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SOME INCLUSION THEOREMS FOR SEQUENCE SPACES

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The purpose of this paper is to study inclusion theorems for some of the more familiar sequence spaces. Necessary and sufficient conditions are given for an FK-space to contain each of the spaces bv_0, bv, ℓ^p, $0 < p < \infty$, and c_0. It is also shown that the Hardy space H^p, $0 < p < 1$, is a barrelled subspace of its containing Banach space B^p. This leads to new results concerning multipliers of H^p and to new estimates on the growth of the Taylor coefficients of B^p functions.

The main results of §2 give necessary and sufficient conditions for an FK-space to contain ℓ, bv_0 or bv. These results were obtained by W. H. Ruckle, A. K. Snyder, and A. Wilansky, and the author in a series of seminars held at Lehigh University during the period 1968–1970. They are rather elementary and probably well-known to other workers in the field; the main reason for including them here (without proof) is that special cases (for example: [7], Theorem 10; [14], Theorems XI (a), (b) and XII (a), (b); [16], Theorem 1 (1); [20], Theorem 1) have been repeatedly established by various authors. Theorems of Dawson [8] and Lorentz and Zeller [19] are extended, and it is shown that ℓ^p is never a closed subspace of any convergence domain. The results of this section also have relevance to Sember's work on conull FK-spaces [22].

FK-spaces containing ℓ^p, $1 < p < \infty$, or c_0 are characterized in § 3. The techniques employed provide an extension to a result of Singer [23] (for Banach spaces with no subspace isomorphic to c_0) to arbitrary sequentially complete locally convex spaces.

Section 4 covers the spaces ℓ^p, $0 < p \leqq 1$. It turns out, rather surprisingly, that if an FK-space contains one of these spaces, then it contains all of them. As a corollary, it is shown that $\bigcap_{p>0} \ell^p$ is a barrelled subspace of ℓ, thus improving a result of Wilansky ([27], p. 45).

Section 5 returns to the study (see [3]) of scarce copies of solid sequence spaces. ℓ and ω enjoy the rather special property of having every scarce copy barrelled, which fact in turn leads to further inclusion theorems. Similar results are obtained for arbitrary solid FK-AK-spaces.

In §6 a study is made of the Hardy spaces H^p, $0 < p < 1$, which can be viewed as sequence spaces by identifying functions with their Taylor series. Using a technique of Wilansky ([27], p. 45), it is shown that H^p is a barrelled subspace of its containing Banach space B^p.

This observation leads to the solution of a problem raised implicitly by Duren, Romberg and Shields in the papers [10], [11] and [12], viz: why do H^p and B^p almost always have the same multipliers? The paper closes with some new results concerning the size of the Taylor coefficients of functions in B^p.

1. **Notation and preliminary ideas.** ω denotes the space of all (real- or) complex-valued sequences, and any vector subspace of ω is called a *sequence space*. A sequence space E with a vector space topology τ is a *K-space* if the inclusion mapping $(E, \tau) \to \omega$ is continuous when ω is endowed with the topology of coordinaterwise convergence. If, in addition, (E, τ) is complete and metrizable, (E, τ) is called a *Frechet K-space*; if τ is locally convex, we abbreviate this to *FK-space*. The basic properties of *FK*-spaces may be found, for example, in [28]. In particular (extending Zeller's theorem 4.5(a) slightly), it follows from the closed graph theorem that if E and F are Frechet K-spaces with $E \subseteqq F$, then E is continuously embedded in F. An *FK*-space whose topology is normable is a *BK-space*.

The following spaces will be important in the sequel: m, the space of all bounded sequences; c, the space of all convergent sequences; c_0, the space of null sequences; ℓ^p, $0 < p < \infty$, the space of all absolutely p-summable sequences;

$$bv = \{x \in \omega: \sum_{j=1}^{\infty} |x_j - x_{j+1}| < \infty\} ,$$

the space of all sequences of bounded variation, which is a *BK*-space under the norm

$$\|x\|_{bv} = \sum_{j=1}^{\infty} |x_j - x_{j+1}| + \lim_{j\to\infty} |x_j| \qquad (x \in bv) .$$

As usual, ℓ^1 is replaced by ℓ, and bv_0 denotes $bv \cap c_0$.

