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THE LEVI PROBLEM FOR A PRODUCT MANIFOLD

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Let S be a Stein manifold, T a one dimensional torus, π a projection of the product $E = S \times T$ onto S and D a subdomain of E. The main object of this paper is to prove that D is a Stein manifold if and only if D is pseudoconvex in the sense of Cartan and $\pi^{-1}(x)$ is not contained in D for any point x of S.

1. A subdomain D of a complex manifold M is called pseudoconvex if, for any boundary point x of D in M, there is a Stein neighborhood U of x in M such that $U \cap D$ is also a Stein manifold. A pair (B, β) is called a domain over M if β is a locally biholomorphic mapping of a complex manifold B in M. A domain (B, β) over C^n is called a domain of holomorphy if there exists a holomorphic function f in B such that the radius of convergence of f at any point x of B is just the boundary distance $d(x)$ of x.

Moreover we recall another definition. Let $\varphi_i: M_i \rightarrow N_i$ be two mappings of a set M_i into a set N_i $(i = 1, 2)$. Then we define the product mapping $\varphi_1 \times \varphi_2$ of the product set $M_1 \times M_2$ into the product set $N_1 \times N_2$ by putting $(\varphi_1 \times \varphi_2)(x, y) = (\varphi_1(x), \varphi_2(y))$ for $(x, y) \in M_1 \times M_2$.

The proof of our theorem falls into two parts. We first prove it in the case of $S = C^n$, where we construct a strongly plurisubharmonic function by means of Hörmander [2] and reduce it to a result of Narasimhan [3]. In general case, using the imbedding of Docquier-Grauert [1], we reduce the theorem to the case of C^n.

2. Let (B, β) be a domain of holomorphy over C^n. In the complex plane C select any two complex numbers ω_1, ω_2 which are linearly independent over the real number field R. The numbers ω_1, ω_2 generate a subgroup Γ of C, namely

$$\Gamma = \{m_1\omega_1 + m_2\omega_2;\ m_1, m_2\ Z = \text{addtive group of integers}\}\,.$$

The quotient $T = C/\Gamma$ is a one dimensional torus. T has a natural complex structure and is a compact Riemann surface. The natural map $\tau: C \rightarrow T$ is a locally biholomorphic map. We denote by $E = B \times T$ the product of two complex manifolds B and T, and by $\pi: E \rightarrow B$ the projection.

We first prove the following lemma:

LEMMA. *Let D be a pseudoconvex open subset of E such that $\pi^{-1}(x)$ is not contained in D for any point x of B. Then D is a Stein manifold.*

Proof. Let $1 \times \tau$ be the product map of the identity 1 of B and the map τ. The map $1 \times \tau$ is a locally biholomorphic map $B \times C$ onto E. If we denote by A the inverse image $(1 \times \tau)^{-1}(D)$ of D, A is pseudoconvex, because D is pseudoconvex. A is Γ-invariant, that is, for any fixed point $\gamma \in \Gamma$, A is invariant under the transformation of $B \times C$: $(y, z) \mapsto (y, z + \gamma)$. Let α be the restriction to A of the product map $\beta \times 1$ of the map β and the identity map 1 of C, that is, $\alpha(y, z) = (\beta(y), z)$ for $(y, z) \in A$. α is a locally biholomorphic map of A into $C^n \times C = C^{n+1}$ and (A, α) is a pseudoconvex domain over C^{n+1}. The distance function $d(y, z)$ of the domain A over C^{n+1} induces the function $d(y, t)$ in D. Indeed, for any point $(y, t) \in D$, $y \in B$, $t \in T$, select two representatives $z, z' \in C$ of the equivalence class t. Then there is $\gamma \in \Gamma$ such that $z' = z + \gamma$. But A is Γ-invariant, and so $d(y, z') = d(y, z)$. Since A is pseudoconvex, by Oka [4], the function $-\log d(y, z)$ is a continuous plurisubharmonic function in A. The function $-\log d(y, t)$ is therefore a continuous plurisubharmonic function in D, and so is the function

$$1/d(y, t) = e^{-\log d(y,t)} .$$

On the other hand, since B is Stein, there is a real analytic strongly plurisubharmonic function $q > 0$ with the following property: for any real number $c > 0$,

$$B_c = \{y \in B; q(y) < c\} \Subset B .$$

The function

$$\gamma(y, t) = \frac{1}{d(y, t)} + q(y)$$

defined in D is a continuous plurisubharmonic function. It holds that

$$\begin{aligned} D_c &= \{(y, t) \in D; \gamma(y, t) < c\} \\ &\subset B_c \times T \cap \left\{(y, t) \in D; d(y, t) > \frac{1}{c}\right\} \Subset D \end{aligned}$$

for any real number $c > 0$.

Since $D = \bigcup_{c>0} D_c$, if we show that D_c is a Stein manifold, we know by Docquier-Grauert [1], that D is itself a Stein manifold.

Fix an arbitrary real number $c > 0$. For any point $y \in B$, we set

$$A(y) = \{z \in C; (y, \tau(z)) \in D\} .$$

By the hypothesis of the lemma, it follows that $A(y) \subsetneqq C$. Select a complex-valued measurable function $a(y)$ in B such that

$$a(y) \in C - A(y) \text{ for any point } y \in B .$$

For sufficiently small number ε with $0 < \varepsilon < 1/(c+1) < 1/c$, we define the function $s(y, t)$ in D_{c+1} as follows:

$$s(y, t) = \frac{1}{\varepsilon^{2n}}\int_{\xi \in B} \rho\left(\frac{y-\xi}{\varepsilon}\right) \sum_{m_1, m_2=-\infty}^{+\infty} \frac{d\lambda(\xi)}{|z - a(\xi) - m_1\omega_1 - m_2\omega_2|^2},$$

where ρ is Friedrichs' modifier, and z in the summation $\sum$ is a representative of t. Clearly the sum $\sum$ converges uniformly, and does not depend on any choice of representative z.

Moreover, we define a function $p(y, t)$ in D_{c+1} by putting

$$p(y, t) = s(y, t) + Kq(y)$$

where K is a sufficient large constant. Since $D_c \Subset D_{c+1}$ and q is a strongly plurisubharmonic function in B, it follows that the function $p(y, t)$ is strongly plurisubharmonic in D_c. By Narasimhan [3], we can conclude that D_c is a Stein manifold.

3. Now we shall prove our main theorem.

THEOREM. *Let E be the product $S \times T$ of a Stein manifold S and a complex torus T, and π be the projection $E \rightarrow S$. Let D be an open subset of E. Then D is a Stein manifold if and only if D is pseudoconvex and $\pi^{-1}(x)$ is not contained in D for any point $x \in S$.*

Proof. By Docquier-Grauert [1], there are a biholomorphic map σ of S onto a regular analytic set of a domain of holomorphy (B, β) over C^n and a holomorphic mapping ρ of B onto $\sigma(S)$ such that the restriction $\rho | \sigma(S)$ is the identiny of $\sigma(S)$. We define a mapping ξ of the product $G = B \times T$ onto $E = S \times T$ by putting $\xi(x, t) = (\sigma^{-1}(\rho(x)), t)$ for $(x, t) \in G$. The inverse image $\xi^{-1}(D)$ of D under the map is a pseudoconvex open subset of G and satisfies the hypothesis of the lemma. $\xi^{-1}(D)$ is therefore a Stein manifold. Since D is a regular analytic subset of the Stein manifold $\xi^{-1}(D)$, D is also a Stein manifold.

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