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THE CLASS OF RECURSIVELY ENUMERABLE SUBSETS OF A RECURSIVELY ENUMERABLE SET

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For any set α, let θA^{α} denote the index set of the class of all recursively enumerable (r.e.) subsets of α (i.e., if $\{W_x\}_{x \geqq 0}$ is a standard enumeration of all r.e. sets, $\theta A^{\alpha} = \{x \mid W_x \subset \alpha\}$.) The purpose of this paper is to examine the possible Turing degrees of the sets θA^{α} when α is r.e. It is proved that if $\boldsymbol{b}$ is any nonrecursive r.e. degree, the Turing degrees of sets θA^{α} for α r.e., $\alpha \in \boldsymbol{b}$, are exactly the degrees $\boldsymbol{c} > \boldsymbol{0}'$ such that $\boldsymbol{c}$ is r.e. in $\boldsymbol{b}$.

Index sets of form θA^{α} appear to have useful properties in the study of the partial ordering of all index sets under one-to-one reducibility. For instance, in the case where α is a nonrecursive incomplete r.e. set, the index set $\overline{\theta A^{\alpha}}$ was used in [1] to provide an example of an index set which is neither r.e. nor productive. In [2] it is shown that if the Turing degree of α is not $\geqq \boldsymbol{0}'$, then the set θA^{α} is at the bottom of c discrete ω-sequences of index sets (i.e., linearly ordered chains of index sets such that no index sets are intermediate between the elements of the chain.) In particular, such a set θA^{α} has at least two nonisomorphic immediate successors in the partial ordering of index sets.

It is natural to ask: What relation, if any, exists between the Turing degree of α and that of θA^{α}? In the case where α is co-r.e., it is easy to see that neither degree determines the other, since $\overline{\theta A^{\alpha}}$ is r.e. and hence has degree $\boldsymbol{0}$ or $\boldsymbol{0}'$ (by Rice's theorem [5]), independently of the degree of α; while both $\boldsymbol{0}$ and $\boldsymbol{0}'$ contain sets θA^{α} for $\alpha \in \boldsymbol{0}$. In this paper it is shown that when α is r.e., the situation is similar, though more complicated. It was shown in [3, Theorem 1] that if β is a complete r.e. set, then θA^{β} is a complete Π_2^0 set. On the other hand, C. G. Jockush, Jr. has constructed an example (unpublished) of an effectively simple set γ such that θA^{γ} has degree $\boldsymbol{0}'$. Since β and γ both have degree $\boldsymbol{0}'$ [4], this shows that when α is r.e., the degree of α need not determine that of θA^{α}. The main result of this paper shows that these examples are extremal cases of the fact that when α is r.e., the degree of θA^{α} can take on all possible values within certain obvious restrictions. More precisely, we prove the following:

THEOREM. *Let $\boldsymbol{b}$ be a nonrecursive* r.e. *degree. Let*

$\mathscr{B}_1 = \{\boldsymbol{c} \mid (\exists\alpha)\ (\alpha \in \boldsymbol{b}$ *and* $\theta A^\alpha \in \boldsymbol{c}\}$,

$\mathscr{B}_2 = \{\boldsymbol{c} \mid (\exists\alpha)\ (\alpha$ *is* r.e. *and* $\alpha \in \boldsymbol{b}$ *and* $\theta A^\alpha \in \boldsymbol{c}\}$,

$\mathscr{B}_3 = \{\boldsymbol{c} \mid \boldsymbol{c} \geqq \boldsymbol{0}'$ *and* $\boldsymbol{c}$ *is* r.e. *in* $\boldsymbol{b}\}$.

Then $\mathscr{B}_1 = \mathscr{B}_2 = \mathscr{B}_3$.

Proof. Clearly $\mathscr{B}_2 \subset \mathscr{B}_1$. It is thus sufficient to prove that $\mathscr{B}_1 \subset \mathscr{B}_3 \subset \mathscr{B}_2$.

$\mathscr{B}_1 \subset \mathscr{B}_3$: Assume $\alpha \in \boldsymbol{b}$ and $\theta A^\alpha \in \boldsymbol{c}$.
Since $\overline{\theta A^\alpha} = \{x \mid W_x \cap \bar{\alpha} \neq \emptyset\}$, $\overline{\theta A^\alpha}$ is r.e. in α so $\boldsymbol{c}$ is r.e. in $\boldsymbol{b}$. Since $\boldsymbol{b} > \boldsymbol{0}$, $\alpha \neq \emptyset$ or N, so θA^α is a nontrivial index set which, by the proof of Rice's theorem [5, Theorem 14-XIV] implies $K \leqq_T \theta A^\alpha$ (where K denotes the complete r.e. set). So $\boldsymbol{c} \geqq \boldsymbol{0}'$. Since $\boldsymbol{c}$ was arbitrary, this shows $\mathscr{B}_1 \subset \mathscr{B}_3$.

The remainder of this paper is devoted to proving that $\mathscr{B}_3 \subset \mathscr{B}_2$. We assume that $\boldsymbol{c} \geqq \boldsymbol{0}'$, $\boldsymbol{c}$ r.e. in $\boldsymbol{b}$, and describe the construction of an r.e. set α such that $\alpha \in \boldsymbol{b}$ and $\theta A^\alpha \in \boldsymbol{c}$.

2. Preliminaries. The notation is that of [5]. Given β r.e., nonrecursive, and γ r.e. in β, $\boldsymbol{0}' \leqq_T \gamma$, we require an r.e. set α such that $\alpha \equiv_T \beta$ and $\theta A^\alpha \equiv_T \gamma$. We attempt to achieve this as follows:

(a) to get $\beta \leqq_T \alpha$, we "code" β into α;

(b) to get $\alpha \leqq_T \beta$, we arrange that an odd integer y is put into α only when some $x \leqq y$ has just appeared in β. (The idea here is similar to that used in the proof of Theorem 2 of [7].);

(c) to get $\gamma \leqq_T \theta A^\alpha$, we define a sequence $\{S_e\}_{e \geqq 0}$ of r.e. sets such that the index of S_e is recursive in θA^α and $e \in \gamma \leftrightarrow S_e \cap \bar{\alpha} \neq \emptyset$;

(d) to get $\theta A^\alpha \leqq_T \gamma$, we try to "preserve" nonempty intersections $W_e \cap \bar{\alpha}$ whenever they occur during the construction.

These requirements evidently conflict, and priorities must be assigned, in the manner of [6].

The fact that γ is r.e. in β will be used in the following way: Let $\{D_i\}_{i \geqq 0}$ be the canonical indexing of finite sets; $\langle x, y\rangle$ is a standard recursive pairing function with recursive inverses π_1, π_2, and $\langle x, y, u, v\rangle = \langle\langle\langle x, y\rangle, u\rangle, v\rangle$.

LEMMA 1. *If* γ *is* r.e. *in a set* β, *then there is a recursive function* f *such that for each* x, $x \in \gamma \leftrightarrow (\exists z)\ (z \in W_{f(x)}$ *and* $D_{\pi_1(z)} \subset \beta$ *and* $D_{\pi_2(z)} \subset \bar{\beta}$).