The sequence $(1, 1, \cdots)$ is denoted by e and it is clear that $bv = bv_0 + \{e\}$. e^j, $j = 1, 2, \cdots$, denotes the sequence $(0, \cdots, 0, 1, 0, \cdots)$ with the 'one' in the jth position, and φ denotes the linear span of $\{e^j: j = 1, 2, \cdots\}$ in ω.

We shall also be concerned with matrix transformations $y = Ax$, where $x, y \in \omega$, $A = \{a_{ij}\}_{i,j=1}^{\infty}$ is an infinite matrix with complex coefficients, and (formally)

$$y_i = \sum_{j=1}^{\infty} a_{ij}x_j \qquad (i = 1, 2, \cdots) .$$

If each of these series converges we say that $y = Ax$ *exists* and write $y \in \omega_A$. More generally, if E is any subset of ω, we denote by E_A the set

$$\{x \in \omega\colon Ax \text{ exists and } Ax \in E\}\ .$$

c_A is called the *convergence domain* of the matrix A and $x \in c_A$ is said to be *A-limitable*. The sequence $\{a_{ij}\}_{j=1}^{\infty}$ is called the ith *row* of A and $\{a_{ij}\}_{i=1}^{\infty}$ the jth *column*.

The following result, proved by Zeller in [28], Theorem 4.10(a), will be useful throughout the paper.

THEOREM 1. *Let E be an FK-space whose topology is given by means of the seminorms $\{q_n\}_{n=1}^{\infty}$ and let A be an infinite matrix. Then E_A is an FK-space when topologized by*

$$x \longrightarrow |x_j| \qquad (j = 1, 2, \cdots)\ ;$$

$$x \longrightarrow \sup_n \Big|\sum_{j=1}^{n} a_{ij}x_j\Big| \qquad (i = 1, 2, \cdots)\ ;$$

$$x \longrightarrow q_n(Ax) \qquad (n = 1, 2, \cdots)\ .$$

Finally, if E and F are vector spaces which form a separated dual pair, we follow the notation of [21] and denote the weak topology on E by $\sigma(E, F)$, the Mackey topology by $\tau(E, F)$, and the strong topology by $\beta(E, F)$. If (E, τ) is a topological vector space, the set of all τ-continuous linear functionals on E is denoted by E'.

2. *FK*-spaces containing ℓ, bv_0 or bv.

THEOREM 2. *An FK-space E contains ℓ if and only if $\{e^j\colon j = 1, 2, \cdots\}$ is a bounded subset of E.*

Our next result follows from Theorems 1 and 2 by using the fact that E is a K-space.

COROLLARY. *Let A be a matrix and E an FK-space. Then A maps ℓ into E if and only if the columns of A belong to E and form a bounded subset there.*

Putting $E = c$ or ℓ^p, $1 \leqq p \leqq \infty$, enables us to give 'quickie' proofs of results of Hahn ([14], Theorems XII (a) and (b)), Cohen and Dunford ([7], Theorem 10), and Knopp and Lorentz ([16], Theorem 1(1)).

Using the result of Hahn, Lorentz and Zeller ([19], Lemma 5) have shown that l cannot be the convergence domain of any matrix A. Motivated by their paper, we introduce the set $c_0(r, s) = \{x \in c_0\colon x_{r_j} = -x_{s_j},\ j = 1, 2, \cdots;\ x_k = 0 \text{ for other } k\}$, where $r = \{r_j\}_{j=1}^{\infty}$ and $s = \{s_j\}_{j=1}^{\infty}$ are sequences of positive integers satisfying $r_1 < s_1 < r_2 < s_2 < \cdots$. $c_0(r, s)$ is a BK-space under the topology generated by the

norm

$$x \longrightarrow \sup_n |x_{r_j}| ,$$

and since $\sum_{k=1}^{s_j} x_k e^k \to x$ as $j \to \infty$, it is clear that $\varphi \cap c_0(r, s)$ is dense in $c_0(r, s)$.

