LOCALLY CIRCULAR MINIMAL SETS

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This paper is devoted to an analysis of the class of mini-
mal sets which are constructed in the following way: Start
with a homeomorphism of the circle without periodic points.
It contains a unique minimal set. Take a finite number of
copies of this minimal set. Define a homeomorphism of this
space onto itself by using the original homeomorphism and
a continuous rule to determine in which copy the image lies.
Finally some of the pairs of doubly asymptotic orbits may
be identified. In other words, minimal skew products of a
specific kind of minimal cascade and a finite permutation group
with some trivial identification are studied.

Their structure group is always a direct product of a circle with
a finite cyclic group, and they can be characterized in terms of their
dynamical properties. If no pairs of doubly asymptotic points are
identified, there is a classification theorem based on four invariants.
One of these invariants is an equivalence class of weakly cohomologous
functions. This weakening of the usual cohomology relation gives an
effective test for isomorphism for this class of minimal sets. More-
over, the minimal cohomology classes contain canonical representatives.

1. Finite skew products. A *cascade* is a topological space to-
gether with a homeomorphism of it onto itself. At times we will use the
more general concept of a transformation group, but the phase space
will always be compact and metric. Since for a cascade we want to
write the homeomorphism on the left, we will write all our actions
on the left. We will assume that the reader is familiar with the
proximal relation, the equicontinuous structure relation, the enveloping
semigroup, and the surrounding ideas. Finite sets will be important
and we will denote their cardinality by $|A|$.

Let (X, T) and (Y, H) be transformation groups and let $F: X \times T \to H$ be a continuous mapping satisfying the condition

$$F(x, st) = F(tx, s)F(x, t)$$

for all x in X and for all s, t $\in T$. Then we can define a new trans-
formation group $(X \times Y, T)$ by

$$t(x, y) = (tx, F(x, t)y) .$$

This transformation group will be called the *skew product of (X, T)
and (Y, H) with respect to F*. Bronstein [3] has established the

following:

THEOREM 1.1. *Let (X, T) and (Y, H) be minimal transformation groups and let F be as above. If the proximal relation of (Y, H) is a closed equivalence relation and if y is a distal point of (Y, H), then (x, y) is an almost periodic point of the skew product $(X \times Y, T)$.*

Let (X, ϕ) be a minimal cascade, let $Y = \{1, \cdots, n\}$ with the descrete topology, and let $H = S_n$ be the permutation group on n-symbols. Then (Y, S_n) is a minimal equicontinuous transformation group. A skew product of (X, ϕ) and (Y, S_n) will be called a *finite skew product of (X, ϕ)*. It follows from Theorem 1.1 that finite skew products are pointwise almost periodic. For finite skew products with $T = Z$, the group of integers, the defining equation takes the form

$$(1) \qquad F(x, m + n) = F(\phi^n(x), m)F(x, n) .$$

For convenience we will denote $X \times Y = X \times \{1, \cdots, n\}$ by X_n. The finite skew of (X, ϕ) and (Y, S_n) with respect to F is also a cascade and its generating homeomorphism ϕ_F is given by

$$\phi_F(x, i) = (\phi(x), F(x, 1)i) .$$

Note that $F(x, 1)$ is simply a continuous function from X into S_n. Conversely, let $F \colon X \to S_n$ continuously and extend F to $X \times Z$ by the formulas

$$F(x, 1) = F(x)$$
$$F(x, 0) = e \ (e \text{ denotes the identity of } S_n)$$
$$F(x, n) = F(\phi^{n-1}(x))F(\phi^{n-2}(x)) \cdot \ \cdots \ \cdot F(\phi(x))F(x) \quad \text{for} \quad n > 1$$
$$F(x, n) = [F(\phi^n(x), -n)]^{-1} \quad \text{for} \quad n < 0 .$$

LEMMA 1.2. *Let $F \colon X \to S_n$ continuously be extended to $X \times Z$ by the above formulas. Then the extension satisfies equation (1) and $\phi_F(x, i) = (\phi(x), F(x)i)$ is the generating homeomorphism of a finite skew product.*

Proof. Equation (1) is established by considering the various cases which arise from the possible signs. If m and n are both non-negative, just use the first three formulas. If they are both negative, use the fourth formula to reduce it to the nonnegative case. For the two cases of mixed signs observe that for $n < 0$

$$F(x, n) = F(\phi^n(x))^{-1}F(\phi^{n+1}(x))^{-1} \cdot \ \cdots \ \cdot F(\phi^{-1}(x))^{-1}$$

and then compute.

Henceforth, F will be a continuous function from X into S_n, and whenever we need the above extension to $X \times Z$ we will simply write $F(x, n)$. The skew product of (X, ϕ) and (Y, S_n) with respect to F will be denoted by (X_n, ϕ_F). Note that $\pi(x, i) = x$ is a homomorphism of (X_n, ϕ_F) onto (X, ϕ).

LEMMA 1.3. *If (x, i) and (y, j) are distinct proximal points of (X_n, ϕ_F), then x and y are proximal in (X, ϕ) and $x \neq y$.*

Proof. Obvious.

We will say (Y, ψ) is a *finite extension* of (X, ϕ) if there exists an open homomorphism θ of (Y, ψ) onto (X, ϕ) such that $|\theta^{-1}(x)|$ is a finite constant independent of x. Note that a finite skew product of (X, ϕ) is always a finite extension of (X, ϕ).

THEOREM 1.4. *Let (X, ϕ) be a minimal cascade on a totally disconnected space, and let (Y, ψ) be a given cascade. Then (Y, ψ) is a finite extension of (X, ϕ) if and only if (X, ψ) is a finite skew product of (X, ϕ).*

Proof. Let θ be an open n to one homomorphism of (Y, ψ) onto (X, ϕ). It suffices to find a homeomorphism γ of Y onto X_n such that $\gamma \circ \psi \circ \gamma^{-1}(x, i) = (\phi(x), F(x)i)$ where $F(x) \in S_n$. The continuity of $\gamma \circ \psi \circ \gamma^{-1}$ will guarantee the continuity of F. Let $\{y_1, \cdots, y_n\} = \theta^{-1}(x_0)$, let $U_1, \cdots, U_n$ be disjoint open neighborhoods of the y_i's, let V be any open and closed subset of X which is contained in $\theta(U_1) \cap \theta(U_2) \cap \cdots \cap \theta(U_n)$, and let $W_i = \theta^{-1}(V) \cap U_i$. Then each W_i is open and closed in Y and θ maps each W_i homeomorphically onto V. Cover X by a finite number of open closed and disjoint subsets constructed as above, call them $V_1, \cdots, V_m$ and let $W_{1j}, \cdots, W_{nj}$ denote the decomposition of $\theta^{-1}(V_j)$ into disjoint open and closed subsets of Y which θ maps homeomorphically onto V_j. Set $\gamma(y) = (\theta(y), i)$ provided $y \in W_{ij}$. Then γ is the desired homeomorphism.

COROLLARY 1.5. *Let (X_n, ϕ_F) be a finite skew product of the minimal cascade (X, ϕ) where X is totally disconnected. Then every minimal set of (X_n, ϕ_F) is also a finite skew product of (X, ϕ).*

2. **Finite skew products of circular cascades.** Let K denote the unit circle in the complex plane and let ϕ be a homeomorphism of K onto K. If ϕ has no periodic points, then (K, ϕ) has exactly one minimal set which is either all of K or a Cantor subset C. $K \backslash C = \bigcup_{n=1}^{\infty} (a_i, b_i)$ where (a_i, b_i) is a counterclockwise open interval in K and

$[a_i, b_i] \cap [a_j, b_j] = \varnothing$ when $i \neq j$. The intervals (a_i, b_i) are called complementary intervals of C. We will call all minimal sets coming from homeomorphisms of K without periodic points *circular cascades*, and we will use the terms *transitive* and *intransitive* to distinguish the two types. If (K, ϕ) is a transitive circular cascade, then (K, ϕ) is isomorphic to (K, M_g) when $M_g(z) = gz$ and $g^n \neq 1$ for $n \neq 0$. If $(C, \phi|C)$ is an intransitive circular cascade, then two distinct points are proximal if and only if they are end points of the same complementary interval, the proximal relation equals the equicontinuous structure relation, and the maximal equicontinuous factor is of the form (K, M_g) where $g^n \neq 1$ for $n \neq 0$. If we choose g with $\operatorname{Im} g > 0$, then it is uniquely determined for any circular cascade and will be denoted by $r(\phi)$. So for any circular cascade there exists a homomorphism p of it onto $(K, M_{r(\phi)})$ and $p(x) = p(y)$ if and only if x and y are proximal. Let $T(\phi)$ equal the image of the distal points under p. The set $T(\phi)$ is uniquely determined up to congruence. (Two sets A and B of K are *congruent*, written $A \equiv B$, if there exists a z such that $M_z(A) = B$.) Clearly $M_{r(\phi)}[T(\phi)] = T(\phi)$ and $K \setminus T(\phi)$ is countable. The author [5] has shown that $r(\phi)$ and $T(\phi)$ form a complete set of invariants for circular cascades.