Proof. Let $\gamma = W_e^\beta$. Then in the notation of Chapter 9 of [5], $x \in \gamma \leftrightarrow x \in W_e^\beta \leftrightarrow \varphi_e^\beta(x)$ is defined $\leftrightarrow (\exists y)(\exists u)(\exists v)(\langle x, y, u, v\rangle \in W_{\rho(e)}$ and $D_u \subset \beta$ and $D_V \subset \bar{\beta})$ where $\rho(e)$ is a recursive function of e. Let

$$V = \{\langle u, v\rangle \mid (\exists y)(< x, y, u, v\rangle \in W_{\rho(e)})\}.$$

Then V is an r.e. set, whose index can be uniformly computed from x; so there is a recursive function f such that $V = W_{f(x)}$, and

$$\begin{aligned} x \in \gamma &\longleftrightarrow (\exists u)\,(\exists v)\,(\langle u, v\rangle \in V \text{ and } D_u \subset \beta \text{ and } D_v \subset \bar{\beta}) \\ &\longleftrightarrow (\exists z)\,(z \in W_{f(x)} \text{ and } D_{\pi_1(z)} \subset \beta \text{ and } D_{\pi_2(z)} \subset \bar{\beta})\,. \end{aligned}$$

DEFINITION 2. Let g be a recursive function such that $\{D_{g(i)}\}_{i \geqq 0}$ is a recursive partitioning of the positive even integers into disjoint finite sets such that $|D_{g(i)}| = i + 1$ for each i (e.g., let $D_{g(i)} = \{i^2 + i + 2k \mid 0 < k \leqq i + 1\}$). Let $e_x = e(x)$ be a recursive function such that $e_x =$ the unique i for which $2x \in D_{g(i)}$.

3. Construction. α will be constructed in stages, $\alpha = \bigcup_s \alpha_s$ where α_s is the finite set of integers which has been put into α by the end of stage s. If W is r.e., W^s will denote the result of performing s steps in some fixed enumeration of W; in particular $W^0 = \varnothing$.

We define α_s and auxiliary recursive functions $y_e^s = y_e(s)$ and $z_e^s = z_e(s)$ and a partial recursive function $h(y)$ by simultaneous recursion. If $y_e^s > 0$, y_e^s serves to witness that $e \in \gamma$, while z_e^s witnesses that $W_e \cap \bar{\alpha} \neq \varnothing$.

Stage 0.

$\alpha_0 = \{0\}$, $y_e^0 = z_e^0 = 0$.

Let $C_s = \{z \mid z > 0 \text{ and } (\exists e)\,(\exists t)_{t<s}\,(z = y_e^s \vee z = z_e^t)\}$; so $C_1 = \varnothing$. Assume inductively that C_s is finite and that $y_e^{s-1} > 0$ implies y_e^{s-1} is odd and $h(y_e^{s-1})$ is defined, for all e.

Stage $s > 0$, $s \equiv 1 \pmod 3$.
Let

$E_s = \{y \mid (\exists x)\,(y = 2x \text{ and } e_x \leqq s \text{ and } e_x \in \beta^s \text{ and } (\forall i)_{i<e_x}(y \neq z_i^{s-1})\}$

$O_s = \{y \mid \exists e)_{e \leqq s}(y = y_e^{s-1} \text{ and } D_{\pi_2(h(y))} \cap \beta^s \neq \varnothing\}\,.$

Let $\alpha_s = \alpha_{s-1} \cup E_s \cup O_s$. If $z_e^{s-1} \in E_s$, let $z_e^s = 0$. Otherwise, let $z_e^s = z_e^{s-1}$. If $y_e^{s-1} \in O_s$, let $y_e^s = 0$. Otherwise, let $y_e^s = y_e^{s-1}$.

Stage $s > 0$, $s \equiv 2 \pmod 3$.
Let $\alpha_s = \alpha_{s-1}$. For each $e \leqq s$ (if any) such that
(a) $z_e^{s-1} = 0$ and
(b) $(\exists z)\,(z \in W_e^s \cap \bar{\alpha}_s$ and $(\forall i)_{i \leqq e}\,(z \neq y_i^{s-1})$ and
$(\forall x)_{x<z}\,(z = 2x \rightarrow e_x > e))$,
let $z_e^s =$ the least such z. For all other e, let $z_e^s = z_e^{s-1}$. If $y_j^{s-1} = z_e^s$

for some $e < j$, let $y_j^s = 0$. Otherwise let $y_j^s = y_j^{s-1}$.

Stage $s > 0$, $s \equiv 0 \pmod 3$.

Let $\alpha_s = \alpha_{s-1}$, $z_e^s = z_e^{s-1}$ for all e. Let $F_s = \{e \mid e \leqq s$ and $y_e^{s-1} = 0$ and $(\exists x)(x \in W_{f(e)}^s$ and $D_{\pi_1(x)} \subset \beta^s$ and $D_{\pi_2(x)} \subset \overline{\beta^s})\}$. If $e \notin F_s$, let $y_e^s = y_e^{s-1}$. If $F_s \neq \varnothing$, let $F_s = \{k_0, k_1, \cdots, k_n\}$, $n \geqq 0$, $k_i < k_j$ for $i < j$. Define $y_{k_i}^s$ inductively as follows: assume $y_{k_j}^s$ has been defined for all $j < i$. Let x_i = least x such that $x \in W_{f(k_i)}^s$ and $D_{\pi_1(x)} \subset \beta^s$ and $D_{\pi_2(x)} \subset \overline{\beta^s}$, $y_{k_i}^s$ = least odd $y \in \overline{\alpha_s}$ such that

$$y > \max (D_{\pi_1(x_i)} \cup D_{\pi_2(x_i)} \cup C_s \cup \{y_{k_j}^s \mid j < i\}) .$$

Define $h(y_{k_i}^s) = x_i$ for each $i \leqq n$.

It is easily verified for all three types of stages that C_s is always finite, since new nonzero values are assigned to z_e^s and y_e^s for at most $s + 1$ values of e. For the second inductive assumption, it suffices to note that new nonzero values of y_e^s are defined only if $s \equiv 0 \pmod 3$, and that each new value $y_e^s > 0$ is odd and $h(y_e^s)$ is defined. That $h(y)$ is well defined will be proved below.

It is clear that $\alpha = \bigcup_s E_s \cup \bigcup_s O_s \cup \{0\}$ is r.e. We note for later use that O_s consists of odd numbers and E_s of even numbers, so that

(i) $\quad 0 < y = 2x \in \alpha \longleftrightarrow (\exists s)(s \equiv 1 \pmod 3$ and $y \in E_s)$,

(ii) $\quad y$ odd, $y \in \alpha \longleftrightarrow (\exists s)(s \equiv 1 \pmod 3$ and $y \in O_s)$.

4. Proof of Theorem.

LEMMA 3. *For all e and s,*

(a) *If $y_e^{s-1} \neq y_e^s$ then either* (i) $s \not\equiv 0 \pmod 3$, $y_e^{s-1} > 0$ *and* $y_e^s = 0$, *or* (ii) $s \equiv 0 \pmod 3$, $y_e^{s-1} = 0$ *and* $y_e^s > 0$.

(b) *If $y = y_e^{s-1} > 0$ and $y_e^s \neq y$, then either $(\exists i)_{i<e}(y = z_i^s)$* or $s \equiv 1 \pmod 3$ *and* $y \in O_s$.

(c) *If $y_e^s > 0$, then either* (i) $(\forall t)(t > s \rightarrow y_e^t = y_e^s)$ *or* (ii) *if t' = least $t > s$ such that $y_e^t \neq y_e^s$ then $y_e^{t'} = 0$ and $(\forall s')(s' > t' \rightarrow y_e^{s'} = 0$ or $y_e^{s'} > y_e^s)$.*

(d) *If $s < t$ and $0 < y_e^s = y_e^t$, then $(\forall t')(s < t' < t \rightarrow y_e^{t'} = y_e^s)$.*

(e) *If $\lim_s y_e^s$ exists and $\lim_s y_e^s = y > 0$, then $(\forall s)(\forall t)(y = y_e^s$ and $s < t \rightarrow y = y_e^t)$.*

Proof. (a) is clear from the construction.

(b) Assume $y = y_e^{s-1} > 0$, $y_e^s \neq y$. Then by (*a*), $s \not\equiv 0 \pmod 3$. If $s \equiv 1 \pmod 3$, then $y_e^s \neq y_e^{s-1}$ only if $y_e^{s-1} \in O_s$. If $s \equiv 2 \pmod 3$, then $y_e^s \neq y_e^{s-1}$ only if $y = y_e^{s-1} = z_i^s$ for some $i < e$.

