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UNIFORM INTEGRABILITY OF DERIVATIVES ON σ-LATTICES

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This paper contains a new derivation of the Radon-Nikodym derivative on a σ-lattice. Absolute continuity is defined in this setting, and the definition is justified by obtaining an extension of a standard result on the uniform integrability of derivatives. An application to mean convergence of martingales and a version of Jensen's inequality are given.

Let $(\Omega, \mathscr{F}, P)$ be a probability space, φ a finite signed measure on $\mathscr{F}$, and $\mathscr{M}$ a σ-lattice of elements of $\mathscr{F}$. By this, we mean $\mathscr{M}$ contains Ω and the empty set $\emptyset$, and is closed under countable unions and countable intersections. We denote by $\mathscr{M}^c$ the collection of complements of elements of $\mathscr{M}$. An extended real valued function X on Ω is said to be measurable with respect to $\mathscr{M}$ (we will write $X \in \mathscr{M}$) if sets of the form $\{X \geqq a\} = \{\omega \mid X(\omega) \geqq a\}$ are in $\mathscr{M}$ for every real a.

In [3] it is shown that there exists an a.s. unique (a.s. with respect to P) function $X \in \mathscr{M}$ such that for every real a,

$$\varphi(A \cap \{X \geqq a\}) \geqq aP(A \cap \{X \geqq a\}) \qquad A \in \mathscr{M}^c$$
$$\varphi(A \cap \{X \leqq a\}) \leqq aP(A \cap \{X \leqq a\}) \qquad A \in \mathscr{M}\ .$$

X is called the derivative of φ with respect to P on $\mathscr{M}$. We denote this by $X = D(\varphi, \mathscr{M})$.

In this paper we present a new proof of the existence of this derivative X, based on the observation that the set $\{X \geqq a\}$ maximizes $\varphi(A) - aP(A)$ as A runs over $\mathscr{M}$. We introduce what appears to be suitable definition of absolute continuity in the present setting. Our main result is that the collection of derivatives $\{D(\varphi, \mathscr{M}_0)\}$, for $\mathscr{M}_0 \subseteqq \mathscr{M}$, is uniformly integrable if and only if φ is absolutely continuous with respect to P on $\mathscr{M}$. As an application of this, we can strengthen a result in [2] concerning mean convergence of martingales over σ-lattices.

Finally, we prove a version of Jensen's inequality for the present setting. This enables one to obtain very easily the results in [2].

The σ-lattice $\mathscr{M}$ is a σ-field if and only if $\mathscr{M} = \mathscr{M}^c$. It will be noted that in all that follows, if $\mathscr{M}$ is a σ-field the standard theory is obtained. We have here direct generalizations of the classical theory. What is remarkable is that little extra work seems to be entailed.

2. The derivative. We begin by defining:

$$\begin{aligned}(1)\qquad \varphi^+ &= \sup\{\varphi(A) \mid A \in \mathscr{M}\}\\ \varphi^- &= \inf\{\varphi(A) \mid A \in \mathscr{M}^c\}\,.\end{aligned}$$

Then the following holds for $A_1 \in \mathscr{M}^c$, $A_2 \in \mathscr{M}$:

$$(2)\qquad \varphi(A_1) - \varphi^- \geqq \varphi(A_1 \cap A_2) \geqq \varphi(A_2) - \varphi^+\,.$$

To establish the right hand inequality, we note that $A_1^c \cap A_2 \in \mathscr{M}$, hence $\varphi(A_1^c \cap A_2) \leqq \varphi^+$. Then $\varphi(A_1 \cap A_2) = \varphi(A_2 - \varphi(A_1^c \cap A_2) \geqq \varphi(A_2) - \varphi^+$. The left hand inequality follows similarly.