Now let (C, ϕ) be a fixed intransitive circular cascade and let F map C continuously into S_n. We can find complementary intervals $(a_1, b_1), \cdots, (a_p, b_p)$ of C arranged in counterclockwise order on K such that F is constant on $[b_1, a_2] \cap C, [b_2, a_3] \cap C, \cdots, [b_p, a_1] \cap C$ and $F(a_i) \neq F(b_i)$ $i = 1, \cdots, p$. In view of Corollary 1.5 and the pointwise almost periodicity (C_n, ϕ_F), we will assume (C_n, ϕ_F) in minimal. In this section we will be concerned with the proximal and the equicontinuous structure relations of (C_n, ϕ_F).

It is clear from Lemma 1.3 that (x, i) is a distal point of (C_n, ϕ_F) if x is not the end point of any complementary interval of C and (C_n, ϕ_F) is point distal. For the end point of a complementary interval of C, the proximal cell is given by the following lemma.

LEMMA 2.1. *Let (a, b) be a complementary interval of C. Then the following statements hold:*

(a) *There exists $A_+(a, b)$ $\{A_-(a, b)\}$ in S_n such that $(b, A_+(a, b)i)$ $\{(b, A_-(a, b)i)\}$ is the unique point of C_n which is positively $\{$negatively$\}$ asymptotic to (a, i).*

(b) *They are the only points proximal to (a, i) and they need not be distinct.*

(c) *For any integer n*

$$A_+(\phi^n(a), \phi^n(b)) = F(b, n)A_+(a, b)F(a, n)^{-1}$$

$$\{A_-(\phi^n(a), \phi^n(b)) = F(b, n)A_-(a, b)F(a, n)^{-1}\} \, .$$

Proof. (a) Let $P(a)$ be the smallest nonnegative integer such that $n \geqq P(a)$ implies $\phi^n(a) \neq a_i$ $i = 1, \cdots, p$. Clearly $P(a) = P(b)$. It follows that

$$F(\phi^{P(a)}(a), n) = F(\phi^{P(b)}(b), n)$$

for all $n \geqq 0$. Let

$$A_+(a, b) = F(b, P(b))^{-1} F(a, P(a)) \ .$$

Then for $n \geqq P(a)$ we have the second coordinates of $\phi_F^n(a, i)$ and $\phi_F^n(b, A_+(a, b)i)$ equal which implies positive asymptoticity. Similarly one obtains $A_-(a, b)$.

(b) This follows from Corollary 1.5.

(c) First observe that $P(\phi^n(a)) = P(a) - n$ and then apply equation (1).

Let Z_q denote the integers modulo q; i.e., Z/qZ.

LEMMA 2.2. *If ψ is an automorphism of the topological group $K \times Z_q$, then*

$$\psi(z, i) = \begin{cases} (\xi^i z, ki(\mathrm{mod}\ q)) \\ (\xi^i \bar{z}, ki(\mathrm{mod}\ q)) \end{cases}$$

where ξ is a qth root of 1 and k and q are relatively prime.

THEOREM 2.3. *Let (K_q, ψ) be a discrete flow on $K_q = K \times \{1, \cdots, q\}$. If (K_q, ψ) has a dense orbit, then (K_q, ψ) is isomorphic to a unique minimal set of the form $(K \times Z_q, M_\delta)$ where $\delta = (g, 1)$, $g^n \neq 1$ for $n \neq 0$, and $g = \exp(\theta i)$ with $\theta \in (0, \pi/q)$.*

Proof. Clearly $\psi(K \times \{i\}) = K \times \{\rho i\}$ where $\rho \in S_q$; and if ρ is not a q-cycle, there is no dense orbit. Thus $\psi^k(K \times \{1\}) = K \times \{1\}$ if and only if q divides k and $(K \times \{1\}, \psi^q)$ has a dense orbit. Therefore, it is equicontinuous and so (K_q, ψ). We now know that K_q can be given the structure of a monothetic group and there is generator δ such that (K_q, ψ) is isomorphic to (K_q, M_δ).

We can assume that $K_0 = K \times \{1\}$ is the identity component, and that the group structure on K_0 is isomorphic to complex multiplication on K [2, Theorem 2, page 19]. The cosets of K_0 in K_q are $K_0, K_0\delta, K_0\delta^2, \cdots, K_0\delta^{q-1}$ because $\delta^k \in K_0$ if and only if q divides k. There exists h in K_0 such that $h^q = \delta^q$. It follows that $h^{-1}\delta$ is an element of K_q of order q and K_q is isomorphic to $K \times Z_q$. Therefore, (K, ψ) is isomorphic to $(K \times Z_q, M_\delta)$.

Using automorphisms of $K \times Z_q$ we can change δ. From Lemma 2.2 it is easy to see that exactly one image of δ under an automor-

phism satisfies the conditions in the theorem.

THEOREM 2.4. *The maximal equicontinuous factor of (C_n, ϕ_F) is isomorphic to $(K \times Z_q, M_\delta)$ where δ is uniquely determined by the condition $\delta = (g, 1)$, $g^n \neq 1$ for $n \neq 0$ and $g = \exp(\theta i)$ where $\theta \in (0, \pi/q)$.*

Proof. By Theorem 2.3, it is sufficient to show that the maximal equicontinuous factor of (C_n, ϕ_F) is a one-dimensional manifold. Let (Y, ψ) be the maximal equicontinuous factor of (C_n, ϕ_F) and let p denote the canonical projection of (C_n, ϕ_F) onto (Y, ψ). There exists a homomorphism π' of (C_n, ϕ_F) onto $(K, M_{r(\phi)})$ and hence a homomorphism ρ of (Y, ψ) onto $(K, M_{r(\phi)})$ such that $\pi' = \rho \circ p$. The homomorphism $\pi' = p' \circ \pi$ where π is the projection of C_n onto C and $p'(x) = p'(y)$ if and only if x and y are proximal in (C, ϕ). Consequently if x is a distal point of (C, ϕ), then $p^{-1}(p(x, i)) \subset \{(x, i) : i = 1, \cdots, n\}$. Let $E_0[x, i] = p^{-1}(p(x, i)) \cap \{(x, i) : i = 1, \cdots, n\}$ for any x in C. Since each $E_0[x, i]$ consists of points which are distal, ξ an element of the enveloping semigroup of (C_n, ϕ_F) will be injective on $E_0[x, i]$ and it is easy to check that $\xi(E_0[x, i]) \subset E_0[\xi(x, i)]$. It follows by using minimality that

$$|E_0[x, i]| = |E_0[y, j]|$$

for all $x, y \in C$ and $1 \leq i, j \leq n$.

Let x_0 be a distal point of (C, ϕ). To show that Y is a one-dimensional manifold it suffices to show that $p(x_0, 1)$ has an open neighborhood homeomorphic to $(-1, 1)$. Let I be an open interval in K with end points in $T(\phi)$ such that $\pi'(x_0, 1) \in I$. Let $V = p'^{-1}(I) \times \{i : (x_0, i) \in E_0[x_0, 1]\}$. For sufficently small V, $p^{-1}(p(V)) = V$ and $p(V)$ is open in Y. Let R be the restriction of the equicontinuous structure relation of (C_n, ϕ_F) to V. It is immediate that V/R is homeomorphic to $p(V)$. Because $|E_0[x, i]|$ is a constant V/R is also homeomorphic to $\pi'(V) = I$ which completes the proof.

3. **Some general lemmas.** Through out this section (X, T) will be a minimal transformation group and E will denote its equicontinuous structure relation. Let $E[x] = \{y : (x, y) \in E\}$ and let D be the set of distal points of (X, T).

