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CONDUCTOR, PROJECTIVITY AND INJECTIVITY

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Here we discuss the role of the conductor of a ring extension vis-a-vis the descent of projectivity and injectivity. Regarding the former, the first result says that an injective homomorphism of commutative rings descends projectivity if it does so modulo the conductor. The cheap version—that with noetherian hypotheses—of the descent of projectivity by a finite homomorphism due to Gruson then follows easily. A carbon copy—with the natural modification—of the descent of injectivity is also proved.

The statement of the results follows very closely the lines of similar work of Ferrand ([2]) on flat modules rather than those of the remarkable [3].

1. Conductor and projectivity. Throughout rings will be commutative with identity element. The price to lift the restriction of commutativity would be to load the exposition with expressions like "two-sided ideal", "bi-module", etc., without any real gain, in view of the fact that for the applications the commutativity is critical.

The main result of this section is, almost word for word, the projective analogue of [2]. Rather than using the results already obtained there, at no cost, we will provide complete proofs based on simple calculations.

THEOREM 1.1. *Let $h: A \to B$ be an injective homomorphism of rings, I an ideal of A and E an A-module. Then E is A-projective if and only if $B \otimes_A E$ is B-projective and E/IE is A/I-projective in the following cases*:

(i) *I is also a B-ideal.*

(ii) *I is nilpotent.*

Proof. (i) (1) $\mathrm{Tor}_1^A(A/I, E) = 0$:

We must show that the natural map $I \otimes_A E \to E$ is injective. This follows from the commutative diagram

$$\begin{array}{ccc} I \otimes_A E & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \\ I \otimes_B B \otimes_A E & \longrightarrow & B \otimes_B B \otimes_A E \end{array}$$

where the vertical map on the left is the natural identification while the lower horizontal map is injective by the B-flatness of $B \otimes_A E$.

(2) $\mathrm{Tor}_1^A(B, E) = 0$:

Let

$$(*) \qquad 0 \longrightarrow G \xrightarrow{j} F \longrightarrow E \longrightarrow 0$$

be exact with F A-free. By tensoring it with B we get (from now on unadorned tensor products are taken over A)

$$(**) \qquad 0 \longrightarrow \mathrm{Tor}_1^A(B, E) \longrightarrow B \otimes G \xrightarrow{1 \otimes j} B \otimes F \longrightarrow B \otimes E \longrightarrow 0 .$$

As $B \otimes E$ is B-projective, this sequence splits piecemeal; by tensoring it with B/I over B, we get the exact sequence

$$\begin{aligned} 0 \longrightarrow B/I \otimes_B \mathrm{Tor}_1^A(B, E) &\longrightarrow B/I \otimes_B B \otimes G \\ \longrightarrow B/I \otimes_B B \otimes F &\longrightarrow B/I \otimes_B B \otimes E \longrightarrow 0 \end{aligned}$$

which can also be written

$$\begin{aligned} 0 \longrightarrow \mathrm{Tor}_1^A(B, E)/I.\,\mathrm{Tor}_1^A(B, E) &\longrightarrow B/I \otimes G/IG \\ \longrightarrow B/I \otimes F/IF &\longrightarrow B/I \otimes E/IE \longrightarrow 0. \end{aligned}$$

By (1) it follows then that $Tor_1^A(B, E) = I.\,\mathrm{Tor}_1^A(B, E)$. Let now $x \in \mathrm{Tor}_1^A(B, E)$: we can write $x = \sum a_i \cdot x_i$ with $a_i \in I$, $x_i \in \mathrm{Tor}_1^A(B, E) \hookrightarrow B \otimes G$. It follows thus that x lies in the image of G in $B \otimes G$. But in the diagram

$$\begin{array}{ccc} G & \longrightarrow & F \\ \downarrow & & \downarrow \\ B \otimes G & \longrightarrow & B \otimes F \end{array}$$

the right vertical map is injective as $h: A \to B$ is injective and F is A-free. We then have $x = 0$.

