PACIFIC JOURNAL OF MATHEMATICS
Vol. 46, No. 1, 1973

FACTORED CODIMENSION ONE CELLS IN EUCLIDEAN n-SPACE

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Seebeck has proved that if the m-cell C in Euclidean n-space E^n factors k times, where $m \leqq n-2$ and $n \geqq 5$, then every embedding of a compact k-dimensional polyhedron in C is tame relative to E^n. In this note we prove the analogous result for the case $m+1=n \geqq 5$ and $n-k \geqq 3$. In addition we show that if C factors 1 time, then each $(n-3)$-dimensional polyhedron properly embedded in C can be homeomorphically approximated by polyhedra in C that are tame relative to E^n.

Following Seebeck [8] we say that an m-cell C in E^n *factors k times* if for some homeomorphism h of E^n onto itself and some $(m-k)$-cell B in E^{n-k}, $h(C) = B \times I^k$, where I^k denotes the k-fold product of the interval I naturally embedded in E^k and where

$$B \times I^k \subset E^{n-k} \times E^k = E^n$$

is the product embedding.

In another paper [6] the author has studied results comparable to Seebeck's for factored cells in E^4, but the techniques employed here differ slightly from those used in [6] and [8]. The main result generalizes work of Bryant [2], and the final section here expands on his methods to obtain a strong conclusion about tameness of all subpolyhedra in certain factored cells.

1. Definitions and Notation. For any point p in a metric space S and any positive number δ, $N_\delta(p)$ denotes the set of points in S whose distance from p is less than δ.

The symbol Δ^2 denotes a 2-simplex fixed throughout this paper, $\partial\Delta^2$ its boundary, and Int Δ^2 its interior.

Let A denote a subset of a metric space X and p a limit point of A. We say that A is *locally simply connected at* p, written 1-LC at p, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that each map of $\partial\Delta^2$ into $A \cap N_\delta(p)$ can be extended to a map of Δ^2 into $A \cap N_\varepsilon(p)$. Furthermore, we say that A is *uniformly locally simply connected*, written 1-ULC, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that each map of $\partial\Delta^2$ into a δ-subset of A can be extended to a map of Δ^2 into an ε-subset of A. Similarly, we say that A is *locally simply connected in X at p*, written 1-LC in X at p, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that each map of $\partial\Delta^2$ into $A \cap N_\delta(p)$ extends to a map of Δ^2 into $N_\varepsilon(p)$, and we say that A is *uniformly locally simply connected in* X (1-ULC in X) if the corresponding uniform property is satisfied.

Suppose f and g are maps of a space X into a space Y that has a metric ρ. The symbol $\rho(f, g) < \varepsilon$ means that $\rho(f(x), g(x)) < \varepsilon$ for each x in X.

A subset S of a metric space is called an *ε-subset* if the diameter of S, written diam S, is less than ε.

A compact 0-dimensional subset X of a cell C is said to be *tame* (*relative to* C) if $X \cap \partial C$ is tame relative to ∂C and $X \cap \operatorname{Int} C$ is tame relative to Int C. In addition, a 0-dimensional F_σ set F in C is said to be *tame* (*relative to* C) if F can be expressed as a countable union of tame (relative to C) compact subsets.

For definitions of other terms used here the reader is referred to such papers as [3, 8].

2. Tame polyhedra in factored cells.

The goal of this section is to show that for any k-dimensional polyhedron P in a cell C that factors k times, $E^n - P$ is 1-*ULC*. However, instead of arguing this directly, we prove first that $E^n - C$ is 1-*ULC* in $E^n - P$.

PROPOSITION 1. *If C is an $(n-1)$-cell in E^n that factors k times ($k \leqq n-3$) and P a k-dimensional polyhedron* (*topologically*) *embedded in C, then $E^n - C$ is 1-ULC in $E^n - P$.*

Proof. Suppose $C = B \times I^k \subset E^{n-k} \times E^k$. Define a subset Z of P as the set of all points p of P for which there exist a neighborhood N_p of p (relative to P) and a point b in B such that $N_p \subset \{b\} \times I^k$, and define $Q = P - Z$. We prove first that, for each point c in C, $E^n - C$ is 1-*LC* in $E^n - Q$ at c.