LEMMA 3.1. *If $E[x_1] \subset D$ for some x_1 in X and $|E[x_2]|$ is finite for some x_2 in X, then*

(a) *$E[x'] \subset D$ if and only if $|E[x']| = \min \{|E[x]| : x \in X\}$ and*

(b) *the canonical projection p of X onto X/E is open at x if and only if $x \in D$.*

Proof. (a) Let ξ be an element of the enveloping semigroup of (X, T). Then ξ maps $E[x]$ into $E[\xi x]$. Since ξ is injective on D, $|E[x]| \leq |E[\xi x]|$ when $E[x] \subset D$. Hence $|E[x]| = \min \{|E[x]|: x \in X\}$ whenever $E[x] \subset D$. If $E[x] \not\subset D$, then there exists $x', x'' \in E[x]$ which are proximal. Choose ξ so that $\xi x_1 = x'$. Then x'' is not in the range of ξ and $|E[x_1]| < |E[x]|$.

(b) Use Proposition 2.3 [8].

The transformation group (X, T) is *locally proximally equicontinuous* if for every x in X there exists a neighborhood U of x such that

$$E \cap U \times U = P \cap U \times U$$

where P denotes the proximal relation. Note that if U satisfies this equation then so does tU where $t \in T$ and V when $V \subset U$. Hence for minimal transformation groups it suffices to check it at one point. Clearly this is a weaker notion then proximal equicontinuity [1].

LEMMA 3.2. *The transformation group (X, T) is locally proximally equicontinuous if and only if there exists $\varepsilon > 0$ such that $(x, y) \in E$ and $d(x, y) < \varepsilon$ implies $(x, y) \in P$.*

We will denote the derived set of A, a subset of X, by A'; i.e., $x \in A'$ if and only if given $\varepsilon > 0$ there exists $y \in A$ such that $0 < d(x, y) < \varepsilon$. As usual $\Delta = \{(x, x) \in X \times X\}$.

LEMMA 3.3. *The following are equivalent:*
(a) *$P' \backslash \Delta$ is a closed set in $X \times X$,*
(b) *there exist $\varepsilon > 0$ such that*

$$P' \cap \{(x, y): d(x, y) < \varepsilon\} \subset \Delta$$

(c) *given x there exists an open neighborhood U of x such that $(P \cap U \times U)' \subset \Delta$.*

LEMMA 3.4. *Let R be a closed invariant equivalence relation on X which is contained in P and let $(Y, T) = (X/R, T)$. Then the following statements hold:*
(a) *If (X, T) is locally proximally equicontinuous, then (Y, T) is locally proximally equicontinuous.*
(b) *If P'_X/Δ_X is closed, then P'_Y/Δ_Y is closed.*

Proof. (a) If (Y, T) is not locally proximally equicontinuous, we can find $(y_n, y'_n) \in E_Y \backslash P_Y$ such that $(y_n, y'_n) \rightarrow (y, y)$. Let ρ denote

the canonical homomorphism of (X, T) onto (Y, T) and let $x_n, x'_n \in X$ such that $\rho(x_n) = y_n$ and $\rho(x'_n) = y'_n$. We can assume that (x_n, x'_n) converges to (x, x'). Because $R \subset P$ it is clear that $(x_n, x'_n) \in E_x$. Since $\rho(x) = y = \rho(x')$, $(x, x') \subset R \subset P_X$. Let U be an open set satisfying $E \cap U \times U = P \cap U \times U$. For some τ, τx, $\tau x' \in U$ and hence for large n, τx_n, $\tau x'_n \in U$ and $(x_n, x'_n) \in P_X$. Thus $(y_n, y'_n) \in P_Y$ for large n which is a contradiction.

(b) If $P'_Y \backslash \Delta_Y$ is not closed, we can find $(y_n, y'_n) \in P'_Y \backslash \Delta_Y$ such that $(y_n, y'_n) \to (y, y)$. Since given $(w_1, w_2) \in P_Y$, we can find $(z_1, z_2) \in P_X$ such that $\rho(z_i) = w_i$, we can find $(x_n, x'_n) \in P'_X$ such that $\rho(x_n) = y_n$ and $\rho(x'_n) = y'_n$. We can assume that $(x_n, x'_n) \to (x, x')$. Then $(x, x') \in R \subset P_X$ and $(x, x') \in P'_X$. Becuase $P'_X \backslash \Delta_X$ is closed and $(x_n, x'_n) \in P'_X$ we see that $x \neq x'$. But $P'_X \backslash \Delta_X$ closed also implies that $P'_X \cap P_X \subset \Delta_X$. Thus $x = x'$ and we have a contradiction.

4. **Locally circular minimal sets.** A minimal cascade (Y, ψ) will be called *locally circular* if there exists a minimal finite skew product (C_n, ϕ_F) of a circular cascade and a homomorphism θ of (C_n, ϕ_F) onto (Y, ψ) such that $\theta(x) = \theta(y)$, $x, y \in C_n$, implies x and y are doubly asymptotic. Locally circular is some what of a misnomer because they are not locally circular in the obvious meaning of this term. However, they can be characterized by using localizations of the properties which characterize circular cascades. This characterization is the following:

THEOREM 4.1. *Let (Y, ψ) be a minimal cascade on a compact metric space. Then (Y, ψ) is locally circular if and only if the maximal equicontinuous factor of (Y, ψ) is a one-dimensional manifold, (Y, ψ) is locally proximally equicontinuous, and $P' \backslash \Delta$ is closed subset of $Y \times Y$.*

Proof. It follows from §2 that minimal skew products of circular cascades satisfy the three conditions, and then by Lemma 3.4 so do locally circular cascades.

Now suppose (Y, ψ) has the three properties stated in the theorem. Choose $\varepsilon > 0$ such that $d(x, y) < \varepsilon$ and $(x, y) \in E$ implies $(x, y) \in P$ and such that $d(x, y) < \varepsilon$ and $(x, y) \in P'$ implies $x = y$. If $1/(n - 1) < \varepsilon$, then $P \cap \{(x, y): 1/n < d(x, y) < 1/(n - 1)\}$ is finite. Therefore, $P \cap \{(x, y): 0 < d(x, y) < 1/N\}$ is countable where $1/N < \varepsilon$, and $P \backslash \Delta$ is countable. Since Y/E is uncountable, there exist points such that $E[y] \subset D$. Next we observe that $E[y]$ is finite for all y in Y. If this were not the case we could find a y and a sequence of distinct points $\{y_n\}$ such that y_n converges to y and $y_n \in E[y]$ for all n. For large n and m we would have $d(y_n, y_m) < \varepsilon/2$ and hence $(y_n, y_m) \in P$.

Holding m fixed and letting n go to infinity would show that $(y, y_m) \in P'$ and $d(y, y_m) < \varepsilon$. Hence we would have the contradiction $y = y_m$ for large m. Let p be the canonical projection of Y onto Y/E. By Lemma 3.1. $E[y'] \subset D$ if and only if $|E[y']| = \min \{|E[y]|: y \in Y\} = \alpha$ and p is open at y if and only if $y \in D$.

Since the maximal equicontinuous factor of (Y, ψ) is a one-dimensional manifold, it has the form $(K \times Z_m, M_\delta)$ where $\delta = (h, 1)$ and $h^n \neq 1$ for $n \neq 0$ by Theorem 2.3. Let π be the composition of p and the projection of $K \times Z_m$ onto K. The map π is a homomorphism of (Y, ψ) onto (K, M_h). We now use a construction due to Veech [8, Theorem 3.1] to obtain minimal cascades (X, ϕ) and $(\tilde{X}, \tilde{\phi})$ and homomorphisms q, $\tilde{q}$ and π^* with the following properties:

(a) the diagram

$$
\begin{array}{ccc}
(Y, \psi) & \xleftarrow{\ \tilde{q}\ } & (\tilde{X}, \tilde{\phi}) \\[4pt]
\pi \downarrow & & \downarrow \pi^* \\[4pt]
(K, M_h) & \xleftarrow{\ q\ } & (X, \phi)
\end{array}
$$

is commutative

(b) the map π^* is open

(c) if π is open at every point in $\pi^{-1}(z)$, then $q^{-1}(z)$ consists of exactly one point

(d) if π is open at every point in $\pi^{-1}(\pi(y))$, then $\tilde{q}^{-1}(y)$ consists of exactly one point.