THEOREM 1. (a) *There is a $A \in \mathscr{M}$ such that $\varphi(A) = \varphi^+$.* (b) *There is a $A \in \mathscr{M}^c$ such that $\varphi(A) = \varphi^-$.*

Proof. For each $n \geqq 1$, choose $A_n \varepsilon\, \mathscr{M}$ such that $\varphi(A_n) \geqq \varphi^+ - 2^{-n}$. Then

$$\begin{aligned}\varphi\Big(\bigcup_{m=n}^{\infty} A_m\Big) &= \varphi(A_n) + \sum_{m=n+1}^{\infty} \varphi(A_m - (A_n \cup \cdots \cup A_{m-1}))\\ &\geqq \varphi(A_n) + \sum_{m=n+1}^{\infty} (\varphi(A_m) - \varphi^+)\\ &\geqq \varphi^+ - \sum_{m=n}^{\infty} 2^{-m} = \varphi^+ - 2^{-m+1}\,.\end{aligned}$$

Let $A = \limsup A_n$, and by continuity of φ, $\varphi(A) = \varphi^+$.

The proof of (b) is similar.

COROLLARY 1. $\varphi(\Omega) = \varphi^+ + \varphi^-$.

Proof. Select $A \in \mathscr{M}$ so that $\varphi(A) = \varphi^+$. Then $\varphi(\Omega) \geqq \varphi^+ + \varphi^-$. The reverse inequality is obtained in an obvious way.

COROLLARY 2. *For $A \in \mathscr{M}$, $\varphi(A) = \varphi^+$ if and only if $\varphi(A^c) = \varphi^-$.*

Proof. If $A \in \mathscr{M}$ and $\varphi(A) = \varphi^+$, then $\varphi(A^c) = \varphi(\Omega) - \varphi(A) = \varphi(\Omega) - \varphi^+ = \varphi^-$.

THEOREM 2. *If $A \in \mathscr{M}$, the following are equivalent:*

(a) $\varphi(A) = \varphi^+$

(b) $\varphi(A_0 \cap A) \geqq 0$ *for all* $A_0 \in \mathscr{M}^c$, *and* $\varphi(A_0 \cap A^c) \leqq 0$ *for all* $A_0 \in \mathscr{M}$.

Proof. Assume (a). Then $\varphi(A^c) = \varphi^-$. The first inequality of (b) follows from the right hand inequality of (2), and the second from the left hand.

Conversely, assume (b) is valid for $\Lambda \in \mathscr{M}$. Select $\Lambda_1 \in \mathscr{M}$ so that $\varphi(\Lambda_1) = \varphi^+$. Then $\varphi(\Lambda^c \cap \Lambda_1) \geqq 0$ from (2) and $\varphi(\Lambda^c \cap \Lambda_1) \leqq 0$ from (b), hence $\varphi(\Lambda^c \cap \Lambda_1) = 0$. Similarly, $\varphi(\Lambda \cap \Lambda_1^c) = 0$, and so $\varphi(\Lambda) = \varphi(\Lambda_1) = \varphi^+$. This completes our proof.

DEFINITION. A function $X \in \mathscr{M}$ is a derivative of φ with respect to P if for every real a,

$$\begin{aligned} \varphi(\Lambda \cap \{X \geqq a\}) &\geqq aP(\Lambda \cap \{X \geqq a\}) && \Lambda \in \mathscr{M}^c \\ \varphi(\Lambda \cap \{X \leqq a\}) &\leqq aP(\Lambda \cap \{X \leqq a\}) && \Lambda \in \mathscr{M} . \end{aligned}$$

For any real a, let $\varphi_a = \varphi - aP$, and define φ_a^+ and φ_a^- by (1), using φ_a. Since for each fixed $\Lambda \in \mathscr{F}$, $\varphi_a(\Lambda)$ is nonincreasing, so are φ_a^+ and φ_a^-.

COROLLARY. *$X \in \mathscr{M}$ is a derivative of φ with respect to P on $\mathscr{M}$ if and only if $\varphi_a(\{X \geqq a\}) = \varphi_a^+$ for every real a.*

The reader is referred to [4], p. 108, for a classical version of the following.