(3) (*) *splits*:

Let ϕ be a splitting for (**), i.e., $\phi(1 \otimes j) = 1$. On the other hand, let ψ be a splitting of

$$0 \longrightarrow G/IG \xrightarrow{j'} F/IF \longrightarrow E/IE \longrightarrow 0 .$$

The projectivity of F yields then a map θ: $F \to G$ such that $\theta j = 1 + g$, with $g: G \to IG$.

From the product

$$(1 - (1 \otimes \theta)(1 \otimes j))(\phi(1 \otimes j) - 1) = 0$$

one gets

$$((1 \otimes g)\phi + (1 \otimes \theta))(1 \otimes j) = 1 .$$

If we restrict the map $(1 \otimes g)\phi + (1 \otimes \theta)$ to F, we get $(1 \otimes g)\phi + \theta)j = 1$ where $(1 \otimes g)\phi + \theta$ is actually a map from F into G. This completes the proof of (i).

(ii) Say $I^n = (0)$ and let $I_i = (I^i . B) \cap A$. Then $I_i^2 \subset I_{i+1}$ and $I_n = (0)$. By passing to A/I_{n-1} ($\hookrightarrow B/I_{n-1}. B$) we reduce the question, by induction, to the case $I^2 = (0)$.

(1) $\mathrm{Tor}_1^A(A/I, E) = 0$:
This can be read off the diagram

$$\begin{tikzcd}
I \otimes E \arrow[r] \arrow[d, equal] & E \arrow[ddd] \\
I \otimes_{A/I} E/IE \arrow[d] & \\
IB \otimes_{A/I} E/IE \arrow[d, equal] & \\
IB \otimes_B B \otimes E \arrow[r] & B \otimes E
\end{tikzcd}$$

where the left vertical map is injective by the A/I-flatness of E/IE while tho lower horizontal map is injective by the B-flatness of $B \otimes E$.

(2) (*) *splits*:

Tensor (*) with A/I and get $\phi: F \to G$ such that $\phi j = 1 + g$, $g: G \to IG$. As $g^2 = 0$, $(1 - g)\phi$ provides the desired splitting map.

REMARKS. (a) In (ii) above it is enough that I be T-nilpotent, for it follows from [3, p. 60] that E is flat and the argument in (ii) yields a splitting. (b) If A is artinian, with I the radical of A, one has that any injective homomorphism descends projectivity. (c) With k a field and $A = k[x_i^2, x_i^3, i = 1, 2, \cdots]$, $B = k[x_i\text{'s}]$ the conductor I of B in A is such that $A/I = k$. Thus the inclusion $A \to B$ descends projectivity.

COROLLARY 1.2. *Let A, B be commutative rings under the conditions of* (i) *above, and let E be an A-module. If $B \otimes E$ (and resp. E/IE) is finitely generated over B (resp.over A/I), then E is finitely generated over A. (Descent of finiteness)*

Proof. Pick n large enough such that there is a sequence

$$A^n \longrightarrow E \longrightarrow C \longrightarrow 0$$

such that $B \otimes C = 0$ and $C/IC = 0$. Then C is A-projective of trace ideal, say, J. However the trace of $B \otimes C$ is then JB. Since $h: A \to B$ is injective, this implies $C = 0$.

THEOREM 1.3. *Let A be a noetherian ring, $h: A \to B$ an injective homomorphism of rings and B finitely generated as A-module. Let E be an A-module; if $B \otimes E$ is B:projective then E is A-projective.*

Proof. It follows the path of [2]. Let I be a largest ideal of A such that E/IE is not A/I-projective. If $I \neq \sqrt{I} = J$, E/JE is A/J-projective and $B/JB \otimes E$ is also B/JB-projective ($A/J \hookrightarrow B/JB$) and by Theorem 1.1 we get a contradiction. Thus we may assume that A is reduced and such that E/LE is A/L-projective for each ideal $L \neq (0)$. We have the canonical inclusion $A \to A' = \Pi(A/P_i)$, P_i running through the minimal primes. Let $B' = \Pi(B/P_iB)$; then $A' \otimes E$ is A'-projective. However the conductor of A'/A is $\neq (0)$ and thus by Theorem 1.1 E is A-projective.