The transformation group (X, ϕ) is constructed by taking the completion of $\{\pi^{-1}(z): \pi^{-1}(z) \subset D\}$ in the Hausdorff metric coming from Y. So a point z^* of X can also be thought of as a subset of Y. In this role we observe that $z^* \subset \pi^{-1}(q(z^*))$ and $|z^*| = \alpha$. Choose $\varepsilon > 0$ such that where $d(z, y) < \varepsilon$ the following hold:

$$(x, y) \in E \quad \text{implies} \quad (x, y) \in P$$

$$(x, y) \in P' \quad \text{implies} \quad x = y$$

$$(x, y) \notin E \quad \text{implies} \quad \pi(x) \neq \pi(y) .$$

Choose $\delta < \varepsilon/2$ such that the open sets $S(y_i, \delta) = \{y: d(y, y_i) < \delta\}$ are disjoint where $\{y_1, \cdots, y_n\} = \pi^{-1}(z) \subset D$ and z is a fixed point of K. Let $U = \bigcap_{i=1}^n \pi(S(y_i, \delta))$ and $U_i = \pi^{-1}(U) \cap S(y_i, \delta)$. Clearly $\pi(U_i) = U$. If $q(z^*) \in U$, then $z^* \cap U_i$ is a point. If $q(z^*) = q(w^*) \in U$, then $(z^* \cap U_i, w^* \cap U_i) \in P$. Suppose that (z_n^*, w_n^*) is a sequence in $X \times X$ converging to $(z^* w^*)$ and $(z_n^*, w_n^*) \in P$. We have $q(z_n^*) = q(w_n^*)$ and $q(z^*) = q(w^*)$. We can assume $q(z^*) \in U$. If $z^* \neq w^*$, then $z^* \cap U_i \neq w^* \cap U_i$ for some i. It is easy to see that $z_n^* \cap U_i \to z^* \cap U_i$ and $w_n^* \cap U_i \to w^* \cap U_i$. Therefore, $(z^* \cap U_i, w^* \cap U_i) \in P'$ which contradicts

$z^* \neq w^*$ because $d(z^* \cap U_i, w^* \cap U_i) < \varepsilon$. Hence for (X, ϕ), $P' \subset \Delta$ and (X, ϕ) is a circular cascade [5, Theorem 4.1]. If (X, ϕ) is transitive, then $(X, \phi)$is equicontinuous, q is an isomorphism, (Y, π) is a covering space of K, (Y, ψ) is isomorphic to some $(K \times Z_k, M_\delta)$ which is locally circular. If (X, ϕ) is intransitive, then π^* is n to one where $n = m\alpha = |\tilde{q}^{-1}(\pi^{-1}(z))|$ if $\pi^{-1}(z) \subset D$. To finish the proof we must show that $\tilde{q}(x_1) = \tilde{q}(x_2)$ implies x_1 and x_2 are doubly asymptotic. If $\tilde{q}(x_1) = \tilde{q}(x_2)$, then either $\pi^*(x_1) = \pi^*(x_2)$ or $\pi^*(x_1)$, $\pi^*(x_2)$ are proximal. The former is impossible because $\tilde{q}$ maps $\pi^{*-1}(z^*)$ homeomorphically onto z^* [8, 219]. (Here z^* is first regarded as a point of X and then as a subset of Y.) Let $A_i = \pi^{*-1}(\pi^*(x_i))$. Then $A_1 \cap A_2 = \varnothing$. There exist $y \in A_2$ such that x_1, y are positively {negatively} asymptotic. (Theorem 1.4 and Lemma 2.1.) We can assume that $\tilde{q}(x_1), \tilde{q}(y) \in U_1$ where $U_1, \cdots, U_n$ are as above. Then $\tilde{q}(y), \tilde{q}(x_2) \in U_1 \cap z_2^*$, $z_2^* = \pi^*(x_2)$ implies $\tilde{q}(y) = \tilde{q}(x_2)$ and hence $y = x_2$. Therefore, (x_1, x_2) are doubly asymptotic, and the proof is completed.

5. **Invariants.** Let (Y, ψ) be a locally circular minimal set. From Theorem 2.4 we know that its maximal equicontinuous factor is of the form $(K \times Z_q, M_\delta)$ where $\delta = (h, 1)$. Moreover, δ is unique if we require that $h = e^{i\theta}$ where $0 < \theta < \pi/q$. We call this h the *rotation number* of (Y, ψ) and denote it by $r(\psi)$. It is a well-defined invariant of (Y, ψ). Clearly q which is the number of components of the maximal equicontinuous factor is an invariant, which we denote by $c(\psi)$. By Lemma 3.1 we know that $|E[y]|$ where $E[y] \subset D$ is also a well-defined invariant of (Y, ψ). We denote it by $e(\psi)$ and we let $n(\psi) = c(\psi)e(\psi)$. The positive integer $n(\psi)$ will be the canonical number of symbols in the permutation group. We also need a canonical circular cascade from which we can recapture (Y, ψ) by taking a finite skew product with $n(\psi)$ symbols and identifying some doubly asymptotic points. A circular cascade (X, ϕ) is uniquely determined up to isomorphism by two invariants-the rotation number $r(\phi)$ and a subset $T(\phi)$ of K [5]. Certainly we want to have $r(\phi) = r(\psi)$. So we must have a way of using (Y, ψ) to determine a suitable subset of K. A slight modification of the original definition of $T(\phi)$ does this.

LEMMA 5.1. *Let ρ_1 and ρ_2 be homomorphisms of (Y, ψ) onto $(K, M_{r(\psi)})$. Then there exists a $w \in K$ such that $\rho_1 = M_w \circ \rho_2$.*

Proof. Let p be a canonical projection of (Y, ψ) onto its maximal equicontinuous factor $(K \times Z_q, M_\delta)$ and let π be the composition of p and the projection of $K \times Z_q$ onto K. It suffices to show that there exists $w \in K$ such that $\pi = M_w \circ \rho_1$. There exists a homomorphism ρ'

of $(K \times Z_q, M_\delta)$ onto $(K, M_{r(\psi)})$ such that $\rho' \circ p = \rho_1$. If we modify ρ_1 by a rotation we can assume ρ' is also a group homomorphism of $K \times Z_q$ onto K. In other words, there exists $w \in K$ and a group homomorphism of $K \times Z_q$ onto K such that $M_w \circ \rho_1 = \rho' \circ p$ and $\rho' \circ M_\delta = M_{r(\psi)} \circ \rho'$. So it suffices to show that $\rho'(z, i) = z$.

It is easy to see that $\rho'(z, i) = \xi^i z^n$ for some $n \in Z$ and ξ a qth root of unity. We also have $\rho'(\delta) = \rho'(r(\psi), 1) = r(\psi)$, and hence we have $\xi r(\psi)^n = r(\psi)$. If $n \neq 1$, $r(\psi)$ is a root of unity which is impossible because $r(\psi)$ is a generator of K (Theorem 2.4). If $n = 1$, $\xi = 1$ and $\rho'(z, i) = z$.

Let ρ be any homomorphism of (Y, ψ) onto $(K, M_{r(\psi)})$ and let $T(\psi) = \{z \in K : \rho^{-1}(z) \subset D\}$. Suppose θ is an isomorphism (Y, ψ) onto (Y', ψ') and ρ' is a homomorphism of (Y', ψ') onto $(K, M_{r(\psi)})$. (We already know that $r(\psi) = r(\psi')$.) Applying Lemma 5.1 to $\rho_1 = \rho$ and $\rho_2 = \rho' \circ \theta$ we obtain $w \in K$ such that $\rho = M_w \circ \rho' \circ \theta$. Since $\rho^{-1}(z) = \theta^{-1}[\rho'^{-1}(w^{-1}z)]$, $\rho^{-1}(z)$ consists of distal points if and only if $\rho'^{-1}(w^{-1}z)$ consists of distal points. Therefore, $\{z : \rho^{-1}(z) \subset D\} \equiv \{z : \rho'^{-1}(z) \subset D\}$ and $T(\psi)$ is a well defined invariant. In addition, $T(\psi)$ is invariant under $M_{r(\psi)}$ and $K \backslash T(\psi)$ is countable. Hence $r(\psi)$ and $T(\psi)$ uniquely determine a circular cascade. Note that if $(Y, \psi) = (C_n, \phi_F)$ where (C, ϕ) is a circular cascade and $F : C \rightarrow S_n$ continuously, it is not true in general that $r(\psi) = r(\phi)$, $T(\psi) = T(\phi)$ and $n(\psi) = n$.