Our next theorem improves the result of Lorentz and Zeller mentioned above and also the results of Dawson [8].

THEOREM 3. *If $\{A^n\}_{n=1}^{\infty}$ is a family of matrices, each mapping $\swarrow$ into c, then there exist sequences r and s of positive integers such that*

$$c_0(r, s) \subseteqq \bigcap_{n=1}^{\infty} c_{A^n} .$$

Proof. This is essentially the same as that of Lemma 5 of [19] and will be omitted.

Our next result, which follows immediately from Theorem 3, should be compared with Theorem 4 of [2].

COROLLARY 1. *If $\{A^n\}_{n=1}^{\infty}$ is a family of matrices, each mapping $\swarrow$ into c, then $\bigcap_{n=1}^{\infty} c_{A^n} \backslash \bigcup_{0<p<\infty} \swarrow^p$ is nonempty. In particular, for $0 < p < \infty$, $\swarrow^p$ is not the convergence domain of any matrix.*

COROLLARY 2. *For $1 \leqq p < \infty$, $\swarrow^p$ is not a closed subspace of any convergence domain.*

Proof. Let $p \geqq 1$ be fixed, and suppose that, for some matrix A, we have $\swarrow^p \subseteqq c_A$. From Theorem 3 and the remarks made immediately before it, there exist sequences r and s of positive integers such that $c_0(r, s) \subseteqq \bar{\varphi}$, the closure of φ in c_A. It follows that $\bar{\varphi} \backslash \swarrow^p$ is nonempty, so that l^p is not closed in c_A.

We note that the statement of Corollary 2 is valid whenever $0 < p \leqq \infty$. For $0 < p < 1$ this follows since $\swarrow^p$ is not even locally convex (see also Theorem 6); for $p = \infty$, $\swarrow^\infty = m$, a separability argument may be used (see [1], Corollary 1 to Theorems 4 and 5).

THEOREM 4. *An FK-space contains $bv_0(bv)$ if and only if $(e \in E)$ and $\{\sum_{j=1}^{n} e^j : j = 1, 2, \cdots\}$ is a bounded subset of E.*

COROLLARY 1. *Let A be a matrix and E and FK-space. Then A maps bv_0 into E if and only if the columns of A belong to E and their partial sums form a bounded subset there.*

Putting $E = c$ or bv, enables us to give 'quickie' proofs of results

of Hahn ([14], Theorems XI (a) and (b)) and Mears ([20], Theorem 1).

The important class of conull FK-spaces was introduced by Snyder in [24]; an FK-space E containing $\varphi \cup \{e\}$ is said to be *conull* if

$$\sum_{j=1}^{\infty} e^j = e \text{ weakly in } E .$$

The next result, which follows at once from Theorem 4, enables us to remove the hypothesis 'containing bv' from several of the theorems in [22]. (See also p. 159 of [22].)

COROLLARY 2. *Any conull FK-space must contain bv.*

3. *FK*-spaces containing ℓ^p $1 < p < \infty$, or c_0. We begin this section in a more general setting by considering a series, $\sum_{i=1}^{\infty} x_i$, of elements from a locally convex space (E, τ). Motivated by [23], p. 131, we say that $\sum_{i=1}^{\infty} x_i$ *is weakly p-unconditionally Cauchy*, $1 < p \leqq \infty$, if $\sum_{i=1}^{\infty} y_i x_i$ converges in (E, τ) whenever $y \in \ell^p$. (Here we use the convention that $\ell^\infty = c_0$.) Our next theorem improves a result of Singer ([23], Theorem 8, where the same statement, with $1 < p < \infty$, is given, but only for E a Banach space containing no subspace isomorphic to c_0).