(c) Assume $y = y_e^s > 0$. If (i) fails to hold, let t' be the least

$t > s$ such that $y_e^t \neq y_e^s$; thus $t' > s$ and $0 < y = y_e^s = y_e^{t'-1} \neq y_e^{t'}$. Then by (a), $y_e^{t'} = 0$. Suppose (ii) fails to hold; then for some $s' > t'$, $0 < y_e^{s'} \leqq y$. Let s' be least. Then $s' - 1 \geqq t'$ and $y_e^{s'-1} = 0$ or $y_e^{s'-1} > y > 0$. So $y_e^{s'} \neq y_e^{s'-1}$, and it follows by (a) that $s \equiv 0 \pmod 3$ and $y_e^{s'-1} = 0$. But $s' > t' > s$ implies that $y = y_e^s \in C_{s'}$, while $y_e^{s'} \neq y_e^{s'-1}$ implies $y_e^{s'} > \max C_{s'}$. So $y_e^{s'} > y$, which is a contradiction. So (ii) must hold.

(d) Let $s < t$ and $0 < y_e^s = y_e^t$. Suppose that $y_e^{t'} \neq y_e^s$ for some t', $s < t' < t$ and let t' be least. Then by (c), $y_e^{t'} = 0$, and $t > t' > s$ implies $y_e^t = 0$ or $y_e^t > y_e^s$, both contrary to hypothesis. So $y_e^s = y_e^t$ implies $y_e^{t'} = y_e^s$ for all t', $s < t' < t$.

(e) Assume $\lim_s y_e^s = y > 0$ and $y = y_e^s$. Suppose that for some $t > s$, $y_e^t \neq y$, and let t be least. Then by (c), $y_e^{s'} \neq y$ for all $s' > t$ which contradicts the assumption that $y = \lim_s y_e^s$.

LEMMA 4. *For all e and s,*
(a) *$y_e^s > 0$ implies $y_e^s \in \overline{\alpha_s}$.*
(b) *If $y_e^s > 0$ and $e' \neq e$, then $y_e^s \neq y_{e'}^t$ for all t.*
(c) *If $y = y_e^s > 0$, then $h(y)$ is well-defined.*

Proof. (a) Assume $y = y_e^s > 0$, and let s' be the least t such that $y = y_e^t$. Then $0 < s' \leqq s$, and $y_e^{s'-1} \neq y_e^{s'} = y > 0$. So by Lemma 3(*a*), $y_e^{s'-1} = 0$ and $s' \equiv 0 \pmod 3$. By the construction, $y_e^{s'} > y_e^{s'-1}$ implies $y_e^{s'} \in \overline{\alpha_{s'}}$. Now assume $y \in \alpha_s$, and consider the least t such that $y \in \alpha_t$. Clearly $s' < t \leqq s$, $y \in \overline{\alpha_{t-1}}$ and $t \equiv 1 \pmod 3$. Now $y = y_e^s > 0$ is odd, so $y \notin E_t \cup \alpha_{t-1}$. So $y \in \alpha_t$ implies $y \in O_t$. By Lemma 3(d), $y_e^s = y_e^{s'}$ and $s' \leqq t - 1 < t \leqq s$ implies $y_e^{t-1} = y_e^t = y$. But by the construction, $y = y_e^{t-1} \in O_t$ implies $y_e^t = 0 \neq y$, which is a contradiction. So $y \in \overline{\alpha_s}$.

(b) Assume $y_e^s > 0$ and $e' \neq e$. Clearly if $y_{e'}^t = 0$ then $y_{e'}^t \neq y_e^s$; so assume $y_{e'}^t > 0$. Consider the least s' such that $y_e^s = y_e^{s'}$ and the least t' such that $y_{e'}^t = y_{e'}^{t'}$. Then $s' \equiv t' \equiv 0 \pmod 3$, $e \in F_{s'}$ and $e' \in F_{t'}$. If $s' < t'$ then $y_e^{s'} \in C_{t'}$, so by the construction,

$$y_{e'}^t = y_{e'}^{t'} > y_e^{s'} = y_e^s \,.$$

If $s' > t'$, then $y_{e'}^{s'} \in C_{s'}$, so $y_e^s = y_e^{s'} > y_{e'}^{t'} = y_{e'}^t$. If $s' = t'$, then $e, e' \in F_{s'}$, $e = k_i$, $e' = k_j$ for $i \neq j$. If $e < e'$ then $i < j$ and $y_e^s = y_e^{s'} \in \{y_{k_i}^{s'} \mid i < j\}$ while $y_{e'}^t = y_{e'}^{s'} > \max\{y_{k_i}^{s'} \mid i < j\}$, so $y_{e'}^t > y_e^s$. By symmetry, if $e' < e$ then $y_e^s > y_{e'}^t$. Thus in any case, $e \neq e'$ implies $y_e^s \neq y_{e'}^t$.

(c) First note that by the construction, $h(y)$ is defined if and only if there exist e, s such that $y_e^{s-1} = 0$ and $y = y_e^s > 0$. In particular, if $y = y_e^s > 0$, $h(y)$ is defined since if s' is the least t such that $y = y_e^t$, then $y_e^{s'-1} = 0$. To show $h(y)$ is well-defined, it suffices to show that there exists at most one pair e, s such that $y = y_e^s$ and $y_e^{s-1} = 0$. Suppose $y = y_e^s = y_i^t$ where $y_e^{s-1} = y_i^{t-1} = 0$. Since $y > 0$, $y_e^s = y_i^t$ implies $i = e$, by part (b) of this lemma. If $s \neq t$, say $s < t$, then $s \leqq t - 1 < t$. But then by Lemma 3(d), $y_e^{t-1} = y_i^{t-1} = 0 \neq y_e^s$ implies $y_e^s \neq y_e^t = y_i^t$, contrary to hypothesis. This completes the proof.

DEFINITION 5. Let $\tau(x) =$ the least t such that

$$(\forall z)_{z \leq x}(z \in \beta \longrightarrow z \in \beta^t) .$$

Then $\tau(x)$ is defined for all x, and $\tau(x)$ is evidently recursive in β.

LEMMA 6. *If* $t \equiv 1 \pmod 3$, $y = y_e^{t-1} > 0$ *and* $D_{\pi_2 h(y)} \cap \beta^t \neq \varnothing$, *then* $t < \tau(y) + 3$.

Proof. Assume the hypothesis. Let s' be the least s such that $y = y_e^s$ and let $x = h(y)$; x is well-defined by Lemma 4(c). Then $e \leqq s' < t$, $s' \equiv 0 \pmod 3$ and, by the construction, $D_{\pi_2(x)} \subset \overline{\beta^{s'}}$ and $y = y_e^{s'} > \max D_{\pi_2(x)}$. By hypothesis, $D_{\pi_2(x)} \cap \beta^t \neq \varnothing$; let z be any element of $D_{\pi_2(x)} \cap \beta^t$. Then $z \in D_{\pi_2(x)}$ implies $z < y$, so that $z \in \beta$, $z \notin \beta^{s'}$ implies $s' < \tau(y)$. Now suppose $t \geqq \tau(y) + 3$, and let $s = t - 3$. Then $s \geqq \tau(y) > s' \geqq e$ and $s \equiv t \equiv 1 \pmod 3$. Also $s \geqq \tau(y)$ implies $z \in \beta^s$, so $D_{\pi_2(x)} \cap \beta^s \neq \varnothing$. By Lemma 3(d), $0 < y = y_e^{s'} = y_e^{t-1}$ and $s' \leqq s - 1 < s < t - 1$ implies $y_e^{s-1} = y_e^s = y$. But by the construction, $e \leqq s$ and $y = y_e^{s-1}$ and $D_{\pi_2 h(y)} \cap \beta^s \neq \varnothing$ implies $y_e^{s-1} \in O_s$, so that $y_e^s = 0 \neq y$. Since this is a contradiction, we conclude $t < \tau(y) + 3$.