Next we want to connect our invariants with the circular cascade (X, ϕ) and the finite skew product $(\tilde{X}, \tilde{\phi})$ which we constructed in the proof of Theorem 4.1. The starting point for the construction was any homomorphism p of (Y, ψ) onto $(K \times Z_q, M_\delta)$, $\delta = (h, 1)$ such that $p(x) = p(y)$ if and only if $(x, y) \in E$. In this section we will always assume that $\delta = (r(\psi), 1)$. Let ρ be any homomorphism of (Y, ψ) onto $(K, M_{r(\psi)})$. It follows from Lemma 5.1 that we can find p as above such that ρ equals the composition of p and the projection of $K \times Z_q$ onto K. Consequently if we apply the Veech construction to any $\rho : (Y, \psi) \rightarrow (K, M_{r(\psi)})$ we will always get a circular cascade (X, ϕ) and a finite skew product $(\tilde{X}, \tilde{\phi})$ of (X, ϕ) using $n(\psi)$ symbols. The homomorphisms q, $\tilde{q}$ and ρ^* satisfy conditions (a), (b), (c), and (d) in the proof of Theorem 4.1 and $\tilde{q}(x) = \tilde{q}(y)$ implies x and y are doubly asymptotic.

THEOREM 5.2. *Let (Y, ψ) be a locally circular minimal set and let ρ be a homomorphism of (Y, ψ) onto $(K, M_{r(\psi)})$. Suppose (X, ϕ) and $(\tilde{X}, \tilde{\phi})$ are the minimal cascades and q, $\tilde{q}$, and ρ^* are the homomorphisms obtained from Veech's construction [8, Theorem 3.1]. Then the following statements are true:*

(a) *The minimal set (X, ϕ) is a circular cascade, $(\tilde{X}, \tilde{\phi})$ is a finite skew product of (X, ϕ) with $n(\psi)$ symbols, and $\tilde{q}(x) = \tilde{q}(y)$ implies x*

and y are doubly asymptotic.

(b) *The invariants are related by the equations*

$$r(\psi) = r(\tilde{\phi}) = r(\phi)$$

$$T(\psi) = T(\tilde{\phi}) = T(\phi)$$

$$n(\psi) = n(\tilde{\phi}) \; .$$

(c) *The minimal sets* (X, ϕ) *and* $(\tilde{X}, \tilde{\phi})$ *are uniquely determined up to isomorphism by the isomorphism class of* (Y, ψ).
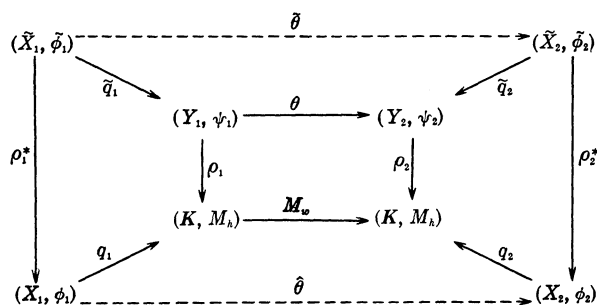
(d) *If* (Y, ψ) *is a finite skew product of a circular cascade, then* $\tilde{q}$ *is an isomorphism of* $(\tilde{X}, \tilde{\phi})$ *onto* (Y, ψ).

Proof. (a) This was established in the preceding paragraph.

(b) Because $\tilde{q}(x) = \tilde{q}(y)$ implies x and y are doubly asymptotic, (Y, ψ) and $(\tilde{X}, \tilde{\phi})$ have the same maximal equicontinuous factor and we can find a homomorphism ρ' such that $\rho' = \rho \circ \tilde{q}$. It follows that $r(\psi) = r(\tilde{\phi})$, $T(\psi) = T(\tilde{\phi})$, and $n(\psi) = n(\tilde{\phi})$.

Since there are points such that $|q^{-1}(z)| = 1$, it follows that $q(x) = q(y)$ if and only if x and y are proximal. Thus $r(\psi) = r(\phi)$. Using Lemma 3.1 we see that $\rho^{-1}(z) \subset D$ if and only if $|q^{-1}(z)| = 1$ which implies $T(\psi) = T(\phi)$.

(c) Let θ be an isomorphism of between the locally circular cascades (Y_1, ψ_1) and (Y_2, ψ_2), and let ρ_i be a homomorphism of (Y_i, ψ_i) onto (K, M_h) where $r(\psi_1) = h = r(\psi_2)$. We must find isomorphisms $\hat{\theta}$ and $\tilde{\theta}$ to complete the following commutative diagram:

$$\begin{array}{ccccc}
(\tilde{X}_1, \tilde{\phi}_1) & \xrightarrow{\quad\tilde{\theta}\quad} & & & (\tilde{X}_2, \tilde{\phi}_2) \\
\end{array}$$

$(\tilde{X}_1, \tilde{\phi}_1) \overset{\tilde{\theta}}{\dashrightarrow} (\tilde{X}_2, \tilde{\phi}_2)$

$\rho_1^* \quad \tilde{q}_1 \quad (Y_1, \psi_1) \xrightarrow{\theta} (Y_2, \psi_2) \quad \tilde{q}_2 \quad \rho_2^*$

$\rho_1 \quad (K, M_h) \xrightarrow{M_w} (K, M_h) \quad \rho_2$

$q_1 \quad q_2$

$(X_1, \phi_1) \overset{\hat{\theta}}{\dashrightarrow} (X_2, \phi_2)$

The M_w is obtained by applying Lemma 5.1. The existence of $\hat{\theta}$ follows from Theorem 2.2 of [5]. Examining Veech's construction one sees that $\tilde{X}$ is the closure of $\{(y, \rho^{-1}[\rho(y)]): \rho^{-1}[\rho(y)] \subset D\}$ in $Y \times Y^Y$ where the metric is the metric on Y plus associated Hausdorff metric and $\tilde{\phi}(y, A) = (\phi(y), \phi(A))$. Finally let $\tilde{\theta}$ be the obvious map.

(d) This follows because the projection of a finite skew product of a circular cascade is either $e(\psi)$ or $2e(\psi)$ to one and hence ρ is either $n(\psi)$ or $2n(\psi)$ to one.

Let (X, ϕ) be a fixed circular cascade, let n be a positive integer, and let $C(X, S_n)$ denote the continuous functions from X into S_n. We say two functions F and G in $C(X, S_n)$ are *cohomologous* if there exists $a \in C(X, S_n)$ such that for all x in X

$$F(x) = a(\phi(x))^{-1} G(x) a(x) \, .$$

It is easy to see that this is an equivalence relation on $C(X, S_n)$. We will call the equivalence classes *cohomology classes of* (X, ϕ) *for* S_n. If F and G are cohomologous, then $\theta(x, i) = (x, a(x)i)$ is an isomorphism of (X_n, ϕ_F) onto (X_n, ϕ_G). To obtain the converse of this we need a slightly weaker notion. One can see this by considering F and $G = F \circ \beta$ where β is an automorphism. Here $\theta(x, i) = (\beta^{-1}(x), i)$ is an isomorphism of (X_n, ϕ_F) onto (X_n, ϕ_G), but if β is properly chosen there is no isomorphism of (X_n, ϕ_F) onto (X_n, ϕ_G) which leaves the x-coordinate fixed. We will say F and G are *weakly cohomologous* if there exists an automorphism β of (X, ϕ) such that F and $G \circ \beta$ are cohomologous. This is also an equivalence relation on $C(X, S_n)$.

THEOREM 5.3. *Let (X, ϕ) be a circular cascade and let $F, G \in C(X, S_n)$ such that (X_n, ϕ_F) and (X_n, ϕ_G) are minimal. Then (X_n, ϕ_F) is isomorphic to (X_n, ϕ_G) if and only if F and G are weakly cohomologous.*

Proof. First suppose that F and G are weakly cohomologous. So there exists $a \in C(X, S_n)$ and β an automorphism such that $F(x) = a(\phi(x))^{-1} G(\beta(x)) a(x)$. Define θ by $\theta(x, i) = (\beta(x), a(x)i)$. Then $\theta \circ \phi_F(x, i) = \theta(\phi(x), F(x)i) = (\beta(\phi(x)), a(\phi(x))F(x)i) = (\phi(\beta(x)), G(\beta(x))a(x)i) = \phi_G(\beta(x), a(x)i) = \phi_G \circ \theta(x, i)$. Clearly θ is a homeomorphism. Hence (X_n, ϕ_F) and (X_n, ϕ_F) are isomorphic.