THEOREM 5. *Let (E, τ) be a sequentially complete locally convex space. Then a series, $\sum_{i=1}^{\infty} x_i$, of elements of E is weakly p-unconditionally Cauchy, $1 < p \leqq \infty$, if and only if*

$$\sum_{i=1}^{\infty} |f(x_i)|^q < \infty \qquad (f \in E') \tag{1}$$

where $q = p/(p-1)$ denotes the dual exponent of p.

Proof. The proof of the necessity of (1) is the same as that given by Singer and will be omitted.

To establish the converse implication, we consider the linear mapping T from φ into E defined by

$$Ty = \sum_{i=1}^{\infty} y_i x_i \qquad (y \in \varphi) .$$

If $F \in E'$, then

$$\langle Ty, F \rangle = F\Big(\sum_{i=1}^{\infty} y_i x_i\Big) = \sum_{i=1}^{\infty} y_i F(x_i)$$

and so, putting $F(x_i) = f_i$, we have $f = \{f_i\}_{i=1}^{\infty} \in \ell^q$ and

$$\langle Ty, F \rangle = \langle y, f \rangle .$$

It follows that T is $\sigma(\mathcal{P}, \ell^q) \to \sigma(E, E')$-continuous and so ([21], p. 62) T is also $\tau(\mathcal{P}, \ell^q) \to \tau$-continuous. Now ℓ^p is $\tau(\ell^p, \ell^q)$-metrizable so that ([21], p. 82)

$$\tau(\mathcal{P}, \ell^q) = \tau(\ell^p, \ell^q) \mid \mathcal{P} .$$

Thus, if $y \in \ell^p$, then $\{\sum_{j=1}^n y_j e^j\}_{n=1}^\infty$ is $\tau(\mathcal{P}, \ell^q)$-Cauchy so that $\{T(\sum_{j=1}^n y_j e^j)\}_{n=1}^\infty$ is τ-Cauchy in E, i.e., $\{\sum_{i=1}^n y_i x_i\}_{n=1}^\infty$ converges in E.

We note that, for $p = \infty$ and E a Banach space, the above result (with different proof) has been given by Bessaga and Pelczynski ([6], Lemma 2).

If E is an FK-space containing $\mathcal{P}$, then putting $x_i = e^i$, $i = 1, 2, \cdots$, gives the following corollary (see also [4], Proposition 5, for the case $p = \infty$).

COROLLARY. *An FK-space E contains ℓ^p, $1 < p \leqq \infty$, if and only if $\mathcal{P} \subseteqq E$ and*

$$\sum_{j=1}^{\infty} |f(e^j)|^q < \infty \qquad (f \in E') ,$$

where $q = p/(p-1)$.

4. *FK-spaces containing ℓ^p, $0 < p \leqq 1$.* For $0 < p < 1$ it is well-known that ℓ^p is a Frechet K-space when topologized by means of the paranorm

$$x \longrightarrow \sum_{j=1}^{\infty} |x_j|^p \qquad (x \in \ell^p) ,$$

so that ([17], §15.12 (1)) ℓ^p is of the first category in ℓ. However, in spite of this, we have the rather surprising

THEOREM 6. *An FK-space contains $\bigcap_{p>0} \ell^p$ (if and) only if it contains ℓ.*

Proof. $\bigcap_{p>0} \ell^p$, regarded as a countable intersection, $\bigcap_{n=1}^\infty \ell^{1/n}$, is a Frechet K-space when endowed with the projective topology, and $\{e^j : j = 1, 2, \cdots\}$, being bounded in each ℓ^p space, is bounded in $\bigcap_{p>0} \ell^p$. If E is an FK-space containing $\bigcap_{p>0} \ell^p$, the inclusion mapping: $\bigcap_{p>0} \ell^p \to E$ is continuous by the closed graph theorem; consequently $\{e^j : j = 1, 2, \cdots\}$ is bounded in E, and the desired conclusion follows from Theorem 2.