LEMMA 7. *Assume y is odd. Then $y \in \alpha$ if and only if*

$$(\exists t)_{t < \tau(y)+3}(\exists e)_{e \leqq t}\ (t \equiv 1 \pmod 3)\ \textit{and}\ y = y_e^{t-1}\ \textit{and}\ D_{\pi_2 h(y)} \cap \beta^t \neq \varnothing).$$

Proof. By the construction, if y is odd then

$$\begin{aligned} y \in \alpha &\longleftrightarrow (\exists t)\ (t \equiv 1 \pmod 3)\ \text{and}\ y \in O_t) \\ &\longleftrightarrow (\exists t)\ (\exists e)_{e \leqq t}\ (t \equiv 1 \pmod 3)\ \text{and}\ y = y_e^{t-1}\ \text{and} \\ &\qquad D_{\pi_2 h(y)} \cap \beta^t \neq \varnothing) . \end{aligned}$$

By Lemma 6, such a t can be bounded by $\tau(y) + 3$, which proves the lemma.

LEMMA 8. *For all i and s,*

(a) *If* $z_i^{s-1} > 0$ *and* $z_i^s \neq z_i^{s-1}$, *then* $s \equiv 1 \pmod 3$, $z_i^s = 0$ *and* $z_i^{s-1} \in E_s \subset \alpha_s$.

(b) *If* $z = 2x > 0$, $i < e_x$ *and* $z = z_i^s$, *then* $z = z_i^t$ *for all* $t \geqq s$.

Proof. (a) Assume $z_i^{s-1} > 0$ and $z_i^s \neq z_i^{s-1}$. If $s \equiv 0 \pmod 3$, then $z_i^s = z_i^{s-1}$ for all i. If $s \equiv 2 \pmod 3$ then $z_i^s \neq z_i^{s-1}$ only if $z_i^{s-1} = 0$. It follows that $s \equiv 1 \pmod 3$. But then $z_i^s \neq z_i^{s-1}$ only if $z_i^{s-1} \in E_s \subset \alpha_s$, and in that case $z_i^s = 0$.

(b) Assume $z = 2x > 0$, $i < e_x$ and $z = z_i^s$. Suppose $(\exists t)(t > s$ and $z_i^t \neq z_i^s)$, and let t be least. Then $0 < z = z_i^{t-1} \neq z_i^t$; so by (a), $t \equiv 1 \pmod 3$ and $z \in E_t$. But this implies that $(\exists x')(z = 2x'$ and $(\forall i)_{i<e_{x'}}(z \neq z_i^{t-1}))$. Clearly $x' = x$, so $(\forall i)_{i<e_x}(z \neq z_i^{t-1})$ which is a contradiction. So $z = z_i^t$ for all $t \geqq s$.

DEFINITION 9. Define functions $\sigma(y)$, $\sigma'(y)$ as follows:

(a) If y is odd, $\sigma'(y) = 0$.

(b) If $y = 2x$ and $e_x \notin \beta$, $\sigma'(y) = 0$.

(c) If $y = 2x$ and $e_x \in \beta$, $\sigma'(y) =$ least s such that $e_x \in \beta^s$.

(d) If $\sigma'(y) = 0$ then $\sigma(y) = 0$.

(e) If $\sigma'(y) > 0$, then $\sigma(y) =$ least $s \geqq \max\{e_x, \sigma'(y)\}$ such that $s \equiv 1 \pmod 3$.

It is clear that σ, σ' are defined for all y and that $\sigma(y) > 0$ if and only if $y = 2x$ and $e_x \in \beta$. Since e_x is a recursive function of x, σ' and σ are recursive in β.

LEMMA 10. *Assume* $y = 2x > 0$. *Then* $y \in \alpha$ *if and only if* $e_x \in \beta$ *and* $\sigma(y) > 0$ *and* $(\forall i)_{i<e_x}(y \neq z_i^{\sigma(y)-1})$.

Proof. ($\leftarrow$). Assume $e_x \in \beta$ and $\sigma(y) > 0$ and $(\forall i)_{i<e_x}(y \neq z_i^{\sigma(y)-1})$. Then by Definition 9, $e_x \leqq \sigma(y)$ and $e_x \in \beta^{\sigma'(y)} \subset \beta^{\sigma(y)}$. Then by the construction, since $\sigma(y) \equiv 1 \pmod 3$, $y \in E_{\sigma(y)} \subset \alpha_{\sigma(y)}$. So $y \in \alpha$.

($\rightarrow$). Assume $y \in \alpha$. Then since $y > 0$ is even, $y \in E_s$ for some s, $s \equiv 1 \pmod 3$; so $e_x \in \beta^s$, $e_x \leqq s$ and $(\forall i)_{i<e_x}(y \neq z_i^{s-1})$. So in particular $e_x \in \beta$, and, by definition of $\sigma(y)$, $0 < \sigma(y) \leqq s$. Suppose that for some $i < e_x$, $y = z_i^{\sigma(y)-1}$. Then by Lemma 8(b), $y = z_i^{s-1}$, since

$$s - 1 \geqq \sigma(y) - 1 .$$

But this is a contradiction, which proves the lemma.

LEMMA 11. $\alpha \leqq_T \beta$.

Proof. We show how to decide membership in α, recursively in β. If $y = 0$, then $y \in \alpha$, by Stage 0 of the construction. Suppose $y > 0$.

Case 1. y is odd.

Then by Lemma 7, $y \in \alpha$ if and only if $(\exists t)_{t<\tau(y)+3}(\exists e)_{e \leqq t}$ ($t \equiv 1$ (mod 3) and $y = y_e^{t-1}$ and $D_{\pi_2 h(y)} \cap \beta^t \neq \varnothing$). Now for fixed t, y it can be decided recursively whether $t \equiv 1 \pmod 3$ and whether $y = y_e^{t-1}$ for some $e \leqq t$, since y_e^s is a recursive function of s. If $y = y_e^{t-1}$, then $h(y)$ is well defined by Lemma 4(c), and it can be decided recursively whether $D_{\pi_2 h(y)} \cap \beta^t \neq \varnothing$. So $y \in \alpha \leftrightarrow (\exists t)_{t<\tau(y)+3} R(t, y)$ where $R(t, y)$ is a recursive predicate. Since, as noted in Definition 5, $\tau(y)$ is recursive in β, the question of whether $y \in \alpha$ can be decided, recursively in β.

Case 2. y is even.

Then by Lemma 10, $y = 2x$ implies that $y \in \alpha$ if and only if $e_x \in \beta$ and $\sigma(y) > 0$ and $(\forall i)_{i<e_x}(y \neq z_i^{\sigma(y)-1})$. For $y = 2x$, whether $e_x \in \beta$ can be decided recursively in β, since e_x is a recursive function of x. If $e_x \notin \beta$, then $y \notin \alpha$. If $e_x \in \beta$, then $\sigma(y) > 0$ can be computed recursively in β, as noted in Definition 9. Since z_i^s is a recursive function of s, the membership of the finite set $D = \{z_i^{\sigma(y)-1} \mid i < e_x\}$ can be completely determined, once $\sigma(y)$ is known. Then

$$y \in \alpha \longleftrightarrow y \notin D .$$

LEMMA 12. *For all* e, $e \in \beta \leftrightarrow D_{g(e)} \cap \alpha \neq \varnothing$.