THEOREM 3. *A derivative of φ with respect to P on $\mathscr{M}$ exists, and is a.s. unique.*

Proof. Define a collection K of functions by setting

$$K = \{X \in \mathscr{M} \mid \varphi_a(\Lambda \cap \{X \geqq a\}) \geqq 0 \text{ all } \Lambda \in \mathscr{M}^c, \text{ all real } a\} .$$

For any real a, we can choose $\Lambda_a \in \mathscr{M}$ so that $\varphi(\Lambda_a) = \varphi_a^+$. Let

$$\begin{aligned} X_a(\omega) &= a && \omega \in \Lambda_a \\ &= -\infty && \omega \in \Lambda_a^c. \end{aligned} \tag{3}$$

Then we claim $X_a \in K$. For suppose $b > a$. Then $\{X_a \geqq b\} = \varnothing$, so $\varphi_a(\Lambda \cap \{X_a \geqq b\}) = 0$. On the other hand, if $b \leqq a$, then $\{X_a \geqq b\} = \Lambda_a$, and $\varphi_b(\Lambda \cap \{X_a \geqq b\}) \geqq \varphi_a(\Lambda \cap \{X_a \geqq b\}) 0$ by Theorem 2.

We now note that if $X \in K$, $Y \in \mathscr{M}$ and $X \geqq Y$, then for any $\Lambda \in \mathscr{M}^c$,

$$\varphi_a(\Lambda \cap \{X \geqq a\}) = \varphi_a(\Lambda \cap \{Y \geqq a\}) + \varphi_a(\Lambda \cap \{a > Y\} \cap \{X \geqq a\}).$$

Since $\Lambda \cap \{a > Y\} \in \mathscr{M}^c$, the right-most term is nonnegative, hence

$$\varphi_a(\Lambda \cap \{X \geqq a\}) \geqq \varphi_a(\Lambda \cap \{Y \geqq a\}) . \tag{4}$$

Now suppose $X_1, X_2, \cdots$ are elements of K. Let

$$Z = \sup (X_1, X_2, \cdots) .$$

Certainly $Z \in \mathscr{M}$. We define

$$\Lambda_1 = \Omega$$

$$\Lambda_k = \{X_1 \leqq a, \cdots, X_{k-1} \leqq a\} \qquad k > 1 .$$

Then $\Lambda_k \in \mathscr{M}^c$ for each $k \geqq 1$, and

$$\{Z > a\} = \bigcup_{k \geqq 1} \Lambda_k \cap \{X_k > a\} .$$

The terms on the right are pair-wise disjoint, so for any $\Lambda \in \mathscr{M}^c$,

$$\varphi_a(\Lambda \cap \{Z > a\}) = \sum_{k \geqq 1} \varphi_a(\Lambda \cap \Lambda_k \cap \{X_k > a\}) \geqq 0 .$$

That $\varphi_a(\Lambda \cap \{Z \geqq a\}) \geqq 0$ follows by continuity, whence $Z \in K$.

Let Q be a countable dense subset of the reals containing the discontinuities of φ_a^+. Define

$$X = \sup \{X_q \mid q \in Q\}$$

where X_q is defined by (3). From the preceding argument $X \in K$, and from (4),

$$\varphi_q^+ \geqq \varphi_q(X \geqq q) \geqq \varphi_q(X_q \geqq q) = \varphi_q^+$$

for all $q \in Q$. It follows that $\varphi_a(X \geqq a) = \varphi_a^+$, hence X is a derivative. Uniqueness a.s. is implied by the following lemma.

LEMMA 1. *Suppose X, Y are measurable with respect to $\mathscr{M}$, and for all real a,*

$$\varphi_a(\Lambda \cap \{Y \geqq a\}) \geqq 0 \qquad \Lambda \in \mathscr{M}^c$$
$$\varphi_a(\Lambda \cap \{X \leqq a\}) \leqq 0 \qquad \Lambda \in \mathscr{M} .$$

Then $Y \leqq X$ a.s.