Now suppose that θ is an isomorphism of (X_n, ϕ_F) onto (X_n, ϕ_G). It follows that their maximal equicontinuous factors are isomorphic. Let $(K \times Z_q, M_\delta)$ denote their maximal equicontinuous factor where $\delta = (h, 1)$, $h = \exp(\alpha i)$, $\alpha \in (0, \pi/q)$. We have the following commutative diagram:

$$
\begin{array}{ccccccc}
(X, \phi) & \xleftarrow{\ \pi\ } & (X_n, \phi_F) & \xrightarrow{\ \theta\ } & (X_n, \phi_G) & \xrightarrow{\ \pi\ } & (X, \phi) \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle p_F} & & \downarrow{\scriptstyle p_G} & & \downarrow{\scriptstyle \pi_2} \\
(K, M_{r(\phi)}) & \xleftarrow{\ \rho_F\ } & (K \times Z_q, M_\delta) & \xrightarrow{\ \tilde{\theta}\ } & (K \times Z_q, M_\delta) & \xrightarrow{\ \rho_G\ } & (K, M_{r(\phi)})
\end{array}
$$

where $\pi(x, i) = x$, p_F, p_G, π_1, and π_2 are canonical projections onto maximal equicontinuous factors, and ρ_F, ρ_G, and $\tilde{\theta}$ are induced by $\pi_1 \circ \pi$, $\pi_2 \circ \pi$ and $p_G \circ \theta$. By changing p_G by a rotation we can assume that $\tilde{\theta}$ is the identity. (If $\tilde{\theta}(1, 0) = (1, 0)$, then $\tilde{\theta}$ is an automorphism

of the topological group $K \times Z_q$ and the identity is the only automorphism which leaves δ fixed.) By changing π_1 and π_2 by rotations we can assume that ρ_F and ρ_G are group homomorphisms. Since $\rho_F(z, 0) = z^k$ for some $k \in Z$ and $\rho_F(1, 1) = \xi$, a qth root of unity, we have $\rho_F(z, i) = z^k \xi^i$. It is easy to see that ρ_F is $|k|q$ to one. Suppose x is a distal point of (X, ϕ). Then $\pi_1^{-1}(\pi_1(x)) = x$ and $\pi^{-1}(x) = (\pi_1 \circ \pi)^{-1}(z)$ consists of n-distal points where $z = \pi_1(x)$. Moreover, $(\pi_1 \circ \pi)^{-1}(z)$ is saturated with respect to the equicontinuous structure relation of (X_n, ϕ_F) and hence ρ_F is $e(\phi_F)$ to one on it. But $(\pi_1 \circ \pi)^{-1}(z) = p_F^{-1}(\rho_F^{-1}(z))$ and hence $|k|qe(\phi_F) = n$ and $|k| = n/qe(\phi_F) = n/c(\phi_F)e(\phi_F) = n/n(\phi_F)$. Similarly $\rho_G(z, i) = z^l \eta^i$ where η is a qth root of unity and $|l| = n/n(\phi_G) = n/n(\phi_F) = |k|$. Since $\rho_F(\delta) = r(\phi) = \rho_G(\delta)$, $\xi h^k = \eta h^k$ or ηh^{-k}. If $\xi h^k = \eta h^{-k}$, then h is a $2kq$th root of unity which is impossible. If $\xi h^k = \eta h^k$, then $\xi = \eta$ and $\rho_F = \rho_G$.

We can write $\theta(x, i) = (\theta_1(x, i), \theta_2(x, i))$ where θ_1 and θ_2 are continuous. We will show that $\theta_1(x, i) = \theta_1(x, j)$ for all x, i and j. Suppose x is a distal point in (X, ϕ). Then $\pi^{-1}(x) = \{x\} \times \{1, \cdots, n\}$ consists of n points and so does $\theta(\pi^{-1}(x))$. Now $\theta(\pi^{-1}(x)) = \theta(\pi_1 \circ \pi^{-1}(z)) = \theta[(\rho_F \circ p_F)^{-1}(z)] = \theta(p_F^{-1}[\rho_F^{-1}(z)] = p_G^{-1}[\rho_F^{-1}(z)] = p_G^{-1}[\rho_G^{-1}(z)] = (\rho_G \circ p_G)^{-1}(z) = (\pi_2 \circ \pi)^{-1}(z) = \pi^{-1}[\pi_2^{-1}(z)]$. From cardinality considerations $\pi_2^{-1}(z)$ is a point. Therefore, $\theta(\pi^{-1}(x)) = \{x'\} \times \{1, 2, \cdots, n\}$ where $x' = \pi_2^{-1}(z)$ for all distal points x. In other words, if x is distal $\theta_1(x, i) = \theta_1(x, j)$ for all i and j. Since the distal points are dense in X and θ_1 is continuous, we obtain $\theta_1(x, i) = \theta_1(x, j)$ for all x, i, and j. We can now write $\theta_1(x, i) = \theta_1(x)$. For each x, $a_1(x)i = \theta_2(x, i)$ defines an element of S_n, and $a_1 \in C(X, S_n)$ because θ_2 is continuous. The equation $\phi_G \circ \theta = \theta \circ \phi_F$ can be written

$$(\phi[\theta_1(x)], G(\theta_1(x))a_1(x)i) = (\theta_1[\phi(x)], a_1(\phi(x))F(x)i) \ .$$

Therefore, $\phi \circ \theta_1 = \theta_1 \circ \phi$ and $G(\theta_1(x))a_1(x) = a_1(\phi(x))F(x)$ which completes the proof.

COROLLARY 5.4. *Let (X, ϕ) be a circular cascade and let F and G be elements of $C(X, S_n)$ such that (X_n, ϕ_F) and (X_n, ϕ_G) are minimal. Then F and G are cohomologous if and only if there exists an isomorphism $\theta: (X_n, \phi_F) \to (X_n, \phi_G)$ of the form $\theta(x, i) = (x, \theta_2(x, i))$.*

Proof. Obvious.

It is clear that an isomorphism between two minimal sets induces a bijection between their weak cohomology classes, and it follows from Theorem 5.3 that two different isomorphisms induce the same bijection. For two weak cohomology classes W_1 and W_2 we will write

$W_1 = W_2$ provided the corresponding ranges are equal, the domains are isomorphic, and W_1 is carried onto W_2 by the induced map.

Let (Y, ψ) be a locally circular cascade and let ρ be a homomorphism of (Y, ψ) onto $(K, M_{r(\psi)})$. Now construct the circular cascade (X, ϕ) and the finite skew product $(\tilde{X}, \tilde{\phi})$ as above. There exists $F \in C(X, S_{n(\psi)})$ such that $(\tilde{X}, \tilde{\phi})$ is isomorphic to $(X_{n(\psi)}, \phi_F)$. Let $W(\psi)$ be the weak cohomology class of F. It follows easily from Theorems 5.2 and 5.3 that $W(\psi)$ is an invariant of (Y, ψ). The invariants $n(\psi)$, $r(\psi)$, $T(\psi)$, and $W(\psi)$ do not uniquely determine (Y, ψ). However, we will show that they do uniquely determine $(\tilde{X}, \tilde{\phi})$, which is itself and invariant of (Y, ψ). In other words, we can classify the minimal finite skew products of circular cascades and hence the locally circular ones except for the identification of some doubly asymptotic orbits.

THEOREM 5.5. *Let (Y_1, ψ_1) and (Y_2, ψ_2) be two minimal finite skew products of circular cascades. Then (Y_1, ψ_1) and (Y_2, ψ_2) are isomorphic if and only if*

$$n(\psi_1) = n(\psi_2)$$
$$r(\psi_1) = r(\psi_2)$$
$$T(\psi_1) = T(\psi_2)$$
$$W(\psi_1) = W(\psi_2) .$$

Proof. If they are isomorphic, then the equations hold because in the course of this section we have established that $n(\psi_i)$, $r(\psi_i)$, $T(\psi_i)$, and $W(\psi_i)$ depend only on the isomorphism class of (Y_i, ψ_i).

Assume the equations hold. Let (X, ϕ) be a circular cascade determined by $r(\psi_1) = r(\psi_2)$ and $T(\psi_1) \equiv T(\psi_2)$. Then $W = W(\psi_1) = W(\psi_2)$ can be realized in $C(X, S_n)$ where $n = n(\psi_1) = n(\psi_2)$ (Theorem 5.2). Let $F \in W$. Then (Y_1, ψ_1) and (Y_2, ψ_2) are both isomorphic to (X_n, ϕ_F).

6. **Normal representatives of cohomology classes.** Let (C, ϕ) denote a fixed circular cascade where C is a Cantor set in K and ϕ is the restriction of a homeomorphism of K without periodic points to its unique minimal subset C. Observe that if (a, b) is a complementary interval of C, then so is $(\phi(a), \phi(b))$. A cohomology class $\mathscr{C}$ is *minimal* if $F \in \mathscr{C}$ implies (C_n, ϕ_F) is minimal. We want to investigate the minimal cohomology classes of (C, ϕ) for the group S_n. (Note that we already know that there is only one minimal cohomology class of (K, M_g) for the group S_n.)