Proof. $(\leftarrow)$. Assume $D_{g(e)} \cap \alpha \neq \varnothing$. Let $y \in D_{q(e)} \cap \alpha$. Then $y \in D_{g(e)}$ implies y is even, $y > 0$, and, as remarked above, $y \in \alpha$ implies $y \in E_s$ for some s. So $y = 2x$ for some x such that $e_x \in \beta$, i.e., $y = 2x \in D_{g(i)}$ where $i \in \beta$. But by definition of g, $D_{g(e)} \cap D_{g(i)} \neq \varnothing$ implies $i = e$; so $e \in \beta$.

$(\rightarrow)$. Assume $e \in \beta$, and let $t \equiv 1 \pmod 3$ be so large that $t \geqq e$ and $e \in \beta^t$. Now by the definition of g, $D_{g(e)}$ has $e + 1$ elements, while $\{z_i^{t-1} \mid i < e\}$ has at most e elements. So $D_{g(e)} - \{z_i^{t-1} \mid i < e\} \neq \varnothing$. Let $y = 2x \in D_{g(e)} - \{z_i^{t-1} \mid i < e_x\}$. Then $y > 0$, and by definition of e_x, $e_x = e$. So $e_x \leqq t$, $e_x \in \beta^t$ and $(\forall i)_{i<e_x}(y \neq z_i^{t-1})$. This implies $y \in E_t \subset \alpha_t$, and $y \in D_{g(e)} \cap \alpha$.

LEMMA 13. $\beta \leqq_T \alpha$.

Proof. By Lemma 12, β is in fact truth-table reducible to α.

LEMMA 14. *For all* i, e *and* s, $z_e^s > 0$ *implies* $z_e^s \neq y_i^s$.

Proof. By induction on s. Since $z_e^0 = 0$, the lemma holds vacuously for $s = 0$. Assume that for all i and e, $z_e^{s-1} > 0$ implies $z_e^{s-1} \neq y_i^{s-1}$, and assume $y_e^s > 0$.

Case 1. $s \equiv 0 \pmod 3$.

Then $z_e^s = z_e^{s-1}$, which implies $z_e^{s-1} > 0$. Then by the induction hypothesis, $z_e^{s-1} \neq y_i^{s-1}$ for all i. If $y_i^s = y_i^{s-1}$, then

$$z_e^s = z_e^{s-1} \neq y_i^{s-1} = y_i^s .$$

If $y_i^s \neq y_i^{s-1}$, then by the construction, $z_e^s = z_e^{s-1} \in C_S$ while $y_i^s > \max C_s$. So in either case, $z_e^s \neq y_i^s$.

Case 2. $s \equiv 1 \pmod 3$.

By the construction, $z_e^s \neq z_e^{s-1}$ implies $z_e^s = 0$ and $y_i^s \neq y_i^{s-1}$ implies $y_i^s = 0$. So $z_e^s > 0$ implies $z_e^s = z_e^{s-1} \neq y_i^{s-1}$, by the induction hypothesis. If $y_i^{s-1} = y_i^s$, clearly $z_e^s \neq y_i^s$. If $y^s \neq y_i^{s-1}$ then $y_i^s = 0$ so $y_i^s \neq z_e^s$ since $z_e^s > 0$.

Case 3. $s \equiv 2 \pmod 3$.

For each i, either $y_i^s = y_i^{s-1}$ or $y_i^s = 0$. If $y_i^s = 0$ then $y_i^s \neq z_e^s$. If $y_i^s = y_i^{s-1}$, then $y_i^s = z_e^s$ implies $y_i^{s-1} = z_e^s$. If $z_e^s = z_e^{s-1}$ this contradicts the induction hypothesis, since in that case $z_e^{s-1} > 0$. So $z_e^s = y_i^s = y_i^{s-1}$ implies $z_e^s \neq z_e^{s-1}$ which by the construction implies z_e^s satisfies

$$(\forall i)_{i \leqq e}(z_e^s \neq y_i^{s-1}) .$$

So $z_e^s = y_i^{s-1}$ implies $i > e$. But by the construction, $z_e^s = y_i^{s-1}$ for $i > e$ implies $y_i^s = 0$ which contradicts the hypothesis that $y_i^s = z_e^s > 0$. So $y_i^s = y_i^{s-1}$ implies $z_e^s \neq y_i^s$. This completes the proof of the lemma.

LEMMA 15. *For all e and t, if $z_e^t > 0$ then*

(a) $z_e^t \in W_e \cap \overline{\alpha_t}$,

and

(b) $z_e^t = 2x$ *implies* $e_x > e$.

Proof. Assume $z = z_e^t > 0$, and let s' be the least s such that $z = z_e^s$. Then $z_e^{s'} = z$, $s' \leqq t$ and, by the construction, $s' \equiv 2 \pmod 3$, $z \in W_e^{s'} \cap \overline{\alpha_{s'}}$ and $z = 2x$ implies $e_x > e$. It remains to show $z \in \overline{\alpha_t}$. Suppose not, and let t' = the least s such that $z \in \alpha_{t'}$, $s' < t' \leqq t$. Then $t' \equiv 1 \pmod 3$ and $z \in E_{t'} \cup O_{t'}$. We claim that if $z \neq z_e^{t'}$ then $z \neq z_e^s$ for all $s \geqq t'$. Assume otherwise; i.e., $z \neq z_e^{t'}$ but $z = z_e^s$ for some $s > t'$. Let s be least. Then $s > t'$, $z_e^{s-1} \neq z = z_e^s > 0$. Then $s \equiv 2 \pmod 3$ and $z_e^s = z \in W_e^s \cap \overline{\alpha_s}$. But $s > t'$ implies $z \in \alpha_{t'} \subset \alpha_s$, which is a contradiction. So $z \neq z_e^{t'}$ implies $z \neq z_e^s$ for all $s \geqq t'$; so in particular $t \geqq t'$ and $z \neq z_e^{t'}$ implies $z \neq z_e^t$. But since by hypothesis $z = z_e^t$, it follows that $z = z_e^{t'}$. Now by the construction, $t' \equiv 1 \pmod 3$ and $z_e^{t'-1} \neq z_e^{t'}$ implies $z_e^{t'} = 0$. Since $z = z_e^{t'} > 0$, if follows that $z_e^{t'-1} = z_e^{t'} = z$. So $z_e^{t'-1} = z \notin E_{t'}$, which implies $z \in O_{t'}$. But then $z = z_e^{t'-1} = y_i^{t'-1}$ for some i, which contradicts Lemma 14. So t' cannot exist, and $z \in \overline{\alpha_t}$.

DEFINITION 16. z is *permanently restrained by* e if $z > 0$ and $z = \lim_s z_e^s$.

LEMMA 17. *For all* e,

(a) *If* z *is permanently restrained by* e, *then* $z \in \bar{\alpha}$.

(b) *If* z *is permanently restrained by* e, *then*

$$(\forall s)\,(\forall t)\,(z = z_e^s \text{ and } t > s \longrightarrow z = z_e^t)\,.$$

(c) *At most one* z *is permanently restrained by* e.

(d) *If* $z = z_e^s$ *and* $z \in \bar{\alpha}$, *then* z *is permanently restrained by* e.

Proof. (a) Assume z is permanently restrained by e. Then $z > 0$ and $z = z_e^s$ for cofinitely many s. By Lemma 15(a), this implies $z \in \overline{\alpha_s}$ for cofinitely many s. So $z \in \bar{\alpha}$.

(b) Assume z is permanently restrained by e and $z = z_e^s$. Suppose $(\exists t)\,(t > s$ and $z \neq z_e^t)$ and let t be least. Then $t > s$ and $z = z_e^{t-1} \neq z_e^t$. Then by Lemma 8(a), since $z_e^{t-1} = z > 0$ this implies $z \in \alpha$, which contradicts (a). So $(\forall t)\,(t > s \rightarrow z = z_e^t)$.