COROLLARY. *$\bigcap_{p>0} \ell^p$ is a barrelled subspace of ℓ.*

Proof. Noting that $\bigcap_{p>0} \ell^p$ is dense in ℓ, the corollary follows

at once from Theorem 1 of [5]. This theorem, however, is rather lengthy, and the following alternative proof of the corollary is perhaps preferable here.

Suppose that $\{a^n\}_{n=1}^{\infty}$ is a $\sigma(m, \bigcap_{p>0} \ell^p)$-bounded sequence of elements of m. If A denotes the matrix whose nth row is a^n, then clearly we have $\bigcap_{p>0} \ell^p \subseteq m_A$. It follows from Theorems 1 and 6 that $\ell \subseteq m_A$ so that $\{a^n\}_{n=1}^{\infty}$ is also $\sigma(m, \ell)$-bounded. Thus, the topologies $\sigma(m, \bigcap_{p>0} l^p)$ and $\sigma(m, l)$ define the same bounded sequences and therefore the same bounded sets; consequently,

$$\beta(\bigcap_{p>0} \ell^p, m) = \beta(\ell, m) \mid \bigcap_{p>0} \ell^p = \|\cdot\|_1\text{-topology} \mid \bigcap_{p>0} \ell^p ,$$

and the desired result follows at once from Corollary 1, p. 66, of [21].

The corollary implies that if X is any subspace of ℓ with $\bigcap_{p>0} \ell^p \subseteq X \subseteq \ell$, then X is a barrelled subspace of ℓ. In particular, we obtain the result ([27], Example 1.2, p. 45) of Wilansky which asserts that $\ell^{1/2}$ is a barelled subspace of ℓ.

The reader should compare the results of this section with Theorem 8 and its corollary.

5. Scarce copies of sequence spaces. In this section we return to our study of scarce copies of sequence spaces (see [3]) and begin by recalling some definitions. A sequence space E is said to be *solid* (respectively *monotone*) if $xy \in E$ whenever $x \in E$ and $y \in m$ (respectively $y_j = \pm 1$ for all j). $r = \{r_n\}_{n=1}^{\infty}$ will always denote a nondecreasing, unbounded sequence of positive integers with $r_1 = 1$ and $r_n = o(n)$. We define, for each $x \in \omega$ and each positive integer n, the *counting function*, $c_n(x)$, as the number of nonzero elements in the set $\{x_1, x_2, \cdots, x_n\}$. Given any subset E of ω, we consider

$$\sigma(E, r) = \{x \in E: c_n(x) \leqq r_n \text{ for } n = 1, 2, \cdots\} .$$

$\sigma(E, r)$ is not, in general, a vector space, and we denote its linear span by $\Sigma(E, r)$. Such a space will be called a *scarce copy* of E; it is clear that if E is solid, monotone, or contains φ, then every scarce copy of E has the same property. Moreover, if (E, τ) is a K-space containing an element x with $x_j \neq 0$, $j = 1, 2, \cdots$, then every scarce copy of E is of the first category in E.

In view of this last observation the following is somewhat unexpected.

THEOREM 7. *Every scarce copy of ω is barrelled.*

Proof. Let $\Sigma = \Sigma(\omega, r)$ be a scarce copy of ω. If K is a $\sigma(\varphi, \Sigma)$-bounded subset of φ, then

$$\sup_{a \in K} |a_j| < \infty \qquad (j = 1, 2, \cdots)$$

since $\Sigma \supseteq \varphi$. Moreover, it is not difficult to see that there exists a positive integer j_0 such that

$$\sup_{a \in K} |a_j| = 0 \qquad (j \geqq j_0) .$$

Thus the topologies $\sigma(\varphi, \Sigma)$ and $\sigma(\varphi, \omega)$ define the same bounded sets and the desired result follows as in the proof of the corollary to Theorem 6.

Theorem 7 may be interpreted as an inclusion result, viz:

COROLLARY. *If an FK-space E contains a scarce copy of ω, then $E = \omega$.*

Proof. Noting that every scarce copy of ω is dense in ω, we may apply Theorem 1, (i) $\Rightarrow$ (ii), of [5] (i.e., the closed graph theorem for barrelled spaces).

