that $a, b \in S_1$, $c, d \in S_2$, $a \neq b$, $c \neq d$. By Lemma 5, we can assume that one of a, b, c, d (say a) is a maximal idempotent. Since $c \mid g$ and $d \mid g$, we have $ca \mid ga = g$ and $da \mid ga = g$. Hence $ca = da = g$. Since $c\pi d$, one must divide the other (say $c \mid d$). Write $d = xc$ for some $x \in S$. Then $g = da = xca = xg$, and, by induction on n, $x^n \notin I$ for all n. Hence by Lemma 4 x is a maximal idempotent. But $x \in S_2$ implies $x \neq a$. Thus we have two maximal idempotents. As in the first part of the proof, their product must be g, and hence $g^2 = g$, contradicting Case IV.

For Case IV, we now set up an isomorphism with a Type IV semigroup. Let the one-element π-class be the h of type IV. The rest of S, by Lemma 8, is isomorphic to $G \cap [1/2, 1]$ with 0 adjoined, for some subgroup G of the positive reals. (By Lemma 1, $g = 1/2$.) It remains only to check that h multiplies in the right may. Since $h \mid g$, $lhlg = g$, and hence $lh = g$. Similarly, $hl = g$. Now suppose $x \neq 1$, Since $h \mid g$, $xh \mid xg$. But $xg = 0$, for otherwise $xg = g$ and hence $x^ng = g$ for all n; since x is not a maximal idempotent, this would contradict Lemma 11. Note first that $gx = 0$ (since otherwise $gx = g$, and hence $gx^n = g$ for all n, so that x would not be nilpotent). Moreover $hx \neq h$, since h is a maximal element not an idempotent. Hence $1 \mid hx$, so that $1(hx) = hx$. On the other hand $(1h)x = gx = 0$, so that $hx = 0$. Similarly $xh = 0$.

4. Corollaries. We say that a semigroup S satisfies maximal condition on principal ideals if every family of principal ideals contains a maximal member, or equivalently if there is no infinite sequence of principal ideals each of which properly contains its predecessor. It is easy to see that this is equivalent to the assertion that there is no infinite sequence of elements of S each of which properly divides its predecessor.

COROLLARY 1. *Suppose S is a semigroup with maximal condition on principal ideals which satisfies* (1) *and* (2). *Then S is isomorphic to a semigroup of one of the following types.*

I. *Let G be an arbitrary group, and g its identity element. Let $n_0, n_1, \cdots$ be a family of integers (with repetitions allowed) indexed by a (possibly uncountable or empty) set $I = \{0, \cdots\}$. Let $S = G \cup \{(i, n): i \in I,\ n \text{ an integer},\ 0 \leqq n < n_i, (i, n) \neq (0, 0)\}$, with an optional zero element adjoined. Define multiplication by: $(i, n)g = g$ if $g \in G$, $(i, n)(j, m) = (i, n+m)$ if $i = j$ and $n+m < n_i$, $(i, n)(j, m) = g$ otherwise.*

II. *Same as Type II in* § 1.

III. *Let g be a nonnegative integer. Let S be either $\{0, 1, \cdots, g + 1\}$ or $\{1, \cdots, g + 1\}$. Define $m \circ n = \min(m + n, g + 1)$.*

IV. *Let g be a nonnegative integer. Let $S = \{0, 1, \cdots, g+1\} \cup \{h\}$. Define $0h = h0 = g$, $xh = hx = g+1$ if $x \neq 0$, $mn = \min(m+n, g+1)$ if $m \neq h \neq n$.*

Proof. Each of the four types in § 1 reduces, in the presence of maximal condition, to the corresponding type in this corollary. For instance, in Type 1, each of the groups G_i, if it contains any elements of the interval (1/2, 1), must contain a maximal such element, r. It is easy to see that $G \cap (1/2, 1)$ consists of powers $r, r^2, \cdots, r^n$ of r. For each i, let us take n_i to be the first power of r not in the interval T_i. If we identify r^m for all $m < n_i$ with the pair (i, m) we have essentially the operation described in the corollary. Type II is the same as in § 1. Types III and IV are handled like Type I, but are easier.

Another way to prove Corollary 1 would be to repeat the proof of the main theorem, using in place of Lemma 8 a theorem of Clifford [2, Lemma 2.5, page 637] which states essentially that every semigroup satisfying (3) and (4) and containing a maximal element relative to | must be a finite cyclic semigroup. For this corollary (as well as the following ones) it is easy to check that the converse holds.

COROLLARY 2. *Suppose S is a semigroup satisfying* (2) *in which each congruence is the Rees congruence modulo some ideal. Then S is isomorphic to a semigroup constructed as in Type* 1, *G being the one-element group, and the optional extra being omitted.*

Proof. The assumption that every congruence is a Rees congruence implies that S contains a zero element (since the identity congruence must be a Rees congruence). Moreover, each congruence is then uniquely determined by the congruence class containing 0.

Thus (1) is satisfied, if we let $g = 0$. Therefore, we need only specialize the main theorem by determining those cases in which g is a zero element. For this to happen in Type 1, we must have $G = \{g\}$, and the optional extra element must be missing. Thus, Type 1 reduces to the situation described in the corollary. In Type 2 also, we must have $G = \{g\}$, so that S must have either one or two elements, without the extra 0; these cases are already included as degenerate forms of Type 1. Finally, in Types 3 and 4, g can never be a zero element.

COROLLARY 3. *Suppose S is a semigroup satisfying* (2) *and containing an identity element e. Suppose that each congruence on S is uniquely determined by its kernel relative to e. Then S is either a group or a group with a zero element adjoined.*

Proof. The hypothesis states that (1), with g replaced by e, is satisfied. Therefore we need only specialize the main theorem by determining those cases in which g is an identity element. In Type 1, we note that g fails to act as an identity element for the elements of the T_i; hence each T_i must be empty, so that the desired conclusion follows. In Type 2, g fails to act as an identity element on the optional element a; hence a must be lacking, so that the conclusion follows. In Type 3, g fails to act as an identity element unless S consists of two elements, so that S must be a one-element group with a zero adjoined. In Type 4, g fails to act as an identity on the element h.

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