(c) Suppose z is permanently restrained by e. Then $z = z_e^s$ for some s, and by (b), $z = z_e^t$ for all $t > s$. If z' is permanently restrained by e, then $z' = z_e^t$ for cofinitely many t. Then in particular $z' = z_e^t$ for some $t > s$, which implies $z' = z$.

(d) Assume $z = z_e^s$ and $z \in \bar{\alpha}$. Since $0 \in \alpha$ it follows that $z > 0$. Suppose that $(\exists t)\,(t > s$ and $z \neq z_e^t)$ and let t be least. Then $0 < z = z_e^{t-1} \neq z_e^t$. Then by Lemma 8(a), $z = z_e^{t-1} \in \alpha$, contrary to hypothesis. So $(\forall t)\,(t > s \rightarrow z = z_e^t)$, i.e., $z = \lim_s z_e^s$ and z is permanently restrained by e.

LEMMA 18. *For all* e *and* z,

(a) $\lim_s z_e^s$ *exists.*

(b) *If* $z(e) = \lim_s z_e^s$, $z(e)$ *is recursive in* $0'$.

(c) z *is permanently restrained by* e *if and only if* $z = z(e)$ *and* $z(e) > 0$.

Proof. Fix e. Since $D_{g(i)}$ is finite for each i, there exists t such that $(\forall i)_{i \leqq e}\,(D_{g(i)} \cap \alpha^t = D_{g(i)} \cap \alpha)$. Let $t(e)$ be the least such t;

$t(e)$ is recursive in $\mathbf{0}'$ since α is r.e. and $\bigcup_{i \leqq e} D_{g(i)}$ is completely known from the canonical indexing. Define a function $r(e)$ by

$$\begin{aligned} r(e) &= 1 \quad \text{if} \quad (\exists s)\,(s \geqq t(e) \text{ and } z_e^s > 0) \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Then since $t(e)$ can be computed recursively in $\mathbf{0}'$ and z_e^s is a recursive function of s, $r(e)$ is also recursive in $\mathbf{0}'$.

Case 1. $r(e) = 0$.
Then $(\forall s)\,(s \geqq t(e) \rightarrow z_e^s = 0)$. Then $0 = \lim_s z_e^s = z_e^{t(e)}$.

Case 2. $r(e) = 1$.
Then $(\exists s)\,(s \geqq t(e)$ and $z_e^s > 0)$. Let $s(e)$ be the least such s. Then $s(e)$ is recursive in $\mathbf{0}'$, since $s(e)$ can in fact be computed recursively, given $t(e)$. Suppose $(\exists t)\,(t > s(e)$ and $z_e^t \neq z_e^{s(e)})$, and let t be least. Then $t > s(e)$, $0 < z_e^{s(e)} = z_e^{t-1} \neq z_e^t$. By Lemma 8(a), this implies $z_e^{s(e)} = z_e^{t-1} \in E_t$. Then by definition of E_t it follows that $(\exists x)(z_e^{s(e)} = 2x$ and $(\forall i)_{i<e_x}(z_e^{s(e)} \neq z_i^{t-1}))$. Since $z_e^{s(e)} = z_e^{t-1}$, this implies $e \geqq e_x$. But by Lemma 15(b), $z_e^{s(e)} = 2x$ implies $e_x > e$, which gives a contradiction. So $z_e^{s(e)} = z_e^t$ for all $t \geqq s(e)$, and $z_e^{s(e)} = \lim_s z_e^s$.

Thus (a) holds in either case. Define

$$\begin{aligned} z(e) &= 0 \quad \text{if} \quad r(e) = 0 \\ &= s(e) \quad \text{if} \quad r(e) = 1\,. \end{aligned}$$

As shown above, $z(e) = \lim_s z_e^s$ for each e, and $z(e)$ is recursive in $\mathbf{0}'$ since $r(e)$ and $s(e)$ are, which proves (b). (c) is an immediate consequence of Definition 16 and the definition of $z(e)$.

LEMMA 19. *For all e, $e \in \gamma$ if and only if $\lim_s y_e^s = y$ exists and $y \in \bar{\alpha}$.*

Proof. $(\leftarrow)$ Assume $y = \lim_s y_e^s$ and $y \in \bar{\alpha}$. Then $y > 0$ since $0 \in \alpha$. Let t be the least s such that $y = y_e^s$. Then by the construction, $t \equiv 0 \pmod 3$, $e \leqq t$ and $x = h(y)$ implies $D_{\pi_1(x)} \subset \beta^t$ and $D_{\pi_2(x)} \subset \overline{\beta^t}$. Suppose $e \notin \gamma$. Then by Lemma 1, $x \in W_{f(e)}$ and $D_{\pi_1(x)} \subset \beta$ implies $D_{\pi_2(x)} \cap \beta \neq \varnothing$, so there exists $s' > t$ such that $D_{\pi_2(x)} \cap \beta^{s'} \neq \varnothing$. Let s' be least, and choose $s'' \geqq s'$ such that $s'' \equiv 1 \pmod 3$. By Lemma 3(e), $0 < y = \lim_s y_e^s$ and $y = y_e^t$ implies $y = y_e^s$ for all $s \geqq t$; so in particular $s'' > s'' - 1 \geqq s' - 1 \geqq t$ implies $y = y_e^{s''-1} = y_e^{s''}$. But $s'' \geqq s' > t$ also implies $e < s''$ and $D_{\pi_2(x)} \cap \beta^{s''} \neq \varnothing$. So by the construction $y = y_e^{s''-1} \in O_{s''}$ and $y_e^{s''} = O \neq y$, which is a contradiction. So $e \in \gamma$.

($\rightarrow$) Assume $e \in \gamma$. Then by Lemma 1,

$(\exists x)$ ($x \in W_{f(e)}$ and $D_{\pi_1(x)} \subset \beta$ ane $D_{\pi_2(x)} \subset \bar{\beta}$). Let x_e be the least such x, and let s' be the least s such that $x_e \in W^s_{f(e)}$ and $D_{\pi_1}(x_e) \subset \beta^s$. Clearly $x_e \in W^s_{f(e)}$ and $D_{\pi_1(x_e)} \subset \beta^\beta$ and $D_{\pi_2(x_e)} \subset \overline{\beta^s}$ for all $s > s'$. Since x_e is least,

$$(\forall x)_{x<x_e}(x \in W_{f(e)} \text{ and } D_{\pi_1(x)} \subset \beta \longrightarrow D_{\pi_2(x)} \cap \beta \neq \emptyset) .$$

Let $s'' \geqq s'$ be large that

$$(\forall x)_{x<x_e}(x \in W_{f(e)} \quad \text{and} \quad D_{\pi_1(x)} \subset \beta \longrightarrow D_{\pi_2(x)} \cap \beta^{s''} \neq \emptyset) .$$

Now by Lemma 18(a), $\lim_s z^s_i$ exists for all i. Let $m(i)$ be a function such that $z^{m(i)}_i = \lim_s z^s_i$, i.e.,

$$(\forall i)\,(\forall s)\,(s \geqq m(i) \longrightarrow z^s_i = z^{m(i)}_i) ,$$

and choose $t \geqq s''$, $\max_{i<e} m(i)$.

Case 1. $(\forall s)\,(s \geqq t \rightarrow y^s_e > 0)$.

Then $y^t_e > 0$ so by Lemma 3(c), for all $t > s$ either $y^s_e = y^t_e$ or $y^{t'}_e = 0$ for some t', $s < t' \leqq t$. Since by assumption the latter does not happen, it follows that $y^t_e = \lim_s y^s_e$, $y^t_e > 0$.

Case 2. $(\exists s)\,(s \geqq t$ and $y^s_e = 0)$.