In a similar fashion it is possible to establish the following results and we leave the details to the reader.

THEOREM 8. *Every scarce copy of $\bigcap_{p>0} \ell^p$ is barrelled as a subspace of ℓ. In particular, every scarce copy of ℓ is barrelled.*

COROLLARY. *An FK-space contains a scarce copy of $\bigcap_{p>0} \ell^p$ if and only if it contains ℓ.*

Unfortunately, none of the other sequence spaces discussed in this paper ($\ell^p, 1 < p < \infty, bv_0, bv, c_0, c, m$) possesses this property enjoyed by ω and ℓ. Indeed, it can readily be shown that if X is one of these spaces, the *no* scarce copy of X is barrelled. This observation, along with Theorem 1, (i) $\Leftrightarrow$ (iii), of [5], leads to the following existence result.

THEOREM 9. *If X is one of the spaces $\ell^p, 1 < p < \infty$, bv_0, bv, c_0, c or m, and $\Sigma(X, r)$ is a scarce copy of X, then there exists a BK-space Y with*

$$\Sigma(X, r) \subsetneqq Y \subsetneqq X .$$

However, we do have the following result for AK-spaces. (Recall that a topological sequence space (E, τ) containing φ is said to be an *AK-space* provided that $\{\sum_{j=1}^{n} x_j e^j\}_{j=1}^{n}$ is τ-convergent to x, for each $x \in E$).

THEOREM 10. *Let E be a solid FK-AK-space and let $\Sigma(E)$ denote the union of all the scarce copies of E. Then $\Sigma(E)$ is a barrelled subspace of E.*

Proof. It is an easy consequence of the Banach-Steinhaus theorem that the dual of E may be identified with the space E^α, where

$$E^\alpha = \{y \in \omega: \sum_{j=1}^{\infty} |x_j y_j| < \infty, \text{ for each } x \in E\} .$$

It follows that the FK-topology on E is just $\tau(E, E^\alpha)$, so that $(E, \tau(E, E^\alpha))$ is barrelled. Furthermore, $\Sigma(E)$ may be identified with the sequences $\{\lambda_j x_j\}_{j=1}^{\infty}$ where $x \in E$ and λ satisfies $\lambda_j = 0$ or 1, $j = 1, 2, \cdots$, and $\lim_{n\to\infty} 1/n \sum_{j=1}^{n} \lambda_j = 0$. Thus, from a result of Webb ([26], Lemma D(1), p. 360), $\Sigma(E)$ is $\tau(\Sigma(E), E^\alpha)$-barrelled. (Note that Webb's result was stated only for perfect sequence spaces—the extension to the solid case is routine.) Now, just as in the proof of Theorem 5, we see that $\tau(\Sigma(E), E^\alpha) = \tau(E, E^\alpha) \mid \Sigma(E)$, and it follows that $\Sigma(E)$ is a barrelled subspace of E.

COROLLARY. *Let E be a given solid FK-AK-space. Then an FK-space contains every scarce copy of E if and only if it contains E.*

6. **H^p, $0 < p < 1$, and its containing Banach space.** In this section we consider the Hardy space H^p, $0 < p < 1$, of functions f which are analytic in the open unit disk and which satisfy

$$\|f\|_{H^p} = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty .$$

It is a well-known result of Walters ([25], Theorem) that H^p is a Frechet space under the paranorm $\|\cdot\|_p$. We shall also be interested in the space B^p, $0 < p < 1$, of functions analytic in the disk which satisfy

$$\|f\|_{B^p} = \frac{1}{2\pi} \int_0^1 (1 - r)^{1/p-2} \left(\int_0^{2\pi} |f(re^{i\theta})| \, d\theta \right) dr < \infty .$$

For $0 < p < 1$, it is known that $H^p \subset B^p$ ([15], p. 412), that B^p is a Banach space under the norm $\|\cdot\|_{B^p}$ ([10], Theorem 3), and that H^p is dense in B^p ([10], Theorem 3 (iii)).