Proof. Choose $a < b$. Then

$$0 \leqq \varphi_b(\{X \leqq a < b \leqq Y\}) \leqq \varphi_a(\{X \leqq a < b \leqq Y\}) \leqq 0 .$$

It follows that

$$bP(\{X \leqq a < b \leqq Y\}) = aP(\{X \leqq a < b \leqq Y\})$$

and so $P(\{X \leqq a < b \leqq Y\}) = 0$, from which our result clearly follows.

We now give some elementary properties of the derivative.

LEMMA 2. *Let φ^1, φ^2 be finite signed measures on $\mathscr{F}$, with $\varphi^1 \leqq \varphi^2$. Then $D(\varphi^1, \mathscr{M}) \leqq D(\varphi^2, \mathscr{M})$ a.s.*

Proof. Let $X^k = D(\varphi^k, \mathscr{M})$, $k = 1, 2$. Then

$$\varphi_a^1(\Lambda \cap \{X^2 \leqq a\}) \leqq \varphi_a^2(\Lambda \cap \{X^2 \leqq a\}) \leqq 0$$

for $\Lambda \in \mathscr{M}$, any real a. Since $\varphi_a^1(\Lambda \cap \{X^1 \geqq a\}) \geqq 0$ for $\Lambda \in \mathscr{M}^c$, any real a, the result follows from Lemma 1.

LEMMA 3.

$$\begin{aligned} D(a\varphi + bP, \mathscr{M}) &= aD(\varphi, \mathscr{M}) + b & a \geqq 0 \\ &= aD(\varphi, \mathscr{M}^c) + b & a < 0 . \end{aligned}$$

Suppose $a < 0$, and let $\Psi = a\varphi + bP$, and $X = D(\varphi, \mathscr{M})$. Since $aX + b \in \mathscr{M}^c$, we must show that $\Psi_t^+ = \Psi_t(aX + b \geqq t)$. But

$$\begin{aligned} \Psi_t^+ &= a\varphi^-_{(t-b)/a} \\ &= a\varphi_{(t-b)/a}(X \leqq (t-b)/a) \\ &= a\varphi_{(t-b)/a}(aX + b \geqq t) \\ &= \Psi_t(aX + b \geqq t) . \end{aligned}$$

This proves our assertion. When $a = 0$ or $a > 0$, similar proofs work.

Finally, we note that $|D(\varphi, \mathscr{M})|$ is a.s. finite. For

$$aP(X \geqq a) \leqq \varphi(X \geqq a) \leqq \varphi^+$$

hence

$$P(X \geqq a) \leqq \frac{\varphi^+}{a}$$

and so

$$P(X = \infty) = \lim_{a \to \infty} P(X \geqq a) = 0 .$$

Similarly we can prove $P(X = -\infty) = 0$.

3. **Absolute continuity and uniform integrability.** When $X \in \mathscr{F}$, $\Lambda \in \mathscr{F}$, we will write

$$E(X; \Lambda) = \int_\Lambda X dP$$

if the right hand side is well defined. The following result is in [3], but since the proof is not long, we will include it here.

THEOREM 4. *If $X = D(\varphi, \mathscr{M})$, then for a, b real,*

$$\begin{aligned} \varphi(\Lambda \cap \{a \leqq X \leqq b\}) &\geqq E(X; \Lambda \cap \{a \leqq X \leqq b\}) & \Lambda \in \mathscr{M}^c \\ &\leqq E(X; \Lambda \cap \{a \leqq X \leqq b\}) & \Lambda \in \mathscr{M} . \end{aligned} \tag{5}$$

Proof. For any $n \geqq 1$, choose $a_0, a_1, \cdots, a_n$, with $a_0 = a - 1/n$, $a_{k+1} - a_k = (b - a_0)/n$, $a_n = b$. For $\Lambda \in \mathscr{M}^c$,

$$\begin{aligned}\varphi(\Lambda \cap \{a - \frac{1}{n} < X \leqq b\}) &= \sum_{k=0}^{n-1} \varphi(\Lambda \cap \{a_k < X \leqq a_{k+1}\}) \\ &\geqq \sum_{k=0}^{n-1} a_k P(\Lambda \cap \{a_k < X \leqq a_{k+1}\}) .\end{aligned}$$

As n increases, the right and left hand terms of this inequality approach the corresponding terms of (5). For $\Lambda \in \mathscr{M}$, a similar proof works.