DEFINITION. Let $F: C \to S_n$ continuously and let (a_i, b_i), $i = 1, \cdots, p$ denote the complementary intervals of C such that $F(a_i) \neq F(b_i)$. We

say F is *normal* if for $n > 0$ and $1 \leqq i, j \leqq p$

$$(\phi^n(a_i), \phi^n(b_i)) \neq (a_j, b_j)$$

with at most one exception of the form

$$(\phi(a_i), \phi(b_i)) = (a_j, b_j) , \quad i \neq j .$$

Let $\mathscr{C}$ be a cohomology class of (C, ϕ) for S_n. We say F is a normal representative of $\mathscr{C}$ if F is in $\mathscr{C}$ and F is normal. One of the advantages of normality is the existence of a simple test for minimality.

THEOREM 6.1. *Let F be normal. Then (C_n, ϕ_F) is minimal if and only if the subgroup of S_n generated by the range of F acts transitively on $\{1, \cdots, n\}$.*

Proof. The subgroup of H of S_n generated by the range of F consists of all finite products of the form $F(x_1)F(x_2) \cdot \cdots \cdot F(x_n)$. It is clear that if (C_n, ϕ_F) is minimal, then H acts transitively on $\{1, \cdots, n\}$. (This observation does not require the normality of F. Just examine the orbit closure of any point in (C_n, ϕ_F).)

Suppose F is normal and H acts transitively on $\{1, \cdots, n\}$. If for each $i, 1 \leqq i \leqq n$, $C \times \{i\}$ is contained in one minimal set of (C_n, ϕ_F), then $C \times \{i\}$ and $C \times \{F(x)i\}$ are in the same minimal set for all x in C. Consequently it suffices to prove that $C \times \{i\}$ is contained in one minimal subset of (C_n, ϕ_F). This is where the normality of F is needed. Let (a_j, b_j) $j = 1, \cdots, p$ be the complementary intervals of C such that $F(a_i) \neq F(b_i)$. If the allowed exception occurs, we can assume by renumbering that

$$(\phi(a_2), \phi(b_2)) = (a_1, b_1) .$$

Let (a, b) be any complementary interval of C except (a_1, b_1). Then it is clear that $\phi_F^m(a, i)$ and $\phi_F^m(b, i)$ belong to the same $C \times \{j(m)\}$ either for all $m > 0$ or for all $m \leqq 0$ when $1 \leqq j(m) \leqq n$ because F is normal. Therefore, they are asymptotic in one sense and in the same minimal set. Since the minimal sets of (C_n, ϕ_F) are open and closed in C_n, it follows that, for each i, $C \times \{i\}$ is contained in one minimal subset of (C_n, ϕ_F).

We will show that every minimal cohomology class has a normal representative and that the complementary intervals (a_i, b_i) can be specified. Let F be a fixed continuous map from C to S_n such that (C_n, ϕ_F) is minimal. We will proceed by taking C_n apart and reassembling it.

DEFINITION. A finite collection of complementary intervals will be called *admissible* if they satisfy the following conditions:

1. If $F(a) \neq F(b)$ for some complementary interval (a, b), then (a, b) is in $\mathscr{A}$.

2. If (a, b) and $(\phi^n(a), \phi^n(b))$ are in $\mathscr{A}$ and $n > 0$, then $(\phi^k(a), \phi^k(b))$ is in $\mathscr{A}$ for $k = 1, \cdots, n - 1$.

Henceforth, $\mathscr{A}$ will be an admissible collection of complementary intervals. We will also assume that $\mathscr{A} = \{(a_1, b_1), \cdots, (a_q, b_q)\}$ is indexed counterclockwise and (a_1, b_1) has been chosen so that $(\phi^n(a_1), \phi^n(b_1)) \notin \mathscr{A}$ for all $n > 0$. Let $I_i = [b_i, a_{i+1}] \cap C$ if $i < q$ and $I_q = [b_q, a_1] \cap C$. Define L_1 to be the identity element of S_n and $L_i = A_-(a_i, b_i)L_{i-1}$. Now let

$$M_j = \bigcup_{i=1}^{q} I_i \times \{L_i j\} .$$

The proof of the next lemma is straight forward and is left to the reader.

LEMMA 6.2. *The following statements are valid:*

(a) $M_i \cap M_j = \varnothing$ *when* $i \neq j$.

(b) $C_n = \bigcup_{i=1}^{n} M_i$.

(c) *The map* θ *defined by* $\theta(x, k) = (x, j)$ *if* $(x, k) \in M_j$ *is a homeomorphism of* C_n *onto* C_n.

The next step is to determine $G: C \to S_n$ such that $\theta \circ \phi_F \circ \theta^{-1} = \phi_G$ and then investigate the intervals on which G is constant. From Corollary 5.4 we know that F and G are cohomologous. So G is a representative of the cohomology class of F. Define G by $G(x)i = j$ if $\phi_F(\theta^{-1}(x, i)) \in M_j$. It is easy to check that $G(x)$ is injective on $\{1, \cdots, n\}$ and hence an element of S_n. If $\phi_F(\theta^{-1}(x, i)) \in M_j$, then $G(x)i = j$ and $\theta \circ \phi_F \circ \theta^{-1}(x, i) = (\phi(x), j) = (\phi(x), G(x)i) = \phi_G(x, i)$. Therefore, $\theta \circ \phi_F \circ \theta^{-1} = \phi_G$. Notice we did not establish the continuity of G, but this follows from the continuity of ϕ_G which we know because $\phi_G = \theta \circ \phi_F \circ \theta^{-1}$.

LEMMA 6.3. *The function* G *constructed as above is normal.*

Proof. We will present only an outline of the proof leaving the details to the reader.

The first step is to show that if (a, b) is a complementary interval of C such that $(a, b) \notin \mathscr{A}$ and $(\phi(a), \phi(b)) \notin \mathscr{A}$, then $G(a) = G(b)$.

Next suppose that (a, b) is a complementary interval such that $(a, b) \notin \mathscr{A}$ and $(\phi^m(a), \phi^m(b)) = (a_{j(m)}, b_{j(m)}) \in \mathscr{A}$ where $j(m) = 2, \cdots, p$ and $1 \leq m \leq N$. The goal here is to show that $G(\phi^m(a)) = G(\phi^m(b))$ for $0 \leq m \leq N - 1$. It suffices to show that $G(a, m) = G(b, m)$ for $0 \leq m \leq N$ because $G(\phi^m(x)) = G(x, m + 1)G(x, m)^{-1}$. Observe that

$G(x, m)k = k'$ if and only if $\phi_F^m \circ \theta^{-1}(x, k)M_{k'}$ and that (a, i) and (b, i) are negatively asymptotic for $1 \leq i \leq m$. This part can now be completed using the definition of M_k.

From the above it follows that $G(a) \neq G(b)$ implies $(a, b) \in \mathscr{A}$ and $(\phi(a), \phi(b)) \notin \mathscr{A}$ or $(\phi(a), \phi(b)) = (a_1, b_1)$ by eliminating the other possibilities. In other words, G is normal.

THEOREM 6.4. *Every minimal cohomology class of (C, ϕ) for S_n has a normal representative.*

Proof. Let $\mathscr{C}$ be a minimal cohomology class of (C, ϕ) for S_n and let $F \in \mathscr{C}$. Then clearly we can find an admissible collection of complementary intervals for F and construct G cohomologous to F as above. By Lemma 6.3 G is normal.

Let F be a normal representative of $\mathscr{C}$, and let $J(F) = \{(a_1, b_1), \cdots, (a_q, b_q)\}$ be the complementary intervals of C satisfying

(a) $F(a_i) \neq F(b_i)$, $i = 1, \cdots, q$.

(b)   If (a, b) is any complementary interval for which $F(a) \neq F(b)$, then either $(a, b) = (a_i, b_i)$ for some i in $\{1, \cdots, q\}$ or $(\phi(a), \phi(b)) = (a_1, b_1)$ but not both.

(c)   They are arranged counterclockwise in C. We will refer to $J(F)$ as the jumps of F and when it happens that $F(a) \neq F(b)$ and $(\phi(a), \phi(b)) = (a_1, b_1)$ we will say that F has an exceptional jump at (a, b).