For a subset E of ω, $\lambda \in \omega$ is said to be a multiplier *of H^p (respectively B^p) into E if whenever*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^p \text{ (respectively } B^p) ,$$

then $\{\lambda_n a_n\}_{n=0}^{\infty} \in E$. The theory of multipliers has been studied extensively by various authors and we refer the reader to [9], §6.4, for the details. The purpose of this section is to show that H^p and B^p 'nearly always' have the same multipliers, thus answering a question raised implicitly by the work of Duren, Romberg and Shields in the papers [10], [11] and [12].

For the remainder of this section it will be convenient for us to regard H^p and B^p as sequence spaces. This we do, as usual, by identifying functions analytic in the disk with their Taylor coefficients.

PROPOSITION 1. *Under the identification mentioned above, B^p is a BK-space and H^p a Frechet K-space.*

Proof. It follows from the remarks made above that B^p is a Banach sequence space. To show that the coordinate mappings are continuous, we consider

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in B^p .$$

Fixing n, we have

$$\begin{aligned} |a_n|\beta\left(n+1, \frac{1}{p}-1\right) &= \int_0^1 r^n (1-r)^{1/p-2}|a_n|\,dr \\ &\leqq \frac{1}{2\pi}\int_0^1 (1-r)^{1/p-2}\left(\int_0^{2\pi}|f(re^{i\theta})|\,d\theta\right)dr \\ &\quad \text{by the Cauchy integral formula} \\ &= \|f\|_{B^p} . \end{aligned}$$

Consequently, B^p is a BK-space.

That H^p is a K-space is a well-known result ([25], Theorem 6) due to Walters.

We note here that the constant $\beta(n+1, 1/p-1)$ given in the proof of Proposition 1 is best possible. This can be seen by considering the functions $f(z) = 1, z, z^2$, etc. As a corollary, we obtain the following slight sharpening of a result of Duren, Romberg and Shields and Shields ([10], Theorem 4).

COROLLARY. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in B^p$, then*

$$|a_n| \leqq \left(\frac{1}{p}-1\right)(n+1)^{1/p-1}\|f\|_{B^p} .$$

Proof. Note that

$$\beta\left(n+1, \frac{1}{p}-1\right) = \frac{\Gamma(n+1/p)}{\Gamma(n+1)\Gamma(1/p-1)}$$
$$= \frac{(1/p-1)1/p(1/p+1)\cdots(1/p+n-1)}{n!} \leqq (1/p-1)(n+1)^{1/p-1}.$$

Our next result is based on idea of Wilansky ([27], Example 1.2, p. 45).

LEMMA 1. *Let (E, τ) be a metrizable locally convex space and let F be a subspace of E. Suppose there exists a Frechet topology $\tau_1 \geqq \tau|F$ on F with the property that every τ_1-continuous linear functional is $\tau|F$-continuous. Then $(F|\tau|F)$ is barrelled.*

Proof. Given a barrel, V, in $(F, \tau|F)$, we are required to show that V is a $\tau|F$-neighborhood of zero. Now V, being absorbent, we have $F = \bigcup_{n=1}^{\infty} nV$. Each nV, being $\tau|F$-closed, must also be τ_1-closed, and it follows from the Baire category theorem that one of them must contain a τ_1-open set. Thus V itself is a convex τ_1-neighborhood of zero.

We complete the proof by showing that every *convex* τ_1-neighborhood of zero is a $\tau|F$-neighborhood of zero. To do this, we consider the locally convex topology τ_2 generated by τ_1. (A base for this topology is formed by taking convex covers of τ_1-neighborhoods of zero.) Clearly, we have

$$\tau|F \leqq \tau_2 \leqq \tau_1 \tag{2}$$

and so our hypothesis imply that $(F, \tau_2)' = (F, \tau|F)'$. But τ is metrizable so that ([21], p. 82) $\tau|F$ is just the Mackey topology. Consequently

$$\tau_2 \leqq \tau|F, \tag{3}$$

and the inclusion relationships (2) and (3) give the desired result.