COROLLARY. *$D(\varphi, \mathscr{M})$ is integrable.*

Proof. Since $\Omega \in \mathscr{M} \cap \mathscr{M}^c$, both inequalities in Theorem 4 hold, and we obtain

$$\varphi(\{a \leqq X \leqq b\}) = E(X; \{a \leqq X \leqq b\}) .$$

Our conclusion follows from the fact that $P(|X| < \infty) = 1$ and φ is finite.

DEFINITION. φ is absolutely continuous with respect to P on $\mathscr{M}$ if the following conditions hold: (a) If $\Lambda \in \mathscr{M}$ and $P(\Lambda) = 0$, then $\varphi(\Lambda) \leqq 0$. (b) If $\Lambda \in \mathscr{M}^c$ and $P(\Lambda) = 0$, then $\varphi(\Lambda) \geqq 0$.

THEOREM 5. *The following are equivalent:*
(a) *φ is absolutely continuous with respect to P on $\mathscr{M}$.*
(b) $\varphi(D(\varphi, \mathscr{M}) = \infty) = \varphi(D(\varphi, \mathscr{M}^c) = -\infty) = 0$
(c) $\lim_{a\to\infty} \varphi_a^+ = \lim_{b\to-\infty} \varphi_b^- = 0$
(d) *There is an $X \in \mathscr{M}$ such that*

$$\begin{aligned}\varphi(\Lambda) &\geqq E(X; \Lambda) & \Lambda \in \mathscr{M}^c \\ &\leqq E(X, \Lambda) & \Lambda \in \mathscr{M} .\end{aligned}$$

Proof. Let $X = D(\varphi, \mathscr{M})$. Then since

$$aP(a \leqq X) \leqq \varphi(a \leqq X < \infty)$$

we always have $\lim_{a\to\infty} aP(a \leqq X) = 0$. Now we note

$$\begin{aligned}0 \leqq \varphi_a^+ &= \varphi_a(X \geqq a) \\ &= \varphi(X \geqq a) - aP(X \geqq a) .\end{aligned}$$

Hence

$$\lim_{a\to\infty} \varphi_a^+ = \varphi(X = \infty) .$$

Similarly,

$$\lim_{b\to-\infty} \varphi_b^- = \varphi(X = -\infty) .$$

Hence, (b) and (c) are equivalent.

Now assume (a). Since $P(X = \infty) = 0$, it follows that $\varphi(X = \infty) \leqq 0$. But since $\varphi(X \geqq n) \geqq nP(X \geqq n) \geqq 0$, $\varphi(X = \infty) \geqq 0$ always holds. Hence $\varphi(X = \infty) = 0$. Similarly we can prove $\varphi(X = -\infty) = 0$, hence (b) holds.

Assume (c). Select $\Lambda \in \mathscr{M}$, and suppose $P(\Lambda) = 0$. Then for all a,

$$\varphi(\Lambda) \leqq \varphi_a^+ + aP(\Lambda) = \varphi_a^+$$

hence $\varphi(\Lambda) \leqq 0$. In a similar way we can prove $\varphi(\Lambda) \geqq 0$ if $\Lambda \in \mathscr{M}^c$ and $P(\Lambda) = 0$, so (c) implies (a).

Clearly (d) implies (a). Conversely, if (b) holds, then (d) follows from Theorem 4 by letting $a \to -\infty, b \to \infty$. This completes the proof.