Consider c to be of the form (b, y), where $b \in B$ and $y \in \operatorname{Int} I^k$ (the case $y \in \partial I^k$ is similar and easier). Suppose N is a neighborhood of (b, y) such that $N \cap (B \times \partial I^k) = \varnothing$. There exist an open subset U of E^{n-k} and a contractible open subset V of I^k such that $(b, y) \in U \times V \subset N$. By the construction of Q there exists a point $y' \in V$ such that $(b, y') \notin Q$. Let U' be an open subset of E^{n-k} such that

$$b \in U' \subset U \text{ and } (U' \times \{y'\}) \cap Q = \varnothing .$$

Now we obtain an open subset W of E^{n-k} such that $b \in W \subset U'$ and the inclusion map $i: W \rightarrow U'$ is homotopic to a constant map.

Let L be a loop in $(W \times V) - C$. Since V is contractible to y', L is homotopic in $(W \times V) - C$ to a loop L' in $W \times \{y'\}$. But L' is contractible in

$$U' \times \{y'\} \subset N - Q .$$

Thus, $E^n - C$ is 1-*LC* in $E^n - Q$ at c.

The definition of Z implies that P is locally tame at each point of Z. Hence, if $f\colon \Delta^2 \to E^n - Q$ is a map such that $f(\partial\Delta^2) \subset E^n - P$, then f can be approximated arbitrarily closely by maps $g\colon \Delta^2 \to E^n$ such that $g \mid \partial\Delta^2 = f \mid \partial\Delta^2$ and $g(\Delta^2) \subset E^n - P$. Thus, $E^n - C$ is 1-*LC* in $E^n - P$ at each point c of C. Since C is compact, the corresponding uniform property holds as well.

There may be some value in observing that this argument also gives the following result.

PROPOSITION 2. *Let* $B \times I^k \subset E^{n-k} \times E^k = E^n$ *be an* m*-cell* $(m < n,\ k \leqq n - 3)$ *and* X *a compactum in* $B \times I^k$ *such that* $\dim (X \cap (\{b\} \times I^k)) < k$ *for each* b *in* B. *Then* $E^n - (B \times I^k)$ *is* 1-*ULC in* $E^n - X$.

THEOREM 3. *If* C *is an* $(n - 1)$*-cell in* E^n *that factors* k *times* $(k \leqq n - 3)$ *and* X *is either a* k*-dimensional polyhedron or a* $(k - 1)$*-dimensional compactum in* C, *then* $E^n - X$ *is* 1-*ULC*.

This theorem follows immediately from [1, Prop. 1] and either Proposition 1 or Proposition 2.

COROLLARY 4. *If* C *is an* $(n - 1)$*-cell in* $E^n (n \geqq 5)$ *that factors* k *times* $(k \leqq n - 3)$, *then each* k*-dimensional polyhedron* P *in* C *is tame.*

The corollary is a straightforward application of the Bryant-Seebeck characterization of tameness [3] for codimension 3 polyhedra in terms of the 1-*ULC* property.

3. Approximations in cells that factor 1 time. This section contains a proof of the analogue of Seebeck's Corollary 5.1 [8] for codimension one cells.

PROPOSITION 5. *If* C *is an* $(n - 1)$*-cell in* E^n *that factors* 1 *time, then there exists a tame* 0*-dimensional* F_σ *set* F *in* $\operatorname{Int} C$ *such that, for each point* c *of* $\operatorname{Int} C$, $E^n - C$ *is* 1-*LC in* $(E^n - C) \cup F$ *at* c.

Proof. Assume $C = B \times I \subset E^{n-1} \times E^1 = E^n$. Let $c = (b, t)$ be a point of $\operatorname{Int} C$ and U a neighborhood of c such that $U \cap C \subset \operatorname{Int} C$. We assume further that U is a product neighborhood $U = U' \times J$, where $U' \subset E^{n-1}$ and $J \subset E^1$. Corresponding to U is a neighborhood V of c such that any map $f'\colon \partial\Delta^2 \to V - C$ extends to a map $f\colon \Delta^2 \to U$ such that $f^{-1}(f(\Delta^2) \cap C)$ is 0-dimensional ([4, Cor. 2C, 2.1] or [5, Th. 3.2]). We can change this map f near C, altering only the E^1 coordinates of points in the range, so that in addition $f(\Delta^2) \cap C \subset B \times \{t\}$. We shall obtain a map $g: \Delta^2 \to U$ satisfying

(i) $g \mid \partial \Delta^2 = f \mid \partial \Delta^2 = f'$,

(ii) $g(\Delta^2) \cap C$ is a tame (relative to C) 0-dimensional subset of Int C.

Let ε be a positive number such that if $g: \Delta^2 \to E^n$ and $\rho(f, g) < \varepsilon$, then $g(\Delta^2) \subset U$.