THEOREM 11. *H^p is a barrelled subspace of B^p, $0 < p < 1$.*

Proof. We take $E = B^p$ and $F = H^p$ in the lemma. That $\tau_1 \geqq \tau|F$ follows from Proposition 1 and the closed graph theorem for complete metric spaces (see, for example, [17], §15.12(3)). That τ_1 and $\tau|F$ define the same continuous linear functionals is just Theorem 7 of [10].

We note that the proof of the lemma also gives the following information—*the locally convex topology on H^p generated by $\|\cdot\|_{H^p}$ is just the subspace topology from B^p.* This observation leads to a slight improvement of a result ([18], p. 614) of Livingston—*the convex cover of a $\|\cdot\|_{H^p}$-ball contains an entire $\|\cdot\|_{B^p}$-ball.*

We are now in a position to state the main result of this section.

THEOREM 12. *An FK-space contains H^p if and only if contains B^p.*

Proof. The sufficiency being obvious, we restrict attention to the necessity. But this follows as in the proof of the corollary to Theorem 7, since, by Theorem 11 and the remarks made at the beginning of this section, H^p is a dense barrelled subspace of B^p.

COROLLARY 1. *Let A be a matrix and E an FK-space. Then A maps H^p into E if and only if A maps B^p into E.*

Proof. A maps H^p into $E \iff H^p \subseteqq E_A$
$\iff B^p \subseteqq E_A$ (by Theorems 1 and 12)
$\iff A$ maps B^p into E.

Our next result answers a question raised implicitly by the work of Duren, Romberg and Shields in [10], [11] and [12].

COROLLARY 2. *H^p and B^p have the same multipliers into any FK-space.*

Proof. This follows at once from Corollary 1 noting that coordinaterwise multiplication by λ is given by means of the matrix A, where $a_{ij} = \delta_{ij}\lambda_j$.

Finally, we indicate how Theorem 12 may be used to give new estimates for the Taylor coefficients of B^p functions. If $0 < p < 1$ and

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^p ,$$

well-known results of Hardy and Littlewood ([9], Theorems 6.2 and 6.4) assert that

$$\sum_{n=1}^{\infty} n^{p-2} |a_n|^p < \infty , \tag{4}$$

and

$$a_n = 0(n^{1/p-1}) . \tag{5}$$

We note here that (5) does *not* follow from (4), contrary to the assertion made on page 421 of [13]. Duren, Romberg and Shields ([11], p. 259) point out that (4) may fail for B^p functions, but ([10], Theorem 4) that (5) carries over to this larger class. (The latter observation follows at once from Theorem 12.) Combining (4) and (5) give a whole family of mutually independent inequalities, namely

$$(6) \qquad \sum_{n=1}^{\infty} n^{(pq-p-q)/p} |a_n|^q < \infty ,$$

whenever $q > p$.

Duren and Shields ([11], p. 259) have observed that (6) carries over to B^p functions if $q = 1$, but fails whenever $q < 1$. We complete the picture by establishing

COROLLARY 3. *If* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in B^p$, *then*

$$(7) \qquad \sum_{n=1}^{\infty} n^{(pq-p-q)/p} |a_n|^q < \infty$$

whenever $q \geqq 1$.

Proof. Letting $E = \{x \in \omega: \sum_{n=1}^{\infty} n^{(pq-p-q)/p} |a_n|^q < \infty\}$, it is clear that E is a BK-space whenever $q \geqq 1$. Inequality (7) asserts that $H^p \subseteqq E$; Theorem 12 gives $B^p \subseteqq E$, and inequality (7) is established.

Added in proof. In a private communication, Professor P. L. Duren has kindly pointed out that Corollary 3 also follows from the special case $q = 1$ and the inequality (5).

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