EXAMPLE. Let $(\Omega, \mathscr{F}, P)$denote the unit interval under Lebesgue measure. Let $\mathscr{M}$ denote the class of all intervals of Ω, each of which contains the number 1. Define $\varphi(\{1\}) = -1, \varphi(\{0\}) = 1$, and $\varphi(\Lambda) = 0$ if $\{0, 1\} \cap \Lambda = \emptyset$. Then φ is absolutely continuous with respect to P on $\mathscr{M}$, but is singular with respect to P on $\mathscr{F}$. It is therefore not the case that there exists a function $X \in \mathscr{F}$ such that $\varphi(\Lambda) = E(X; \Lambda)$ for all $\Lambda \in \mathscr{F}$. It is not hard to verify that $D(\varphi, \mathscr{M}) = 0$. a.s.

In what follows, let $\mathscr{M}$ be a fixed σ-lattice, and let $\mathscr{D}$ denote the class of derivatives $X_0 = D(\varphi, \mathscr{M}_0)$, where $\mathscr{M}_0 \subseteqq \mathscr{M}$. For each $X_0 \in \mathscr{D}$, define $\varphi_a^{\pm}(X_0)$ by (1) using $\mathscr{M}_0$. Then

$$\begin{aligned} \varphi_a^+(X_0) &= \varphi_a(X_0 \geqq a) \\ \varphi_a^-(X_0) &= \varphi_a(X_0 \leqq a) \end{aligned}$$

and from the definition of φ^+, it is clear that $\varphi_a^+(X_0) \leqq \varphi_a^+(X)$ and $\varphi_a^-(X_0) \geqq \varphi_a^-(X)$, where $X = D(\varphi, \mathscr{M})$. We now prove our main result:

THEOREM 6. *$\mathscr{D}$ is a uniformly integrable class if and only if φ is absolutely continuous with respect to P on $\mathscr{M}$.*

Proof. For $X_0 \in \mathscr{D}$, for $a > 0$,

$$E(|X_0|; |X_0| \geqq a) = E(X_0; X_0 \geqq a) - E(X_0; X_0 \leqq -a) .$$

Since $\varphi(X_0 = \infty) = \lim_{n\to\infty} \varphi(X_0 \geqq n) \geqq 0$, we obtain from Theorem 4,

$$E(X_0; X_0 \geqq a) \leqq \varphi(\infty > X_0 \geqq a) \leqq \varphi(X_0 \geqq a) .$$

Now,

$$\begin{aligned}\varphi(X_0 \geqq a) &= \varphi(X_0 \geqq a,\ X \geqq b) + \varphi(X_0 \geqq a,\ X < b) \\ &\leqq \varphi(X_0 \geqq a,\ X \geqq b) - bP(X_0 \geqq a,\ X \geqq b) \\ &\quad + bP(X_0 \geqq a,\ X \geqq b) + bP(X_0 \geqq a, \\ &\qquad X < b) \leqq \varphi_b^+(X) + bP(X_0 \geqq a)\ .\end{aligned}$$

Now,

$$aP(X_0 \geqq a) \leqq \varphi(X_0 \geqq a) \leqq \varphi_0^+(X_0) \leqq \varphi_0^+(X)$$

hence,

$$E(X_0,\ \{X_0 \geqq a\}) \leqq \varphi_b^+(X) + \frac{b}{a}\varphi_0^+(X)\ . \tag{6}$$

If φ is absolutely continuous with respect to P on $\mathscr{M}$, then $\lim_{b\to\infty}\varphi_b^+(X) = 0$, and we can get a uniform bound on $E(X_0, \{X_0 \geqq a\})$ for large enough a. In a similar fashion, we can prove $-E(X_0; \{X_0 \leqq -a\})$ is uniformly bounded if φ is absolutely continuous with respect to P on $\mathscr{M}$, proving sufficiency of this condition.