Let $t' - 1$ be the least such s; so $t' > t$, $y^{t'-1}_e = 0$. By the construction, if $t' \not\equiv 0 \pmod 3$ then $y^{t'}_e = 0$ also; so we may as well assume $t' \equiv 0 \pmod 3$. Then since $t' > t \geqq s''$, x_e is the least x satisfying: $x \in W^{t'}_{f(e)}$ and $D_{\pi_1(x)} \subset \beta^{t'}$ and $D_{\pi_2(x)} \subset \overline{\beta^{t'}}$. Since $y^{t'-1}_e = 0$ then by the construction $x^{t'}_e = x_e$, $y^{t'}_e > 0$ and $x_e = h(y^{t'}_e)$; let $y = y^{t'}_e$. Assume $y \neq \lim_s y^s_e$; i.e., $y^s_e \neq y^{t'}_e$ for some $s > t'$. Let s be least; then $s > t'$, and $y = y^{t'}_e = y^{s-1}_e \neq y^s_e$. Then by Lemma 3 (b) either $y \in O_s$ or $y = z^s_i$ for some $i < e$. But $y \in O_s$ only if $D_{\pi_2(h(y))} \cap \beta^s \neq \emptyset$. Since $h(y) = x_e$ and $D_{\pi_2(x)} \subset \bar{\beta}$ this cannot happen. So $y \notin O_s$ which implies $y = z^s_i$ for some $i < e$. But by Lemma 14, $0 < y = y^{s-1}_e$ implies $y \neq z^{s-1}_i$. So $i < e$ and $z^{s-1}_i \neq z^s_i$. But $s > t' > t > \max_{i<e} m(i)$, so $s - 1 \geqq m(i)$ which implies $z^s_i = z^{s-1}_i = z^{m(i)}_i$ which is a contradiction. It follows that $y = \lim_s y^s_e$, where $y > 0$.

Thus in either case $e \in \gamma$ implies $\lim_s y^s_e$ exists and $\lim_s y^s_e = y > 0$. To show $y \in \bar{\alpha}$, it suffices to observe that by Lemma 4 (a), $y^s_e > 0$ implies $y^s_e \in \overline{\alpha_s}$; so $y \in \overline{\alpha_s}$ for cofinitely many s, which can happen only if $y \in \bar{\alpha}$.

LEMMA 20. *Let*

$$\begin{aligned} y(e) &= 0 \qquad \text{if } e \notin \gamma, \\ &= \lim_s y_e^s \quad \text{if } e \in \gamma . \end{aligned}$$

Then $y(e)$ is defined for all e, and $y(e)$ is recursive in γ.

Proof. It follows from Lemma 19 that $y(e)$ is defined for all e. To compute $y(e)$ recursively in γ, ask first whether $e \in \gamma$. If not, $y(e) = 0$. If $e \in \gamma$, it remains to compute $\lim_s y_e^s$. But this can be done recursively in $\mathbf{0}'$, since y_e^s is a recursive function of s, and

$$\begin{aligned} y = \lim_s y_e^s &\longleftrightarrow (\exists s)\,(\forall t)\,(t \geqq s \longrightarrow y = y_e^t) \\ &\longleftrightarrow (\forall s)\,(\exists t)\,(t \geqq s \text{ and } y = y_e^t) . \end{aligned}$$

Since by hypothesis $\gamma \in \boldsymbol{c}$ and $\mathbf{0}' \leqq \boldsymbol{c}$, $y(e)$ can be computed recursively in γ.

LEMMA 21. *$W_e \cap \bar{\alpha} \neq \varnothing$ if and only if there is a z in W_e satisfying one of the following conditions*:

(a) $(\exists i)_{i \leqq e}$ (*z is permanently restrained by i*),
(b) $(\exists i)_{i \leqq e}$ (*$i \in \gamma$ and $z = \lim_s y_i^s$*),
(c) $(\exists i)_{i \leqq e}$ (*$i \in \bar{\beta}$ and $z \in D_{g(i)}$*).

Proof. ($\longleftarrow$) Assume that for some z in W_e, (a), (b) or (c) holds. If z is permanently restrained by i, then $z \in \bar{\alpha}$ by Lemma 17(a). If $z = \lim_s y_i^s$ for some $i \in \gamma$, then $z \in \bar{\alpha}$ by Lemma 19. If $z \in D_{g(i)}$ for $i \in \bar{\beta}$, $z = 2x$ for some $x > 0$ and $i = e_x \in \bar{\beta}$ by Definition 2. Now $0 < z \in \alpha$ only if $z \in E_s \cup O_s$ for some s. But $z \in E_s$ implies $e_x \in \beta$, and $z \in O_s$ implies z is odd. Since neither is the case, $z \notin E_s \cup O_s$ and $z \in \bar{\alpha}$. Thus in all cases, $z \in \bar{\alpha} \cap W_e$ and $W_e \cap \bar{\alpha} \neq \varnothing$.

($\longrightarrow$) Assume $W_e \cap \bar{\alpha} \neq \varnothing$. Let z be the least element of $W_e \cap \bar{\alpha}$, and let s' be the least s such that $z \in W_e^s$, then $z \in W_e^s \cap \overline{\alpha_s}$ for all $s \geqq s'$. Since $0 \in \alpha$, $z > 0$ and since z is least, it follows that $(\forall y)_{y<z}\,(y \in W_e \rightarrow y \in \alpha)$. Thus for each $y < z$ such that $y \in W_e$ there is a stage $s(y)$ such that $y \in \alpha^{s(y)}$. Choose $t \geqq s'$, $\max_{y<z} s(y)$.

Case 1. $(\forall s)\,(s \geqq t \rightarrow z_e^s > 0)$.

Let $y = z_e^t$. Then $y > 0$ and by Lemma 15(a), $y \in W_e$. Suppose that $z_e^s \neq y$ for some $s > t$, and let s be least. Then

$$0 < y = z_e^{s-1} \neq z_e^s$$

which by Lemma 8(a) implies $z_e^s = 0$, which contradicts the assumption since $s > t$. It follows that $y = \lim_s z_e^s$ so that $y \in W_e$ is permanently restrained by e. So (a) holds.

Case 2. $(\exists s)(s \geqq t$ and $z_e^s = 0)$.

Let $t' - 1$ be any such s; so $t' > t$ and $z_e^{t'-1} = 0$. If $t' \not\equiv 2 (\mathrm{mod}\ 3)$ then by the construction $z_e^{t'} = 0$ also, so we may as well assume $t' \equiv 2 (\mathrm{mod}\ 3)$. Since $t' - 1 \geqq t \geqq s'$ we have $z \in W_e^{t'} \cap \overline{\alpha_{t'}}$. We consider several subcases.

Subcase 2.1. $(\forall i)_{i \leqq e} (z \neq y_i^{t'-1})$ and $(\forall x)_{x<z} (z = 2x \to e_x > e)$.

Now since $t' > t > \max_{y<z} s(y)$ it follows that no $y < z$ is in $W_e^{t'} \cap \overline{\alpha_{t'}}$. So z is the least z satisfying the requirements of the construction for choosing a new value of $z_e^{t'}$ when $t' \equiv 2 (\mathrm{mod}\ 3)$. So $z = z_e^{t'}$. Since by hypothesis $z \in \bar{\alpha}$, this implies by Lemma 17(d) that z is permanently restrained by e. So (a) holds.

Subcase 2.2. $(\exists i)_{i \leqq e} (z = y_i^{t'-1})$ or $(\exists x)_{x<z} (z = 2x$ and $e_x \leqq e)$.

Subcase 2.2.1. $(\exists i)_{i \leqq e} (z = y_i^{t'-1})$.