Assume then A to be a domain. We may take $B = A[y]$ and also a domain. Let $ay^n + \cdots + b = 0$ be a least degree equation satisfied by y. Then with $z = ay$ $A \to B_0 = A[z] \to A[y]$, B_0 is a free extension of A and hence it is enough to show that $B_0 \otimes E$ is B_0-projective. From the hypothesis on A it follows that for every ideal $M \neq (0)$ of B_0, $M \cap A \neq (0)$ and $B_0/MB_0 \otimes E$ is B_0/MB_0-projective. Also, since the conductor of B/B_0 is nonzero, a final application of Theorem 1.1 yields the desired conclusion.

2. Conductor and injectivity.

Now we discuss the injective analogue of Theorem 1.1. The existence of injective envelopes is quite essential for the proof.

THEOREM 2.1. *Let $h: A \to B$ be an injective homomorphism of rings, I an ideal of A and E an A-module. Then E is A-injective if and only if $\mathrm{Hom}_A(B, E)$ is B-injective and $\mathrm{Hom}_A(A/I, E)$ is A/I-injective in the following cases:*

(i) *I is also a B-ideal.*

(ii) *I is nilpotent.*

Proof. (i) (1) $\mathrm{Ext}^1_A(A/I, E) = 0$:

Let

$$(***) \qquad 0 \longrightarrow E \xrightarrow{j} F \xrightarrow{\pi} C \longrightarrow 0$$

be exact with F an injective envelope of E. It yields

$$0 \longrightarrow \mathrm{Hom}\,(A/I, E) \longrightarrow \mathrm{Hom}\,(A/I, F) \longrightarrow \mathrm{Hom}\,(A/I, C) \longrightarrow \mathrm{Ext}^1\,(A/I, E) \longrightarrow 0$$

(where the lower A's are dropped without risk of confusion). But $\operatorname{Hom}(A/I, F) = {}_IF = \{x \in F, Ix = 0\}$ is the injective envelope of ${}_IE$ as an A/I-module and thus ${}_IC = \operatorname{Ext}^1(A/I, E)$. We also have the diagram

$$\begin{tikzcd}
\operatorname{Hom}(A, E) \arrow[r] & \operatorname{Hom}(I, E) \arrow[r] & \operatorname{Ext}^1(A/I, E) \arrow[r] & 0 \\
\operatorname{Hom}(B, E) \arrow[r] \arrow[u] & \operatorname{Hom}_B(I, \operatorname{Hom}(B, E)) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

with the lower sequence exact by the B-injectivity of $\operatorname{Hom}(B, E)$. It thus follows that ${}_IC = \operatorname{Ext}^1(A/I, E) = 0$.

(2) $\operatorname{Ext}^1(B, E) = 0$:
The sequence

$$0 \longrightarrow \operatorname{Hom}(B, E) \longrightarrow \operatorname{Hom}(B, F) \longrightarrow \operatorname{Hom}(B, C) \longrightarrow \operatorname{Ext}^1(B, E) \longrightarrow 0$$

splits piecemeal as B-modules. Applying $\operatorname{Hom}_B(B/I, -)$ to it we get

$$0 \longrightarrow \operatorname{Hom}(B/I, E) \longrightarrow \operatorname{Hom}(B/I, F) \longrightarrow \operatorname{Hom}(B/I, C) \longrightarrow \operatorname{Hom}_B(B/I, \operatorname{Ext}^1(B, E)) \longrightarrow 0 .$$

Since ${}_IC = 0$, ${}_I\operatorname{Ext}^1(B, E) = 0$ also. Let $\alpha \in \operatorname{Hom}(B, C)$; say $\alpha(1) = c$. Pick $b \in F$ with $\pi(b) = c$ and choose $\beta\colon A \to F$ such that $\beta(1) = b$. Let now $\phi\colon B \to F$ extend β. We claim that $\pi\phi = \alpha$. For $a \in I$, $x \in F$, $a(\pi\phi - \alpha)(x) = \pi\phi(ax) - \alpha(ax) = 0$. Thus $I.\operatorname{image}(\pi\phi - \alpha) = 0$ and $\pi\phi = \alpha$ as ${}_IC = 0$.