Cover $f^{-1}(f(\Delta^2) \cap C)$ by the interiors of a collection of small, pairwise disjoint 2-cells ${}_1D_1, {}_1D_2, \cdots, {}_1D_{k(1)}$ in Int Δ^2. Slide the sets $f({}_1D_i)$ vertically to define a map $g_1: \Delta^2 \to E^n$ satisfying

(A_1) $g_1 \mid \Delta^2 - \bigcup {}_1D_i = f \mid \Delta^2 - \bigcup {}_1D_i$,

(B_1) $\rho(g_1, f) < \varepsilon/2$,

(C_1) $g_1({}_1D_i) \cap C \subset B \times \{{}_1t_i\}$, where ${}_1t_i \neq {}_1t_j$ whenever $i \neq j$,

(D_1) $g_1^{-1}(g_1(\Delta^2) \cap C)$ is 0-dimensional.

The ${}_1D_i$'s must be chosen with sufficiently small diameters that each set $f({}_1D_i) \cap C$ is contained in the interior of a small $(n-2)$-cell in $B \times \{t\}$. Thus,

(E_1) there exist pairwise disjoint $(n-1)$-cells ${}_1K_1, {}_1K_2, \cdots, {}_1K_{k(1)}$ in Int C, each of diameter $< \varepsilon/2$, such that $\bigcup \operatorname{Int} {}_1K_i \supset g_1(\Delta^2) \cap C$.

The remaining approximations g_j will be so close to g_1 that $\bigcup \operatorname{Int} {}_1K_i \supset g_j(\Delta^2) \cap C$.

Let $\varepsilon_2 = \min \{\varepsilon/4, 1/2\rho(g_1(\Delta^2) \cap C, C - \bigcup {}_1K_i)\}$. To repeat this process, cover $g_1^{-1}(g_1(\Delta^2) \cap C)$ by the interiors of a collection of a very small, pairwise disjoint 2-cells ${}_2D_1, {}_2D_2, \cdots, {}_2D_{k(2)}$ in $\bigcup \operatorname{Int} {}_1D_i \subset \operatorname{Int} \Delta^2$. Slide the sets $g_1({}_2D_i)$ vertically to define a map $g_2: \Delta^2 \to E^n$ satisfying

(A_2) $g_2 \mid \Delta^2 - \bigcup {}_2D_i = g_1 \mid \Delta^2 - \bigcup {}_2D_i$,

(B_2) $\rho(g_2, g_1) < \varepsilon_2$,

(C_2) $g_2({}_2D_i) \cap C \subset B \times \{{}_2t_i\}$, where ${}_2t_i \neq {}_2t_j$ whenever $i \neq j$,

(D_2) $g_2^{-1}(g_2(\Delta^2) \cap C)$ is 0-dimensional.

The ${}_2D_i$'s must be chosen with sufficiently small diameters that each set $g_1({}_2D_i)$ is contained in a small $(n-2)$-cell in some $(B \times \{{}_1t_j\}) \cap (\bigcup \operatorname{Int} {}_1K_i)$. Thus,

(E_2) there exist pairwise disjoint $(n-1)$-cells ${}_2K_1, {}_2K_2, \cdots, {}_2K_{k(2)}$ in $\bigcup \operatorname{Int} {}_1K_i$, each of diameter $< \varepsilon_2$, such that $\bigcup \operatorname{Int} {}_2K_i \supset g_2(\Delta^2) \cap C$.

By continuing in this manner we construct a sequence of maps $g_n: \Delta^2 \to E^n$ satisfying analogous conditions $(A_n) - (B_n)$ and an associated sequence of collections $\{{}_nK_i\}$ of $n-1$ cells in C satisfying an analogous condition (E_n). The restrictions of condition (B_n) guarantee that $g = \lim g_n$ is a continuous function of Δ^2 into U, and the restrictions of (E_n) guarantee that

$$g(\Delta^2) \cap C \subset \bigcap_{n=1}^{\infty} \left(\bigcup_{i=1}^{k(n)} \operatorname{Int}_n K_i \right).$$

Thus, $g(\Delta^2) \cap C$ is a tame (relative to C) 0-dimensional subset of C [7, Lemma 2].

To prove the theorem from this fact, observe that for each $\varepsilon > 0$ there exists a countable collection $\{V_i\}$ of open sets covering Int C such that any map $f'\colon \partial\Delta^2 \to V_i - C$ extends to a map g of Δ^2 into an ε-subset of E^n such that $g(\Delta^2) \cap C$ is a tame 0-dimensional subset of Int C. Since there are only countably many homotopy classes of maps of $\partial\Delta^2$ into $V_i - C$, the desired set F can be defined as the countable union of sets $g(\Delta^2) \cap C$.