Let i be least, $i \leqq e$. If $z = \lim_s y_i^s$ then by Lemma 19, $z \in \bar{\alpha}$ implies $i \in \gamma$, in which case (b) holds. If $z \neq \lim_s y_i^s$, then $(\exists s)(s \geqq t'$ and $z \neq y_i^s)$; let s be least. Then $0 < z = y_i^{s-1} \neq y_i^s$ which by Lemma 3(b) implies $z \in O_s \subset \alpha$ or $z = z_j^s$ for some $j < i$. The former cannot happen, since $z \in \bar{\alpha}$; so $z = z_j^s$ for $j < i \leqq e$. Then by Lemma 17(d), z is permanently restrained by $j < e$, and (a) holds.

Subcase 2.2.2. $(\forall i)_{i \leqq e} (z \neq y_i^{t'-1})$ and $(\exists x)_{x<z} (z = 2x$ and $e_x \leqq e)$.

Let $x = z/2$; then $e_x \leqq e$ and $z \in D_{g(e_x)}$ by Definition 2. If $e_x \in \bar{\beta}$, then (c) holds with $i = e_x$. Now assume $e_x \in \beta$. Then $e_x \in \beta^s$ for some s; let s'' be the least such s. Choose t'' so large that

$$t'' > \max\{s'', t', e_x\}$$

and $t'' \equiv 1 (\mathrm{mod}\ 3)$. Suppose that $(\forall i)_{i<e_x} (z \neq z_i^{t''-1})$. Then since $e_x \leqq t''$ and $e_x \in \beta^{s''} \subset \beta^{t''}$, if follows that $z \in E_{t''} \subset \alpha$, which contradicts the hypothesis that $z \in \bar{\alpha}$. So $z = z_i^{t''-1}$ for some $i < e_x$. But then it again follows by Lemma 17(d) that z is permanently restrained by $i < e_x \leqq e$, and (a) holds.

This completes the proof of the lemma.

LEMMA 22. $\theta A^{\alpha} \leqq_T \gamma$.

Proof. We show how to decide for each e whether $W_e \subset \alpha$, recursively in γ. By Lemma 18, there is a function $z(i)$ recursive in

$\mathbf{0}'$ such that z is permanently restrained by i if and only if

$$z = z(i) > 0 .$$

By Lemma 20, there is a function $y(i)$ recursive in γ such that for each i, $y(i) = \lim_s y_i^s$ if $i \in \gamma$. Let

$$T_e = \{z(i) \mid i \leqq e \quad \text{and} \quad z(i) > 0\} ,$$

$$U_e = \{y(i) \mid i < e \quad \text{and} \quad i \in \gamma\} ,$$

$$V_e = \{z \mid (\exists i)_{i<e} (i \in \bar{\beta} \quad \text{and} \quad z \in D_{g(i)}\} .$$

The sets T_e, U_e, V_e are evidently finite. The membership of T_e can be completely determined recursively in $\mathbf{0}'$ and hence in γ since $\gamma \in \boldsymbol{c}$ and $\mathbf{0}' \leqq \boldsymbol{c}$. The membership of U_e can be determined recursively in γ. The membership of V_e can be obtained recursively in β and hence recursively in γ, since $\beta \in \boldsymbol{b} \leqq \mathbf{0}' \leqq \boldsymbol{c}$. Since this can all be done uniformly given e, there is a function $q(e)$, recursive in γ, such that $T_e \cup U_e \cup V_e = D_{q(e)}$, for each e. Now by Lemma 21,

$$\begin{aligned} W_e \cap \bar{\alpha} \neq \varnothing &\longleftrightarrow (\exists z)\,(z \in W_e \quad \text{and} \quad z \in T_e \cup U_e \cup V_e) \\ &\longleftrightarrow V_e \cap D_{q(e)} \neq \varnothing . \end{aligned}$$

But once $q(e)$ is known, the question of whether $W_e \cap D_{q(e)} = \varnothing$ can be answered recursively in $\mathbf{0}'$, and hence recursively in γ.

LEMMA 23. *For each e, let*

$$T_e = \{z \mid z \textit{ is permanently restrained by some } i < e\} ,$$

$$R_e = \{y \mid y > 0 \textit{ and } (\exists s)\,(y = y_e^s)\} .$$

Then $e \in \gamma$ if and only if $R_e \cap \bar{\alpha} \cap \bar{T}_e \neq \varnothing$.

Proof. ($\rightarrow$). Asume $e \in \gamma$. Then by Lemma 19, $\lim_s y_e^s = y$ exists and $y \in \bar{\alpha}$. Clearly $y \in R_e$, so it suffices to show $y \notin T_e$. Suppose y is permanently restrained by some $i < e$. Then $y = z_i^s$ for cofinitely many s. But $y = \lim_s y_e^s$ implies $y = y_e^s$ for cofinitely many s; while by Lemma 14, $0 < y = z_i^s$ implies $y \neq y_e^s$ for all s, which is a contradiction. So $y \in R_e \cap \bar{\alpha} \cap \bar{T}_e$.

($\leftarrow$) Assume $R_e \cap \bar{\alpha} \cap \bar{T}_e \neq \varnothing$, and let y be the least element of $R_e \cap \bar{\alpha} \cap \bar{T}_e$. Now $y \in R_e$ implies $0 < y = y_e^t$ for some t. Since $y \in \bar{\alpha}$ it suffices by Lemma 19 to show $y = \lim_s y_e^s$. Suppose not; then for some $s > t$, $y_e^s \neq y_e^t = y$; and if s is least, $0 < y = y_e^{s-1} \neq y_e^s$. Then by Lemma 3(b), either $y \in O_s \subset \alpha_s$ or $y = z_i^s$ for some $i < e$. But $y \in \alpha_s$ contradicts the hypothesis that $y \in \bar{\alpha}$; while $0 < y = z_i^s$ for $i < e$

implies by Lemma 17(d) that y is permanently restrained by $i < e$, contradicting the hypothesis that $y \notin T_e$. So $y = \lim_s y_e^s$ and $y \in \bar{\alpha}$ which implies $e \in \gamma$.

LEMMA 24. $\gamma \leqq_T \theta A^\alpha$.

Proof. Let

$$T_e = \{z \text{ is permanently restrained by some } i < e\} ,$$

$$R_e = \{y \mid y > 0 \quad \text{and} \quad (\exists s)\,(y = y_e^s)\} .$$

By Lemma 23, $e \in \gamma$ if and only if $R_e \cap \bar{\alpha} \cap \bar{T}_e \neq \varnothing$. So it suffices to show that we can decide whether $R_e \cap \bar{\alpha} \cap \bar{T}_e = \varnothing$, recursively in θA^α. By Lemma 13, $\beta \leqq_T \alpha$. This implies α is nonrecursive since β is nonrecursive. In particular, $\alpha \neq \varnothing$ or N, so A^α is nontrivial and by Rice's theorem, [4], $\mathbf{0}'$ is recursive in θA^α. Since y_e^s is a recursive function of s, R_e is an r.e. set. By Lemma 18,

$$T_e = \{z(i) \mid i < e \quad \text{and} \quad z(i) > 0\}$$

where $z(i)$ is recursive in $\mathbf{0}'$ and thus in θA^α. Thus the membership of the finite set T_e can be completely determined, recursively in θA^α, and since this can be done uniformly in e, there is a function $p(e)$ recursive in θA^α such that $W_{p(e)} = R_e - T_e$, for each e. Then

$$\begin{aligned} e \in \gamma &\longleftrightarrow W_{p(e)} \cap \bar{\alpha} \neq \varnothing \\ &\longleftrightarrow p(e) \notin \theta A^\alpha , \end{aligned}$$

which can be decided recursively in θA^α once $p(e)$ is known. This completes the proof of the lemma.

It follows from Lemmas 11 and 13 that $\alpha \in \boldsymbol{b}$, and by Lemmas 22 and 24 that $\theta A^\alpha \in \boldsymbol{c}$. Since α is r.e., this completes the proof of the theorem.

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