(3) (***) *splits*:
Consider the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}(B, E) \arrow[r, "j'"] \arrow[d] & \operatorname{Hom}(B, F) \arrow[r, "\pi'"] \arrow[d, "\eta"] & \operatorname{Hom}(B, C) \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & \operatorname{Hom}(A, E) \arrow[r, "j"] & \operatorname{Hom}(A, F) \arrow[r, "\pi"] & \operatorname{Hom}(A, C) \arrow[r] & 0
\end{tikzcd}$$

where η and θ are onto by the injectivity of F. Let ϕ be a splitting for the upper sequence; let $c \in C$ and $\alpha \in \operatorname{Hom}(B, C)$ be such that $\theta(\alpha) = C$ that is, α is such that $\alpha(1) = c$. Define $\psi(c) \in \operatorname{Hom}(A, F)$ to be $\eta(\phi(\alpha))$. We claim that ψ is well defined and provides a splitting for the lower sequence. Assume first that $\alpha(1) = 0$: then $I.\alpha(B) = 0$ and thus $\alpha = 0$, and ψ is well defined. Next, we have $\pi(\phi(c)) = \pi(\eta(\phi(\alpha))) = \theta(\pi'(\phi(\alpha))) = c$.

(ii) Reduce first to the case $I^2 = (0)$. It is enough now to show that $\operatorname{Ext}^1(A/I, E) = 0$ for ${}_IE$ is essential in ${}_IF$ and thus ${}_IC = 0$ implies $C = 0$. Just read this off the commutative diagram

$$
\begin{tikzcd}
\operatorname{Hom}(A, E) \arrow[r] & \operatorname{Hom}(I, E) \arrow[r] \arrow[d, equal] & \operatorname{Ext}^1(A/I, E) \arrow[r] & 0 \\
 & \operatorname{Hom}_{A/I}(I, {}_IE) & & \\
 & \operatorname{Hom}_{A/I}(IB, {}_IE) \arrow[u] \arrow[d, equal] & & \\
\operatorname{Hom}_B(B, \operatorname{Hom}(B, E)) \arrow[uuu] \arrow[r] & \operatorname{Hom}_B(IB, \operatorname{Hom}(B, E)) \arrow[r] & 0 &
\end{tikzcd}
$$

where the right vertical map is surjective by the A/I-injectivity of ${}_IE$ and the lower horizontal map is exact by the B-injectivity of $\operatorname{Hom}(B, E)$.

COROLLARY 2.2. *Let $h: A \to B$ be an injective homomorphism of rings, I the conductor of B/A. Then if A/I and B are both noetherian rings and B/I is finitely generated as an A-module, A is noetherian (and hence B is finitely generated as A-module).*

Proof. According to [1, p. 60] it is enough to show that a direct sum $E = \bigoplus E_i$ of injective A-modules is injective. Clearly $\operatorname{Hom}(A/I, \bigoplus E_i)$ is A/I-injective. If $f \in \operatorname{Hom}(B, \bigoplus E_i)$, $f(I) \subset f(A)$ and the finite generation of B/I implies that $f(B)_i$ (=projection of $f(B)$ in E_i) is trivial for almost all i's. Thus $\operatorname{Hom}(B, \bigoplus E_i) = \bigoplus \operatorname{Hom}(B, E_i)$ and Theorem 2.1 takes over.

REFERENCES

1. H. Bass, *Injective dimension in noetherian rings*, Trans. Amer. Math. Soc., **102** (1962), 18-29.

2. D. Ferrand, *Descent de la platitude par un homomorphisme fini.*, C. R. Acad. Sc. Paris, **269** (1969), 946-949.

3. L. Gruson and M. Raynaud, *Critères de platitude et de projectivité*, Inventines Math., **13**, (1971), 1-89.

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