THEOREM 6. *Suppose C is an $(n-1)$-dimensional cell in E^n that factors 1 time, P is an $(n-3)$-dimensional polyhedron properly embedded in C, and $\varepsilon > 0$. There exists an ε-push h of (C, P) such that $h(P)$ is tame relative to E^n.*

Proof. The case $n = 4$ is trivial, and no push is needed [6]; hence, we assume $n \geqq 5$. By [8, Cor. 5.1] there exists an $\varepsilon/2$ push h_1 of (C, P) such that $h_1(P \cap \partial C)$ is tame. Let F denote the 0-dimensional F_σ set of Proposition 5. There exists an $\varepsilon/2$ push h_2 of $(C, h_1(P))$ such that $h_2h_1(P) \cap F = \varnothing$ and $h_2 \mid \partial C = 1$. Let h denote the ε-push h_2h_1. It follows that $E^n - C$ is 1-LC in $E^n - h(P)$ at each point of Int C, and in stronger form, as shown in § 2, that $E^n - h(P)$ is 1-LC at each point of Int C. The tameness of $h(P) \cap \partial C$ then implies that $E^n - h(P)$ is 1-LC at every point of $h(P)$. Thus, $h(P)$ is tame [3].

COROLLARY 7. *Let S denote an $(n-2)$ sphere in S^{n-1}, the $(n-1)$-sphere, and Σ the suspension of S in S^n, the suspension of S^{n-1}. Then there exists a tame (relative to Σ) 0-dimensional F_σ set F in Σ such that $S^n - \Sigma$ is 1-ULC in $(S^n - \Sigma) \cup F$. Furthermore, if P is an$(n-3)$-dimensional polyhedron in Σ and $\varepsilon > 0$, there exists an ε-push h of (Σ, P) such that $h(P)$ is tame relative to S^n.*

4. Factored cells in which all lower dimensional compacta are locally nice. Let $C = B \times I^k \subset E^{n-k} \times E^k = E^n$ be an r-cell $(r < n)$. Although the low dimensional polyhedra in C are nicely embedded, some $(k+1)$-cell in C may be wild. In this section we mention a property of certain cells B that implies every $(r-1)$-dimensional polyhedron in C is nicely embedded.

THEOREM 8. *Let B denote an m-cell in $E^n(m \leqq n-2)$ such that, for each $(m-1)$-dimensional compactum $X \subset B$, $E^n - X$ is 1-ULC, and let C denote $B \times I^k$, contained in $E^n \times E^k = E^{n+k}$. Then, for each $(m+k-1)$-dimensional compactum $Y \subset C$, $E^{n+k} - Y$ is 1-ULC.*

Proof. It suffices to consider only the case $k = 1$. Let $\varepsilon > 0$ and $w \in \operatorname{Int} I$. We shall costruct an ε-push h of (E^{n+1}, Y) such that

$(E^n \times \{w\}) - h(Y)$ is 1-ULC. Let V denote the ε-neighborhood of Y, $\{b_i\}_1^\infty$ a countable dense subset of B, and π the natural projection of $E^{n+1} = E^n \times E^1$ onto the first factor. For any open subset N of E^{n+1} containing $(b, w) \in B \times I$ there exists a point $(b', w') \in N \cap (B \times I - Y)$. If N is a connected open set of the form $N = W \times J$, then there exists a homeomorphism g of E^{n+1} onto E^{n+1} such that (a) $g \mid E^{n+1} - N =$ identity, (b) $g((b', w')) = (b', w)$, (c) $g(C) = C$ and (d) $\pi g = g$. Consequently, there exist a sequence $\{h_i\}$ of homeomorphisms of E^{n+1} onto itself and a sequence of points $\{b'_i\}$ in B such that for $i = 1, 2, \cdots$

(0) $\rho(x, h_i(x)) < \varepsilon/2^i$ for all x in E^n,

(1) $\rho(b_i, b'_i) < 1/i$,

(2) $(b'_i, w) \notin h_i \circ h_{i-1} \circ \cdots \circ h_1(Y)$,

(3) $h_{i+k}((b'_i, w)) = (b'_i, w)$ for all $k > 0$,

(4) $h_i(C) = C$

(5) $\pi h_i = h_i$.

(6) $h_i \mid E^{n+1} - V =$ identity.

Furthermore, using Condition (a) and careful epsilonics we can construct the sequence $\{h_i\}$ so that the function $h = \lim_{n\to\infty} h_n \circ \cdots \circ h_1$ is an ε-homeomorphism of E^{n+1} onto itself. Then Condition (6) implies that h is an ε-push of (E^{n+1}, Y).