Now if $\mathscr{D}$ is not uniformly integrable, there is an $\varepsilon > 0$ such that for any $a > 0$, there is an $X_0 \in \mathscr{D}$ such that $E(|X_0|, |X_0| > a) > \varepsilon$. Without loss of generality, we can assume this statement holds for the left hand side of (6), hence $\lim_{b\to\infty}\varphi_b^+(X) > 0$, so φ is not absolutely continuous with respect to D on $\mathscr{M}$. This completes our proof.

Application. If $\mathscr{M}_1 \subseteqq \mathscr{M}_2 \subseteqq \cdots$ are σ-lattices, it is shown in [1] that $D(\varphi, \mathscr{M}_n) \to D(\varphi, \mathscr{M})$ a.s., where $\mathscr{M}$ is the minimal σ-lattice containing each $\mathscr{M}_n$, $n \geqq 1$. It follows from Theorem 6 that if φ is absolutely continuous with respect to D on $\mathscr{M}$, this convergence holds in the mean as well. This strengthens a result in [2], where mean convergence of such a sequence is shown to hold when φ is a absolutely continuous with respect to P on $\mathscr{F}$. However, our example above shows this is not the most general situation.

4. Jensen's inequality. Suppose φ is a absolutely continuous with respect to P on $\mathscr{F}$, so that there is a $Y \in \mathscr{F}$ such that $\varphi(\Lambda) = E(Y; \Lambda)$, $\Lambda \in \mathscr{F}$. Let $\mathscr{M} \subseteqq \mathscr{F}$ be a σ-lattice. We define

$$E(Y \mid \mathscr{M}) = D(\varphi, \mathscr{M})\ .$$

and call $E(Y \mid \mathscr{M})$ the conditional expectation of Y given $\mathscr{M}$.

THEOREM 7. *Let F be a nondecreasing, convex function. If $F(Y)$ is integrable, then*

$$F(E(Y \mid \mathscr{M})) \leqq E(F(Y) \mid \mathscr{M}) .$$

If G is a nonincreasing convex function, and $G(Y)$ is integrable, then

$$G(E(Y \mid \mathscr{M})) \leqq E(G(Y) \mid \mathscr{M}^c) .$$

Proof. We prove the second assertion—the first being similar. Let $\{a_n x + b_n; n \geqq 1\}$ be an affine lower envelope for G, with $a_n \leqq 0$ all n. From Lemma 3,

$$a_n E(Y \mid \mathscr{M}) + b_n = E(a_n Y + b_n \mid \mathscr{M}^c)$$

and since $a_n Y + b_n \leqq G(Y)$, it follows from Lemma 2 that

$$a_n E(Y \mid \mathscr{M}) + b_n \leqq E(G(Y) \mid \mathscr{M}^c)) .$$

Our result follows by taking the supremum of the left hand side.

Suppose H is a nonnegative convex function, with $H(0) = 0$. We say $X \in \mathscr{L}(H)$ if $X \in \mathscr{F}$ and $E(H(X)) < \infty$. If we apply Theorem 7 to the functions

$$\begin{aligned} F(x) &= H(x) & x &\geqq 0 \\ &= \ \ 0 & x &< 0 \\ G(x) &= H(x) - F(x) \end{aligned}$$

then the first assertion of the following theorem clearly holds. The second assertion is implied by Theorem 6.

THEOREM 8. If $X \in \mathscr{L}(H)$, then so is $E(X \mid \mathscr{M})$. If $X \in \mathscr{L}(H)$, then the collection $\{H(E(X \mid \mathscr{M}_0))\}$, as $\mathscr{M}_0$ runs over the σ-lattices in $\mathscr{F}$, is uniformly integrable.

Integrability of conditional expectation $E(X \mid \mathscr{M}_0)$, and mean convergence of martingales of form $E(X \mid \mathscr{M}_n)$, where $\mathscr{M}_1 \subseteqq \mathscr{M}_2 \subseteqq \cdots$, can be obtained Theorems 7 and 8, thus obtaining results in [2].

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