THEOREM 6.5. *Let F be a normal representative of the minimal cohomology class $\mathscr{C}$. Let $\mathscr{A} = \{(c_1, d_1), \cdots, (c_p, d_p)\}$ be a collection of complementary intervals of C numbered counterclockwise and satisfying the following conditions:*

(a) $\phi(c_i, d_i) \neq (c_j, d_j)$ *for all $n > 0, 1 \leq i, j \leq p$.*

(b)   *For each i there exists n such that $F(\phi^n(c_i)) \neq F(\phi^n(d_i))$.*

(c)   *For each complementary interval (a, b) such that $F(a) \neq F(b)$, there exists n and i such that $(\phi^n(a), \phi^n(b)) = (c_i, d_i)$.*
Then there exists G a normal representative of $\mathscr{C}$ such that

$$J(G) \subset \{(c_1, d_1), \cdots, (c_p, d_p)\} \, .$$

Proof. First suppose that for each (a_i, b_i) the integer $n(i)$ such that $(\phi^{n(i)}(a_i), \phi^{n(i)}(b_i)) = (c_j, d_j)$ for some j is positive. Here we can apply the construction of G preceding Lemma 6.3 to $\mathscr{A}' = \{(\phi^k(a_i), \phi^k(b_i)): 0 \leq k \leq n(i)\} \cup \{(\phi^{-1}(a_1), \phi^{-1}(b_1))\} \supset J(F)$, and if we start numbering $\mathscr{A}'$ with (c_1, d_1) we get $J(G) \subset \{(c_1, d_1), \cdots, (c_p, d_p)\}$. In the general case we first construct an F' cohomologous to F to which the above argu-

ment applies. We obtain F'' by applying our initial argument to $F(\phi^{-1}(x))^{-1}$ which determines ϕ_F^{-1}.

COROLLARY 6.6. *If there exist complementary intervals* (a_1, b_1), $\cdots$, (a_p, b_p) *such that every complementary interval* (a, b) *has the form* $(\phi^k(a_i), \phi^k(b_i))$ *for some i and k, then there are only a finite number of minimal cohomology class of* (C, ϕ) *for* S_n.

Proof. For every minimal cohomology class $\mathscr{C}$ there exists $F \in \mathscr{C}$ such that $J(F) \subset \{(a_1, b_1), \cdots, (a_p, b_p)\}$.

The problem of computing the number of minimal cohomology classes for a given p and n will be the topic of a future paper with William N. Anderson, Jr.

The indexing of $J(F)$ is only crucial when there is an exceptional jump; i.e., there exists a complementary (a, b) such that $F(a) \neq F(b)$ and $F(\phi^{-1}(a)) \neq F(\phi^{-1}(b))$. In this case we always want $(a, b) = (a_1, b_1)$. When this is not the case we change the indexing freely, but for convenience we always index counterclockwise. We will write $J(F) \sim J(G)$ provided $J(F)$ and $J(G)$ consist of the same complementary intervals; and if both F and G have an exceptional jump, it occurs at the same complementary interval for both of them. We will say F and G are *conjugate* if there exists $\sigma \in S_n$ such that $F(x) = \sigma^{-1}G(x)\sigma$ for all x in C. The second advantage of normal functions in the following:

THEOREM 6.7. *Let* F *and* G *be normal representatives of minimal cohomology classes. If* $J(F) \sim J(G)$, *then* F *and* G *are cohomologous if and only if* F *and* G *are conjugate.*

Proof. Let $\alpha: C \to S_n$ continuously such that $G(x) = \alpha(\phi(x))^{-1}F(x)\alpha(x)$ and let (a, b) be a complementary interval such that $\alpha(a) \neq \alpha(b)$ and $(a, b) \neq (a_1, b_1)$ where $J(F) = \{(a_1, b_1), \cdots, (a_q, b_q)\}$. Then $F(\phi^n(a)) = F(\phi^n(b))$ for either all $n \geq 0$ or all $n < 0$. In the first case it follows that $\alpha(\phi(a)) \neq \alpha(\phi(b))$ and hence by induction $\alpha(\phi^n(a)) \neq \alpha(\phi^n(b))$ for all $n \geq 0$ which contradicts the continuity of α. In the second case we obtain the contradiction by using $G(\phi^{-1}(x)) = \alpha(x)^{-1}F(\phi^{-1}(x))\alpha(\phi^{-1}(x))$. Thus $\alpha(a) = \alpha(b)$ unless $(a, b) = (a_1, b_1)$. Since α is finitely valued, α is not constant if and only if there exist two complementary intervals with $\alpha(a) \neq \alpha(b)$. Therefore, α is constant and F and G are conjugate.

Let F be a normal representative of a minimal cohomology class. The permutations $A_+(a, b)$ and $A_-(a, b)$ are computable. Let $J(F) = \{(a_1, b_1), \cdots, (a_p, b_p)\}$, and let (a, b) be a complementary interval. If $(\phi^n(a), \phi^n(b)) \notin J(F)$ for all n, then clearly $A_+(a, b) = e$ and $A_-(a, b) =$

e. If $(\phi^n(a), \phi^n(b)) = (a_i, b_i)$ where $n \neq 0$, then Lemma 2.1 (c) can be applied and we get

$$A_+(a, b) = F(b_i, -n)A_+(a_i, b_i)F(a_i, -n)^{-1}$$

and

$$A_-(a, b) = F(b_i, -n)A_-(a_i, b_i)F(a_i, -n)^{-1}.$$

Now consider $(a_i, b_i) \in J(F)$. Then $A_+(a_i, b_i) = F(b_i)^{-1}F(a_i)$ and $A_-(a_i, b_i) = e$ for $i > 1$. If there is no exceptional jump, then $A_-(a_1, b_1) = e$. When there is an exceptional jump

$$A_-(a_1, b_1) = F(\phi^{-1}(b_1))F(\phi^{-1}(a_1))^{-1}$$

because $A_-(\phi^{-1}(a_1), \phi^{-1}(b_1)) = e$. It follows that for $i > 1$, (a_i, j) and (b_i, j) are doubly asymptotic if and only if $A_+(a_i, b_i)j = j$. But this is certainly not ture for all j, and for some j the proximal cell of (a_i, j) contains two points which are not proximal. Thus if $J(F)$ contains more than one complementary interval, (C, ϕ_F) is not locally almost periodic. When F does not have an exceptional jump the same comments can be made about (a_1, j). If there is an exceptional jump, then $(\phi^{-1}(a_1), j)$ and $(\phi^{-1}(b_1), j)$ are boubly asymptotic if and only if $F(\phi^{-1}(b_1))^{-1}F(b_1)^{-1}F(a_1)F(\phi^{-1}(a_1))j = j$. We will say (C_n, ϕ_F) is *trivial* if there exists a circular cascade (C', ϕ') and $r > 0$ such that (C_n, ϕ_F) is isomorphic to (C'_r, ϕ'_G) where $G(x) = (1, 2, \cdots, r)$ for all x.

THEOREM 6.8. *Let F be a normal representative of a minimal cohomology class. Then (C_n, ϕ_F) is trivial if and only if $|J(F)| = 1$ and $F(x)F(y) = F(y)F(x)$ for all x and y in C.*

Proof. First suppose (C_n, ϕ_F) is trivial. It follows that it is locally almost periodic and $(\phi^{-1}(a_i), j)$ and $(\phi^{-1}(b_i), j)$ are doubly asymptotic for all j. Consequently $|J(F)| = 1$ and $F(\phi^{-1}(b_1))^{-1}F(b_1)^{-1}F(a_1)F(\phi^{-1}(a_1)) = 1$. Since $|J(F)| = 1$, $F(x)$ takes on only two values, $F(b_1) = F(\phi^{-1}(a_1))$, and $F(a_1) = F(\phi^{-1}(b_1))$.

Now suppose $|J(F)| = 1$ and $F(x)F(y) = F(y)F(x)$ for all x and y. Hence (a, j) and (b, j) are doubly asymptotic where (a, b) is any complementary interval except (a_1, b_1) which is the only complementary interval in $J(F)$. It follows that $P' \subset \Delta$. If we can find N such that the minimal sets of (C_n, ϕ_F^N) are circular cascade, we will be done. Let $N = c(\phi_F)$. We can apply Theorem 4.1 [5] to finish the proof provided (C_n, ϕ_F) and (C_n, ϕ_F^N) have the same equicontinuous structure relation. Because (C_n, ϕ_F) is minimal we know that its equicontinuous structure relation coincides with its regionally proximal relation i.e., $E(\phi_F) = Q(\phi_F)$ [6, Theorem 1.1 and Theorem 1.2 or 4, Theorem 2.12].

Since we know that $Q(\phi_F) = Q(\phi_F^N) \subset E(\phi_F^N) \subset E(\phi_F)$, it follows that $E(\phi_F) = E(\phi_F^N)$.

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