Condition (1) implies that $\{b'_i\}$ is a dense subset of B, and Conditions (2) and (3) yield that $(b'_i, w) \notin h(Y)$ $(i = 1, 2, \cdots)$. Thus, $h(Y) \cap (B \times \{w\})$ is nowhere dense in $B \times \{w\}$. Consequently, $E^n \times \{w\} - h(Y)$ is 1-ULC by hypothesis (since $h(Y) \subset B \times I$), and we obtain the desired conclusion by appealing to Theorem 1 of [1].

We exploit the construction of the push h a second time in proving the following:

THEOREM 9. *Let B denote an $(n-1)$-cell in E^n such that, for each $(n-2)$-dimensional compactum $X \subset B$, $E^n - B$ is 1-ULC in $E^n - X$, and let C denote $B \times I^k$, contained in $E^n \times E^k$. Then for each $(n+k-2)$-dimensional compactum $Y \subset C$, $E^{n+k} - C$ is 1-ULC in $E^{n+k} - Y$.*

Proof. Simplifying as before, we consider $k = 1$ and $c \in C$ a point of the form (b, w), where $b \in B$ and $w \in \operatorname{Int} I$, and we shall show that $E^{n+1} - C$ is 1-LC in $E^{n+1} - Y$ at c.

Let $\varepsilon > 0$. Choose a countable dense subset $\{b_i\}$ of B. Then reapplying the techniques found in the proof of Theorem 8, we find an $(\varepsilon/6)$-homeomorphism h of of E^{n+1} onto itself and a sequence $\{b'_i\}$ of points in B satisfying Conditions (0)–(6) stated there. Let U denote the $\varepsilon/6$-neighborhood of b in E^n and V the $(\varepsilon/3)$-neighborhood of w in

Int I. Then both (b, w) and $h((b, w))$ are contained in $U \times V$, and diam $(U \times V) < \varepsilon/2$. Since B is an $(n-1)$-cell there exists a neighborhood U' of b in E^n such that $b \in U' \subset U$ and each map f of $\partial\Delta^2$ into $U' - B$ can be extended to a map F of Δ^2 into U such that $F^{-1}(F\Delta^2) \cap B)$ is 0-dimensional.

In this paragraph we prove that $U' \times V$ is a neighborhood of $h(c)$ such that any loop in $(U' \times V) - C$ is contractible in an $\varepsilon/2$-subset of $E^{n+1} - h(Y)$. If $f: \partial\Delta^2 \to (U' \times V) - C, f$ is homotopic in $(U' \times V) - C$ to a map $f': \partial\Delta^2 \to U' \times \{w\}$. Let $F: \Delta^2 \to U \times \{w\}$ be an extension of f' such that $F^{-1}(F(\Delta^2) \cap (B \times \{w\})$ is 0-dimensional. Once again $h(Y) \cap (B \times \{w\})$ is nowhere dense in $B \times \{w\}$, which means that $(E^n - B) \times \{w\}$ is 1-ULC in $((E^n) \times \{w\}) - h(Y)$. Cover $F^{-1}(F(\Delta^2) \cap (B \times \{w\})$ by finitely many pairwise disjoint 2-cells $D_1, \cdots, D_t$ in Int Δ^2 such that $F \mid \partial D_i$ can be extended to a map G_i of D_i into $(U \times \{w\}) - h(Y)$. By redefining F as G_i on $D_i (i = 1, \cdots, t)$ one can easily see that $f \mid \partial\Delta^2 \colon \partial\Delta^2 \to (U \times V) - h(Y)$ is homotopic to a constant map.

Because h^{-1} is an $(\varepsilon/6)$-homeomorphism and diam $U \times V < \varepsilon/2$, diam $h^{-1}(U \times V) < \varepsilon$. In addition, $h^{-1}(U' \times V)$ is a neighborhood of c such that any map $g \colon \partial\Delta^2 \to h^{-1}(U' \times V) - C$ can be extended to a map $G: \Delta^2 \to h^{-1}(U \times V) - Y$. This completes the proof.

COROLLARY 10. *Let B denote an m-cell in $E^n (m < n)$ such that, for each $(m-1)$-dimensional compactum $X \subset B$, $E^n - B$ is 1-ULC in $E^n - X$. Then each p-dimensional polyhedron P in $B \times I^k \subset E^n \times E^k$ $(p + 3 \leqq n + k, p < m + k)$ is tame.*

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Received January 5, 1972. Partially supported by NSF Grant GP-19966.

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